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On the refinements of some important inequalities via (p,q)-calculus and their applications

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Abstract

We establish some interesting refinements of the (p,q)-Hölder integral inequality and the (p,q)-power-mean integral inequality. As applications, we show that some existing (p,q)-integral inequalities can be improved by the results obtained in this paper.

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1 Introduction and preliminaries

Let us recall the definition of convex mappings in the classic sense: A mapping $f:\mathcal{I}\subseteq\mathbb{R}^n\to\mathbb{R}$ is called a convex mapping if for all $\upsilon_1,\upsilon_2\in\mathcal{I}$ and $t\in[0,1]$,

$$f\big(t\upsilon_1+(1-t)\upsilon_2\big)\leq tf(\upsilon_1)+(1-t)f(\upsilon_2).$$

Based on the convexity of mappings, many mathematicians have established different classes of inequalities, such as the Hardy-type inequality [13], Ostrowski-type inequality [4], midpoint-type inequality [5], trapezoidal-type inequality [29], Simpson-type inequality [32], and so on, among which the most famous is the Hermite–Hadamard inequality

$$f\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\leq\frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2}f(t)\,\mathrm{d}t\leq\frac{f(\upsilon_1)+f(\upsilon_2)}{2},\tag{1.1}$$

where $f:\mathcal{K}\subseteq\mathbb{R}\to\mathbb{R}$ is a convex mapping, and $\upsilon_1,\upsilon_2\in\mathcal{K}$ with $\upsilon_1<\upsilon_2$.

A number of interesting generalizations of (1.1) have been proposed in the theory of mathematical inequalities. For instance, see [7–9, 18, 19, 34, 35, 37] and the references therein. The application of (1.1) to the error estimates for interpolatory approximation and approximate multivariate integration can be found in [10–12]. An important mathematical tool concerning inequalities related to convex mappings is the Hölder inequality, which is also used widely in many other disciplines of applied mathematics.

Theorem 1.1 (Hölder inequality for sums [21]) *Let* $a = (a_1, a_2, \dots, a_n)$ *and* $b = (b_1, b_2, \dots, b_n)$ *be two positive n-tuples, and let* $\rho, \sigma > 1$ *with* $\rho^{-1} + \sigma^{-1} = 1$. *Then we have*

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^{\rho}\right)^{\frac{1}{\rho}} \left(\sum_{i=1}^{n} b_i^{\sigma}\right)^{\frac{1}{\sigma}}.$$

Theorem 1.2 (Hölder inequality for integrals [21]) *Let* $\sigma > 1$ *and* $\rho^{-1} + \sigma^{-1} = 1$. *If* f *and* w *are real mappings defined on* $[\upsilon_1, \upsilon_2]$ *and if* $|w|^{\rho}$ *and* $|f|^{\sigma}$ *are integrable on* $[\upsilon_1, \upsilon_2]$, *then*

$$\int_{\upsilon_1}^{\upsilon_2} \left|w(x)f(x)\right| \,\mathrm{d}x \le \left(\int_{\upsilon_1}^{\upsilon_2} \left|w(x)\right|^{\rho} \,\mathrm{d}x\right)^{\frac{1}{\rho}} \left(\int_{\upsilon_1}^{\upsilon_2} \left|f(x)\right|^{\sigma} \,\mathrm{d}x\right)^{\frac{1}{\sigma}}.$$

A different form of Hölder inequality was given as follows.

Theorem 1.3 (Power-mean integral inequality) *Let* $\sigma \ge 1$. *If* f *and* w *are real mappings defined on* $[\upsilon_1, \upsilon_2]$ *and if* $|w|$ *and* $|w||f|^{\sigma}$ *are integrable on* $[\upsilon_1, \upsilon_2]$, *then*

$$\int_{\upsilon_1}^{\upsilon_2} \left|w(x)f(x)\right| \,\mathrm{d}x \le \left(\int_{\upsilon_1}^{\upsilon_2} \left|w(x)\right| \mathrm{d}x\right)^{1-\frac{1}{\sigma}} \left(\int_{\upsilon_1}^{\upsilon_2} \left|w(x)\right| \left|f(x)\right|^{\sigma} \,\mathrm{d}x\right)^{\frac{1}{\sigma}}.$$

In 2019, İşcan [14] established an improved version of the Hölder inequality.

Theorem 1.4 (Hölder–İşcan integral inequality) *Let* $\sigma > 1$ *and* $\rho^{-1} + \sigma^{-1} = 1$. *If* f *and* w *are real mappings defined on* $[\upsilon_1, \upsilon_2]$ *and if* $|w|^{\rho}$ *and* $|f|^{\sigma}$ *are integrable on* $[\upsilon_1, \upsilon_2]$, *then*

$$\begin{aligned}
\int_{\upsilon_1}^{\upsilon_2} \left|w(x)f(x)\right| \,\mathrm{d}x \le \frac{1}{\upsilon_2 - \upsilon_1} \Bigg\{ &\left(\int_{\upsilon_1}^{\upsilon_2} (\upsilon_2 - x)\left|w(x)\right|^{\rho} \,\mathrm{d}x\right)^{\frac{1}{\rho}} \left(\int_{\upsilon_1}^{\upsilon_2} (\upsilon_2 - x)\left|f(x)\right|^{\sigma} \,\mathrm{d}x\right)^{\frac{1}{\sigma}} \\
&+ \left(\int_{\upsilon_1}^{\upsilon_2} (x - \upsilon_1)\left|w(x)\right|^{\rho} \,\mathrm{d}x\right)^{\frac{1}{\rho}} \left(\int_{\upsilon_1}^{\upsilon_2} (x - \upsilon_1)\left|f(x)\right|^{\sigma} \,\mathrm{d}x\right)^{\frac{1}{\sigma}} \Bigg\} \\
\le &\left(\int_{\upsilon_1}^{\upsilon_2} \left|w(x)\right|^{\rho} \,\mathrm{d}x\right)^{\frac{1}{\rho}} \left(\int_{\upsilon_1}^{\upsilon_2} \left|f(x)\right|^{\sigma} \,\mathrm{d}x\right)^{\frac{1}{\sigma}}.
\end{aligned}$$

In [15], a different version of Hölder–İşcan inequality was provided as follows.

Theorem 1.5 (Improved power-mean integral inequality) *Let* $\sigma \ge 1$. *If* f *and* w *are real mappings defined on* $[\upsilon_1, \upsilon_2]$ *and if* $|w|$ *and* $|w||f|^{\sigma}$ *are integrable on* $[\upsilon_1, \upsilon_2]$, *then*

$$\begin{aligned}
&\int_{\upsilon_1}^{\upsilon_2} \left|w(x)f(x)\right| \,\mathrm{d}x \\
&\quad \le \frac{1}{\upsilon_2 - \upsilon_1} \Bigg\{ \left(\int_{\upsilon_1}^{\upsilon_2} (\upsilon_2 - x)\left|w(x)\right| \,\mathrm{d}x\right)^{1-\frac{1}{\sigma}} \left(\int_{\upsilon_1}^{\upsilon_2} (\upsilon_2 - x)\left|w(x)\right|\left|f(x)\right|^{\sigma} \,\mathrm{d}x\right)^{\frac{1}{\sigma}} \\
&\quad\quad + \left(\int_{\upsilon_1}^{\upsilon_2} (x - \upsilon_1)\left|w(x)\right| \mathrm{d}x\right)^{1-\frac{1}{\sigma}} \left(\int_{\upsilon_1}^{\upsilon_2} (x - \upsilon_1)\left|w(x)\right|\left|f(x)\right|^{\sigma} \,\mathrm{d}x\right)^{\frac{1}{\sigma}} \Bigg\} \\
&\quad \le \left(\int_{\upsilon_1}^{\upsilon_2} \left|w(x)\right| \,\mathrm{d}x\right)^{1-\frac{1}{\sigma}} \left(\int_{\upsilon_1}^{\upsilon_2} \left|w(x)\right|\left|f(x)\right|^{\sigma} \,\mathrm{d}x\right)^{\frac{1}{\sigma}}.
\end{aligned}$$

In the rest of this section, we review some preliminaries of (p,q)-calculus. Throughout this paper, let $[a,b] \subset \mathbb{R}$, and let p, q be two constants such that $0 < q < p \leq 1$. The existence of (p,q)-derivatives and (p,q)-integrals is required, and the convergence of the corresponding series mentioned later in the proofs are assumed. Now we recall the theory of (p,q)-calculus. These concepts and results related to the (p,q)-derivative and (p,q)-integral are mainly due to Tunç and Göv [30].

Definition 1.1 ([30]) Let $f : [a,b] \to \mathbb{R}$ be a continuous mapping. The (p,q)-derivative of f at $x \in [a,b]$ is defined as

$$ {}_aD_{p,q}f(x) = \frac{f(px+(1-p)a) - f(qx+(1-q)a)}{(p-q)(x-a)}, \quad x \neq a, \qquad {}_aD_{p,q}f(a) = \lim_{x\to a} {}_aD_{p,q}f(x). $$

Example 1.1 Define the mapping $f : [a,b] \to \mathbb{R}$ by $f(x) = x^2$. Let $0 < q < p \leq 1$. For $x \neq a$, we have

$$ \begin{aligned} {}_aD_{p,q}x^2 &= \frac{(px+(1-p)a)^2 - (qx+(1-q)a)^2}{(p-q)(x-a)} \\ &= \frac{(p+q)x^2 + (1-(p+q))2ax + (p+q-2)a^2}{x-a} \\ &= (p+q)x + (2-p-q)a. \end{aligned} $$

For $x = a$, we have ${}_aD_{p,q}a^2 = \lim_{x\to a}({}_aD_{p,q}x^2) = 2a$.

Definition 1.2 ([30]) Let $f : [a,b] \to \mathbb{R}$ be a continuous mapping. If ${}_aD_{p,q}f$ is (p,q)-differentiable on $[a,b]$, then the second-order (p,q)-derivative of f is defined as ${}_aD^2_{p,q}f$ together with ${}_aD_{p,q}({}_aD_{p,q}f) : [a,b] \to \mathbb{R}$. Similarly, the higher-order (p,q)-derivatives of f are defined as ${}_aD^n_{p,q}f : [a,b] \to \mathbb{R}$.

Considering Example 1.1 again, for $x \neq a$, we have

$$ \begin{aligned} {}_aD^2_{p,q}x^2 &= \frac{{}_aD_{p,q}(px+(1-p)a)^2 - {}_aD_{p,q}(qx+(1-q)a)^2}{(p-q)(x-a)} \\ &= \frac{q(p^2x+(1-p^2)a)^2 - (p+q)(pqx+(1-pq)a)^2 + p(q^2x+(1-q^2)a)^2}{pq(p-q)^2(x-a)^2} \\ &= p+q. \end{aligned} $$

Again, for $x = a$, we have ${}_aD^2_{p,q}a^2 = \lim_{x\to a}({}_aD^2_{p,q}x^2) = p+q$.

Definition 1.3 ([30]) Let $f : [a,b] \to \mathbb{R}$ be a continuous mapping. The (p,q)-integral on $[a,b]$ is defined as

$$ \int_a^x f(t)\, {}_a\mathrm{d}_{p,q}t = (p-q)(x-a)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}x + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right) $$

for $x \in [a,b]$. Moreover, if $c \in (a,x)$, then the (p,q)-integral on $[c,x]$ is defined as

$$ \int_c^x f(t)\, {}_a\mathrm{d}_{p,q}t = \int_a^x f(t)\, {}_a\mathrm{d}_{p,q}t - \int_a^c f(t)\, {}_a\mathrm{d}_{p,q}t. $$

Example 1.2 Define the mapping $f : [a,b] \to \mathbb{R}$ by $f(x) = x - \lambda$ with a constant $\lambda \in [a,b]$. Let $0 < q < p \leq 1$. Then

$$\begin{aligned}\int_\lambda^b (x-\lambda)\,{}_a\mathrm{d}_{p,q}x &= \int_a^b (x-\lambda)\,{}_a\mathrm{d}_{p,q}x - \int_a^\lambda (x-\lambda)\,{}_a\mathrm{d}_{p,q}x \\ &= (p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}b + \left(1-\frac{q^n}{p^{n+1}}\right)a - \lambda\right) \\ &\quad - (p-q)(\lambda-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}\lambda + \left(1-\frac{q^n}{p^{n+1}}\right)a - \lambda\right) \\ &= \frac{(b+(p+q-1)a-(p+q)\lambda)(b-a)+(p+q-1)(\lambda-a)^2}{p+q}.\end{aligned}$$

Note that when $p = 1$ and $q \to 1^-$, the above integral reduces to the classic integral

$$\int_\lambda^b (t-\lambda)\,\mathrm{d}t = \frac{(b-\lambda)^2}{2}.$$

Theorem 1.6 ([30]) *Let $f, g : [a,b] \to \mathbb{R}$ be two continuous mappings. Then*

$$\int_a^x \bigl|f(u)\bigr|\bigl|g(u)\bigr|\,{}_a\mathrm{d}_{p,q}u \leq \left(\int_a^x \bigl|f(u)\bigr|^\rho\,{}_a\mathrm{d}_{p,q}u\right)^{\frac{1}{\rho}}\left(\int_a^x \bigl|g(u)\bigr|^\sigma\,{}_a\mathrm{d}_{p,q}u\right)^{\frac{1}{\sigma}}$$

for all $x \in [a,b]$ and $\rho, \sigma > 1$ with $\rho^{-1} + \sigma^{-1} = 1$.

In 2018, Kunt et al. [17]generalized the Hermite–Hadamard inequality to (p,q)-integrals as follows.

Theorem 1.7 ([17]) *Let $f : [a,b] \to \mathbb{R}$ be convex and (p,q)-differentiable on $[a,b]$. Then we have*

$$f\left(\frac{qa+pb}{p+q}\right) \leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\,{}_a\mathrm{d}_{p,q}x \leq \frac{qf(a)+pf(b)}{p+q}. \qquad (1.2)$$

For more details on the (p,q)-integrals, we refer the interested readers to [2, 16, 27, 31]. Note that if we take $p = 1$ in Theorem 1.7, then we have the q-Hermite–Hadamard inequality; for more detail, see [20, 22, 23, 25, 26]. Besides, we are also directed to some recent work related to other type quantum integral inequalities; see, for instance, [1, 3, 6, 24, 28, 33, 36] and the references therein.

This paper is mainly devoted to investigating (p,q)-integral inequalities via (p,q)-calculus. For this purpose, we extend some of important integral inequalities of analysis to (p,q)-calculus such as Hermite–Hadamard, Hölder, and the power-mean integral inequalities. To present some applications of our main results, we establish an identity to express the difference between the middle part and the right-hand side of the analogue of (p,q)-Hermite–Hadamard inequality (1.2). Based on this identity, we give several estimates for (p,q)-integral inequalities via convexity. Meanwhile, we compare some of the derived results in this paper, in an interesting way, with the known works.

2 Main results

In this section, we establish several integral inequalities concerning the quantum version of (p,q)-Hermite–Hadamard inequality, the improved (p,q)-Hölder–İşcan integral inequality, and the refinement of (p,q)-power-mean integral inequality. The first result is as follows.

Theorem 2.1 *Let $f:[a,b]\to\mathbb{R}$ be convex and (p,q)-differentiable on $[a,b]$, and let m be an integer. Then we have*

$$\begin{aligned} f\left(\frac{(p+q-p^m)a+p^m b}{p+q}\right) &\le \frac{1}{p^m(b-a)}\int_a^{p^m b+(1-p^m)a} f(x)\,{}_a\mathrm{d}_{p,q}x \\ &\le \frac{(p+q-p^m)f(a)+p^m f(b)}{p+q}. \end{aligned} \tag{2.1}$$

Proof Using the identity $\sum_{n=0}^{\infty}(1-\frac{q}{p})\frac{q^n}{p^n}=1$, $0<q<p\le 1$, Jensen's inequality for infinite sums, and Definition 1.3, we have that

$$\begin{aligned} &f\left(\frac{(p+q-p^m)a+p^m b}{p+q}\right) \\ &\quad= f\left(\sum_{n=0}^{\infty}\left(1-\frac{q}{p}\right)\frac{q^n}{p^n}\left(\frac{q^n}{p^{n-m+1}}b+\left(1-\frac{q^n}{p^{n-m+1}}\right)a\right)\right) \\ &\quad\le \sum_{n=0}^{\infty}\left(1-\frac{q}{p}\right)\frac{q^n}{p^n}f\left(\frac{q^n}{p^{n-m+1}}b+\left(1-\frac{q^n}{p^{n-m+1}}\right)a\right) \\ &\quad= \frac{1}{p^m(b-a)}\int_a^{p^m b+(1-p^m)a} f(x)\,{}_a\mathrm{d}_{p,q}x. \end{aligned}$$

Using Definition 1.3 and the convexity of f, we get

$$\begin{aligned} &\frac{1}{p^m(b-a)}\int_a^{p^m b+(1-p^m)a} f(x)\,{}_a\mathrm{d}_{p,q}x \\ &\quad= \sum_{n=0}^{\infty}\left(1-\frac{q}{p}\right)\frac{q^n}{p^n}f\left(\frac{q^n}{p^{n-m+1}}b+\left(1-\frac{q^n}{p^{n-m+1}}\right)a\right) \\ &\quad\le \sum_{n=0}^{\infty}\left(1-\frac{q}{p}\right)\frac{q^n}{p^n}\left(\frac{q^n}{p^{n-m+1}}f(b)+\left(1-\frac{q^n}{p^{n-m+1}}\right)f(a)\right) \\ &\quad= \frac{(p+q-p^m)f(a)+p^m f(b)}{p+q}. \end{aligned}$$

The proof is completed. □

Remark 2.1 If we take $m=1$ in Theorem 2.1, then we obtain Theorem 1.7 established by Kunt et al. [17].

Example 2.1 Let $f(x) = x^2$ on $[1,4]$. Applying Theorem 2.1 with $a = 1$, $b = 4$, $p = \frac{1}{2}$, $q = \frac{1}{4}$, and $m = 3$, the left-hand side of (2.1) becomes

$$\begin{aligned}
&f\left(\frac{(p+q-p^m)a+p^m b}{p+q}\right)-\frac{1}{p^m(b-a)}\int_a^{p^m b+(1-p^m)a} f(x)\,{}_a\mathrm{d}_{p,q}x \\
&=\left(\frac{(\frac{1}{2}+\frac{1}{4}-\frac{1}{2^3})+\frac{4}{2^3}}{\frac{1}{2}+\frac{1}{4}}\right)^2 \\
&\quad-\frac{1}{\frac{3}{2^3}}\times\left(\frac{1}{2}-\frac{1}{4}\right)\times(4-1)\times 2^3\sum_{n=0}^{\infty}\frac{4^{-n}}{2^{-n-1}}\left(\frac{4^{-n}}{2^{-n+3-1}}\times 3+1\right)^2 \\
&=\frac{9}{4}-\frac{65}{28}=-\frac{1}{14}<0.
\end{aligned}$$

For the right-hand side of (2.1), we have

$$\begin{aligned}
&\frac{1}{p^m(b-a)}\int_a^{p^m b+(1-p^m)a} f(x)\,{}_a\mathrm{d}_{p,q}x-\frac{(p+q-p^m)f(a)+p^m f(b)}{p+q} \\
&=\frac{1}{\frac{3}{2^3}}\times\left(\frac{1}{2}-\frac{1}{4}\right)\times(4-1)\times 2^3\sum_{n=0}^{\infty}\frac{4^{-n}}{2^{-n-1}}\left(\frac{4^{-n}}{2^{-n+3-1}}\times 3+1\right)^2 \\
&\quad-\frac{(\frac{1}{2}+\frac{1}{4}-\frac{1}{2^3})+\frac{1}{2^3}\times 4^2}{\frac{1}{2}+\frac{1}{4}} \\
&=\frac{65}{28}-\frac{7}{2}=-\frac{33}{28}<0.
\end{aligned}$$

We next establish an important refinement of the (p,q)-Hölder inequality.

Theorem 2.2 ((p,q)-Hölder–İşcan integral inequality) *Let $\gamma_1 > 1$ and $\gamma_1^{-1} + \gamma_2^{-1} = 1$. If f and w are real mappings on $[a,b]$ such that $|f|^{\gamma_1}$ and $|w|^{\gamma_2}$ are integrable on $[a,b]$, then*

$$\begin{aligned}
&\int_a^b \left|w(x)f(x)\right|\,{}_a\mathrm{d}_{p,q}x \\
&\le\frac{1}{b-a}\left\{\left(\int_a^b (b-x)\left|f(x)\right|^{\gamma_1}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}\left(\int_a^b (b-x)\left|w(x)\right|^{\gamma_2}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_2}}\right. \\
&\quad\left.+\left(\int_a^b (x-a)\left|f(x)\right|^{\gamma_1}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}\left(\int_a^b (x-a)\left|w(x)\right|^{\gamma_2}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_2}}\right\} \\
&\le\left(\int_a^b \left|f(x)\right|^{\gamma_1}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}\left(\int_a^b \left|w(x)\right|^{\gamma_2}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_2}}.
\end{aligned}\tag{2.2}$$

Proof First, by Definition 1.3 and the discrete Hölder inequality we obtain

$$\begin{aligned}
&\int_a^b \left|w(x)f(x)\right|\,{}_a\mathrm{d}_{p,q}x \\
&=(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|\left|w\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{b-a}\Biggl\{(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)(b-a) \\
&\quad \times \left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|\left|w\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right| \\
&\quad + (p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\frac{q^n}{p^{n+1}}(b-a) \\
&\quad \times \left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|\left|w\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|\Biggr\} \\
&= \frac{1}{b-a}\Biggl\{(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\left(1-\frac{q^n}{p^{n+1}}\right)(b-a)\right)^{\frac{1}{\gamma_1}} \\
&\quad \times \left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right| \\
&\quad \times \left(\left(1-\frac{q^n}{p^{n+1}}\right)(b-a)\right)^{\frac{1}{\gamma_2}}\left|w\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right| \\
&\quad + (p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}(b-a)\right)^{\frac{1}{\gamma_1}}\left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right| \\
&\quad \times \left(\frac{q^n}{p^{n+1}}(b-a)\right)^{\frac{1}{\gamma_2}}\left|w\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|\Biggr\} \\
&\le \frac{1}{b-a}\Biggl\{\Biggl[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\left(1-\frac{q^n}{p^{n+1}}\right)(b-a)\right) \\
&\quad \times \left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_1}\Biggr]^{\frac{1}{\gamma_1}} \\
&\quad \times \Biggl[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\left(1-\frac{q^n}{p^{n+1}}\right)(b-a)\right) \\
&\quad \times \left|w\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_2}\Biggr]^{\frac{1}{\gamma_2}} \\
&\quad + \left[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}(b-a)\right)\left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_1}\right]^{\frac{1}{\gamma_1}} \\
&\quad \times \left[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}(b-a)\right)\left|w\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\Biggr\} \\
&= \frac{1}{b-a}\Biggl\{\left(\int_a^b (b-x)\left|f(x)\right|^{\gamma_1}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}\left(\int_a^b (b-x)\left|w(x)\right|^{\gamma_2}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_2}} \\
&\quad + \left(\int_a^b (x-a)\left|f(x)\right|^{\gamma_1}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}\left(\int_a^b (x-a)\left|w(x)\right|^{\gamma_2}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_2}}\Biggr\}.
\end{aligned}
$$

This completes the proof for the first part of inequality (2.2). For the second part, using Definition 1.3 again yields that

$$\begin{aligned}
&\frac{1}{b-a}\left\{\left(\int_a^b (b-x)\left|f(x)\right|^{\gamma_1}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}\left(\int_a^b (b-x)\left|w(x)\right|^{\gamma_2}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_2}}\right.\\
&\quad\left.+\left(\int_a^b (x-a)\left|f(x)\right|^{\gamma_1}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}\left(\int_a^b (x-a)\left|w(x)\right|^{\gamma_2}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_2}}\right\}\\
&=\frac{1}{b-a}\left\{\left[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\left(1-\frac{q^n}{p^{n+1}}\right)(b-a)\right)\right.\right.\\
&\quad\left.\times\left|f\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_1}\right]^{\frac{1}{\gamma_1}}\\
&\quad\times\left[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\left(1-\frac{q^n}{p^{n+1}}\right)(b-a)\right)\right.\\
&\quad\left.\times\left|w\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\\
&\quad+\left[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}(b-a)\right)\left|f\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_1}\right]^{\frac{1}{\gamma_1}}\\
&\quad\left.\times\left[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}(b-a)\right)\left|w\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\right\}\\
&\leq\frac{1}{b-a}\left\{\left[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left|f\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_1}\right]^{\frac{1}{\gamma_1}}\right.\\
&\quad\times\left(\sum_{n=0}^{\infty}\left(1-\frac{q^n}{p^{n+1}}\right)(b-a)\right)^{\frac{1}{\gamma_1}}\\
&\quad\times\left[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left|w\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\\
&\quad\times\left(\sum_{n=0}^{\infty}\left(1-\frac{q^n}{p^{n+1}}\right)(b-a)\right)^{\frac{1}{\gamma_2}}\\
&\quad+\left[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left|f\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_1}\right]^{\frac{1}{\gamma_1}}\\
&\quad\times\left(\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}(b-a)\right)^{\frac{1}{\gamma_1}}\\
&\quad\times\left[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left|w\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\\
&\quad\left.\times\left(\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}(b-a)\right)^{\frac{1}{\gamma_2}}\right\}
\end{aligned}$$

$$= \left[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left|f\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_1}\right]^{\frac{1}{\gamma_1}}$$
$$\times \left[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left|w\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}$$
$$= \left(\int_a^b |f(x)|^{\gamma_1}\, {}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}\left(\int_a^b |w(x)|^{\gamma_2}\, {}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_2}}.$$

Thus the proof of Theorem 2.2 is completed. □

As a generalization of the (p,q)-Hölder inequality, we give the following improvement of (p,q)-power-mean integral inequality.

Theorem 2.3 (Improved (p,q)-power-mean integral inequality) *Let $\gamma_1 \geq 1$. If f and w are real mappings on $[a,b]$ such that $|f|$ and $|f||w|^{\gamma_1}$ are integrable on $[a,b]$, then*

$$\begin{aligned}
&\int_a^b |w(x)f(x)|\, {}_a\mathrm{d}_{p,q}x \\
&\leq \frac{1}{b-a}\Bigg\{\left(\int_a^b (b-x)|f(x)|\, {}_a\mathrm{d}_{p,q}x\right)^{1-\frac{1}{\gamma_1}}\left(\int_a^b (b-x)|f(x)||w(x)|^{\gamma_1}\, {}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}} \\
&\quad + \left(\int_a^b (x-a)|f(x)|\, {}_a\mathrm{d}_{p,q}x\right)^{1-\frac{1}{\gamma_1}}\left(\int_a^b (x-a)|f(x)||w(x)|^{\gamma_1}\, {}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}\Bigg\} \\
&\leq \left(\int_a^b |f(x)|\, {}_a\mathrm{d}_{p,q}x\right)^{1-\frac{1}{\gamma_1}}\left(\int_a^b |f(x)||w(x)|^{\gamma_1}\, {}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}.
\end{aligned} \tag{2.3}$$

Proof We can easily see that inequality (2.3) holds for $\gamma_1 = 1$. Now we suppose that $\gamma_1 > 1$. Using Definition 1.3 and the discrete Hölder inequality, we have that

$$\int_a^b |w(x)f(x)|\, {}_a\mathrm{d}_{p,q}x$$
$$= (p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|\left|w\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|$$
$$= \frac{1}{b-a}\Bigg\{(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)(b-a)$$
$$\times \left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|\left|w\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|$$
$$+ (p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\frac{q^n}{p^{n+1}}(b-a)$$
$$\times \left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|\left|w\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|\Bigg\}$$
$$= \frac{1}{b-a}\Bigg\{(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\left(1-\frac{q^n}{p^{n+1}}\right)(b-a)\right)^{1-\frac{1}{\gamma_1}}$$

$$
\begin{aligned}
&\times \left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{1-\frac{1}{\gamma_1}} \\
&\times \left(\left(1-\frac{q^n}{p^{n+1}}\right)(b-a)\right)^{\frac{1}{\gamma_1}} \\
&\times \left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\frac{1}{\gamma_1}}\left|w\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right| \\
&+(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}(b-a)\right)^{1-\frac{1}{\gamma_1}}\left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{1-\frac{1}{\gamma_1}} \\
&\times \left(\frac{q^n}{p^{n+1}}(b-a)\right)^{\frac{1}{\gamma_1}}\left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\frac{1}{\gamma_1}}\left|w\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|\Bigg\} \\
\le{}& \frac{1}{b-a}\Bigg\{\Bigg[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\left(1-\frac{q^n}{p^{n+1}}\right)(b-a)\right) \\
&\times \left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|\Bigg]^{1-\frac{1}{\gamma_1}} \\
&\times \Bigg[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\left(1-\frac{q^n}{p^{n+1}}\right)(b-a)\right) \\
&\times \left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|\left|w\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_1}\Bigg]^{\frac{1}{\gamma_1}} \\
&+\Bigg[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}(b-a)\right)\left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|\Bigg]^{1-\frac{1}{\gamma_1}} \\
&\times \Bigg[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}(b-a)\right) \\
&\times \left|f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|\left|w\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_1}\Bigg]^{\frac{1}{\gamma_1}}\Bigg\} \\
={}& \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\left|f(x)\right|\,{}_a\mathrm{d}_{p,q}x\right)^{1-\frac{1}{\gamma_1}}\left(\int_a^b (b-x)\left|f(x)\right|\left|w(x)\right|^{\gamma_1}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}\right. \\
&\left.+\left(\int_a^b (x-a)\left|f(x)\right|\,{}_a\mathrm{d}_{p,q}x\right)^{1-\frac{1}{\gamma_1}}\left(\int_a^b (x-a)\left|f(x)\right|\left|w(x)\right|^{\gamma_1}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}\right\}.
\end{aligned}
$$

This completes the proof for the first part of inequality (2.3). Now let us invoke Definition 1.3 again to prove the second part. Specifically, we have

$$
\begin{aligned}
&\frac{1}{b-a}\left\{\left(\int_a^b (b-x)\left|f(x)\right|\,{}_a\mathrm{d}_{p,q}x\right)^{1-\frac{1}{\gamma_1}}\left(\int_a^b (b-x)\left|f(x)\right|\left|w(x)\right|^{\gamma_1}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}\right. \\
&\quad\left.+\left(\int_a^b (x-a)\left|f(x)\right|\,{}_a\mathrm{d}_{p,q}x\right)^{1-\frac{1}{\gamma_1}}\left(\int_a^b (x-a)\left|f(x)\right|\left|w(x)\right|^{\gamma_1}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{b-a}\Biggl\{\Biggl[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\biggl(\biggl(1-\frac{q^n}{p^{n+1}}\biggr)(b-a)\biggr) \\
&\quad\times \biggl|f\biggl(\frac{q^n}{p^{n+1}}x+\biggl(1-\frac{q^n}{p^{n+1}}\biggr)a\biggr)\biggr|\Biggr]^{1-\frac{1}{\gamma_1}} \\
&\quad\times \Biggl[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\biggl(\biggl(1-\frac{q^n}{p^{n+1}}\biggr)(b-a)\biggr) \\
&\quad\times \biggl|f\biggl(\frac{q^n}{p^{n+1}}x+\biggl(1-\frac{q^n}{p^{n+1}}\biggr)a\biggr)\biggr|\biggl|w\biggl(\frac{q^n}{p^{n+1}}x+\biggl(1-\frac{q^n}{p^{n+1}}\biggr)a\biggr)\biggr|^{\gamma_1}\Biggr]^{\frac{1}{\gamma_1}} \\
&\quad+ \Biggl[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\biggl(\frac{q^n}{p^{n+1}}(b-a)\biggr)\biggl|f\biggl(\frac{q^n}{p^{n+1}}x+\biggl(1-\frac{q^n}{p^{n+1}}\biggr)a\biggr)\biggr|\Biggr]^{1-\frac{1}{\gamma_1}} \\
&\quad\times \Biggl[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\biggl(\frac{q^n}{p^{n+1}}(b-a)\biggr) \\
&\quad\times \biggl|f\biggl(\frac{q^n}{p^{n+1}}x+\biggl(1-\frac{q^n}{p^{n+1}}\biggr)a\biggr)\biggr|\biggl|w\biggl(\frac{q^n}{p^{n+1}}x+\biggl(1-\frac{q^n}{p^{n+1}}\biggr)a\biggr)\biggr|^{\gamma_1}\Biggr]^{\frac{1}{\gamma_1}}\Biggr\} \\
&\le \frac{1}{b-a}\Biggl\{\Biggl[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\biggl|f\biggl(\frac{q^n}{p^{n+1}}b+\biggl(1-\frac{q^n}{p^{n+1}}\biggr)a\biggr)\biggr|\Biggr]^{1-\frac{1}{\gamma_1}} \\
&\quad\times \Biggl(\sum_{n=0}^{\infty}\biggl(1-\frac{q^n}{p^{n+1}}\biggr)(b-a)\Biggr)^{1-\frac{1}{\gamma_1}} \\
&\quad\times \Biggl[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\biggl|f\biggl(\frac{q^n}{p^{n+1}}b+\biggl(1-\frac{q^n}{p^{n+1}}\biggr)a\biggr)\biggr| \\
&\quad\times \biggl|w\biggl(\frac{q^n}{p^{n+1}}b+\biggl(1-\frac{q^n}{p^{n+1}}\biggr)a\biggr)\biggr|^{\gamma_1}\Biggr]^{\frac{1}{\gamma_1}} \\
&\quad\times \Biggl(\sum_{n=0}^{\infty}\biggl(1-\frac{q^n}{p^{n+1}}\biggr)(b-a)\Biggr)^{\frac{1}{\gamma_1}} \\
&\quad+ \Biggl[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\biggl|f\biggl(\frac{q^n}{p^{n+1}}b+\biggl(1-\frac{q^n}{p^{n+1}}\biggr)a\biggr)\biggr|\Biggr]^{1-\frac{1}{\gamma_1}} \\
&\quad\times \Biggl(\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}(b-a)\Biggr)^{1-\frac{1}{\gamma_1}} \\
&\quad\times \Biggl[(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\biggl|f\biggl(\frac{q^n}{p^{n+1}}b+\biggl(1-\frac{q^n}{p^{n+1}}\biggr)a\biggr)\biggr| \\
&\quad\times \biggl|w\biggl(\frac{q^n}{p^{n+1}}b+\biggl(1-\frac{q^n}{p^{n+1}}\biggr)a\biggr)\biggr|^{\gamma_1}\Biggr]^{\frac{1}{\gamma_1}}
\end{aligned}
$$

$$\times \left(\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}(b-a)\right)^{\frac{1}{\gamma_1}} \Bigg\}$$

$$= \left[(p-q)(b-a)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left| f\left(\frac{q^n}{p^{n+1}}b + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right)\right|\right]^{1-\frac{1}{\gamma_1}}$$

$$\times \left[(p-q)(b-a)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left| f\left(\frac{q^n}{p^{n+1}}b + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right)\right|\right.$$

$$\left.\times \left| w\left(\frac{q^n}{p^{n+1}}b + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right)\right|^{\gamma_1}\right]^{\frac{1}{\gamma_1}}$$

$$= \left(\int_a^b \left|f(x)\right| {}_a\mathrm{d}_{p,q}x\right)^{1-\frac{1}{\gamma_1}} \left(\int_a^b \left|f(x)\right|\left|w(x)\right|^{\gamma_1} {}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_1}}.$$

The proof of Theorem 2.3 is completed. □

3 Applications

In this section, we present some interesting applications of the results developed in Sect. 2. To this end, we consider the difference between the middle part and the right-hand side of the analogue of (p,q)-Hermite–Hadamard inequality (1.2) and propose the following lemma.

Lemma 3.1 *Let $f:[a,b]\to\mathbb{R}$ be twice (p,q)-differentiable mapping on (a,b), and let ${}_aD^2_{p,q}f$ be continuous and integrable on $[a,b]$. Then we have*

$$\frac{qf(a)+pf(pb+(1-p)a))}{p+q} - \frac{1}{p^2(b-a)}\int_a^{p^2b+(1-p^2)a} f(x)\,{}_a\mathrm{d}_{p,q}x$$
$$= \frac{pq^2(b-a)^2}{p+q}\int_0^1 t(1-qt)\,{}_aD^2_{p,q}f\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_{p,q}t.$$

Proof From Definitions 1.1 and 1.2 we have

$${}_aD^2_{p,q}f\big(tb+(1-t)a\big)$$
$$= {}_aD_{p,q}\big({}_aD_{p,q}f\big(tb+(1-t)a\big)\big)$$
$$= \frac{{}_aD_{p,q}f(ptb+(1-pt)a) - {}_aD_{p,q}f(qtb+(1-qt)a)}{t(p-q)(b-a)}$$
$$= \frac{f(p^2tb+(1-p^2t)a) - f(pqtb+(1-pqt)a)}{pt^2(p-q)^2(b-a)^2}$$
$$- \frac{f(pqtb+(1-pqt)a) - f(q^2tb+(1-q^2t)a)}{qt^2(p-q)^2(b-a)^2}$$
$$= \frac{qf(p^2tb+(1-p^2t)a) - (p+q)f(pqtb+(1-pqt)a) + pf(q^2tb+(1-q^2t)a)}{pqt^2(p-q)^2(b-a)^2}.$$

Utilizing this calculation and Definition 1.3, we have

$$\begin{aligned}
&\int_0^1 t_a D_{p,q}^2 f\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_{p,q}t \\
&= \int_0^1 \frac{qf(p^2tb+(1-p^2t)a)-(p+q)f(pqtb+(1-pqt)a)+pf(q^2tb+(1-q^2t)a)}{pqt(p-q)^2(b-a)^2} \\
&\quad \times {}_0\mathrm{d}_{p,q}t \\
&= \int_0^1 \frac{qf(p^2tb+(1-p^2t)a)-qf(pqtb+(1-pqt)a)}{pqt(p-q)^2(b-a)^2}\,{}_0\mathrm{d}_{p,q}t \\
&\quad + \int_0^1 \frac{pf(q^2tb+(1-q^2t)a)-pf(pqtb+(1-pqt)a)}{pqt(p-q)^2(b-a)^2}\,{}_0\mathrm{d}_{p,q}t \\
&= \frac{q[\sum_{n=0}^\infty f(\frac{q^n}{p^{n-1}}b+(1-\frac{q^n}{p^{n-1}})a)-\sum_{n=0}^\infty f(\frac{q^{n+1}}{p^n}b+(1-\frac{q^{n+1}}{p^n})a)]}{pq(p-q)(b-a)^2} \\
&\quad + \frac{p[\sum_{n=0}^\infty f(\frac{q^{n+2}}{p^{n+1}}b+(1-\frac{q^{n+2}}{p^{n+1}})a)-\sum_{n=0}^\infty f(\frac{q^{n+1}}{p^n}b+(1-\frac{q^{n+1}}{p^n})a)]}{pq(p-q)(b-a)^2} \\
&= \frac{q[f(pb+(1-p)a)-f(a)]+p[f(a)-f(qb+(1-q)a)]}{pq(p-q)(b-a)^2} \\
&= \frac{qf(pb+(1-p)a)-pf(qb+(1-q)a)+(p-q)f(a)}{pq(p-q)(b-a)^2}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 t^2{}_a D_{p,q}^2 f\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_{p,q}t \\
&= \int_0^1 \frac{qf(p^2tb+(1-p^2t)a)-(p+q)f(pqtb+(1-pqt)a)+pf(q^2tb+(1-q^2t)a)}{pq(p-q)^2(b-a)^2} \\
&\quad \times {}_0\mathrm{d}_{p,q}t \\
&= \frac{\sum_{n=0}^\infty \frac{q^{n+1}}{p^{n+1}}f(\frac{q^n}{p^{n-1}}b+(1-\frac{q^n}{p^{n-1}})a)-\sum_{n=0}^\infty \frac{q^{n+1}}{p^{n+1}}f(\frac{q^{n+1}}{p^n}b+(1-\frac{q^{n+1}}{p^n})a)}{pq(p-q)(b-a)^2} \\
&\quad + \frac{\sum_{n=0}^\infty \frac{q^n}{p^n}f(\frac{q^{n+2}}{p^{n+1}}b+(1-\frac{q^{n+2}}{p^{n+1}})a)-\sum_{n=0}^\infty \frac{q^n}{p^n}f(\frac{q^{n+1}}{p^n}b+(1-\frac{q^{n+1}}{p^n})a)}{pq(p-q)(b-a)^2} \\
&= \frac{\sum_{n=0}^\infty \frac{q^{n+1}}{p^{n+1}}f(\frac{q^n}{p^{n-1}}b+(1-\frac{q^n}{p^{n-1}})a)-\frac{p}{q}\sum_{n=0}^\infty \frac{q^{n+2}}{p^{n+2}}f(\frac{q^{n+1}}{p^n}b+(1-\frac{q^{n+1}}{p^n})a)}{pq(p-q)(b-a)^2} \\
&\quad + \frac{\frac{p}{q}\sum_{n=0}^\infty \frac{q^{n+1}}{p^{n+1}}f(\frac{q^{n+2}}{p^{n+1}}b+(1-\frac{q^{n+2}}{p^{n+1}})a)-\sum_{n=0}^\infty \frac{q^n}{p^n}f(\frac{q^{n+1}}{p^n}b+(1-\frac{q^{n+1}}{p^n})a)}{pq(p-q)(b-a)^2} \\
&= \frac{(1-\frac{p}{q})\sum_{n=0}^\infty \frac{q^{n+1}}{p^{n+1}}f(\frac{q^n}{p^{n-1}}b+(1-\frac{q^n}{p^{n-1}})a)+f(pb+(1-p)a)}{pq(p-q)(b-a)^2} \\
&\quad + \frac{(\frac{p}{q}-1)\sum_{n=0}^\infty \frac{q^n}{p^n}f(\frac{q^{n+1}}{p^n}b+(1-\frac{q^{n+1}}{p^n})a)-\frac{p}{q}f(qb+(1-q)a)}{pq(p-q)(b-a)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\frac{1}{p}f(pb+(1-p)a)) + (1-\frac{p}{q})\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+2}}f(\frac{q^{n+1}}{p^n}b+(1-\frac{q^{n+1}}{p^n})a)}{p(p-q)(b-a)^2} \\
&\quad + \frac{\frac{p}{q}(\frac{p}{q}-1)\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+2}}f(\frac{q^{n+1}}{p^n}b+(1-\frac{q^{n+1}}{p^n})a)) - \frac{1}{q}f(qb+(1-q)a))}{q(p-q)(b-a)^2} \\
&= \frac{\frac{1}{p}f(pb+(1-p)a) + (1-\frac{p}{q})[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f(\frac{q^n}{p^{n-1}}b+(1-\frac{q^n}{p^{n-1}})a) - \frac{1}{p}f(pb+(1-p)a)]}{p(p-q)(b-a)^2} \\
&\quad + \left(\frac{p}{q}\left(\frac{p}{q}-1\right)\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n-1}}b+\left(1-\frac{q^n}{p^{n-1}}\right)a\right) - \frac{1}{p}f\big(pb+(1-p)a\big)\right]\right. \\
&\quad \left. - \frac{1}{q}f\big(qb+(1-q)a\big)\right) \Big/ \big(q(p-q)(b-a)^2\big) \\
&= \frac{q^2-p^2+pq}{pq^3(p-q)(b-a)^2}f\big(pb+(1-p)a\big) - \frac{1}{q^2(p-q)(b-a)^2}f\big(qb+(1-q)a\big) \\
&\quad + \frac{p+q}{p^3q^3(b-a)^3}\int_a^{p^2b+(1-p^2)a} f(x)\,{}_a\mathrm{d}_{p,q}x.
\end{aligned}
$$

After suitable arrangement, we get the desired result. Thus the proof is completed. □

Remark 3.1 In Lemma 3.1, choosing $p=1$, we obtain Lemma 4.1 in [20].

Using Lemma 3.1, we derive the following theorem.

Theorem 3.1 *Let $f:[a,b]\to\mathbb{R}$ be twice (p,q)-differentiable mapping on (a,b), and let ${}_aD_{p,q}^2 f$ be continuous and integrable on $[a,b]$. If $|{}_aD_{p,q}^2 f|^{\gamma_2}$ is convex for $\gamma_2>1$ on $[a,b]$ and $\gamma_1^{-1}+\gamma_2^{-1}=1$, then we have*

$$
\begin{aligned}
&\left|\frac{qf(a)+pf(pb+(1-p)a)}{p+q} - \frac{1}{p^2(b-a)}\int_a^{p^2b+(1-p^2)a} f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\quad \le \frac{pq^2(b-a)^2}{p+q}\Big\{\mathcal{Q}_1^{\frac{1}{\gamma_1}}\big[\Psi_1\big|{}_aD_{p,q}^2 f(a)\big|^{\gamma_2} + \Psi_2\big|{}_aD_{p,q}^2 f(b)\big|^{\gamma_2}\big]^{\frac{1}{\gamma_2}} \\
&\qquad + \mathcal{Q}_2^{\frac{1}{\gamma_1}}\big[\Psi_2\big|{}_aD_{p,q}^2 f(a)\big|^{\gamma_2} + \Psi_3\big|{}_aD_{p,q}^2 f(b)\big|^{\gamma_2}\big]^{\frac{1}{\gamma_2}}\Big\},
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
\mathcal{Q}_1 &= (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^2\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma_1}, \\
\mathcal{Q}_2 &= (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^3\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma_1}, \\
\Psi_1 &= \frac{1}{p^3+p^2q+pq^2+q^3} - \frac{2}{p^2+pq+q^2} + \frac{1}{p+q}, \\
\Psi_2 &= \frac{1}{p^2+pq+q^2} - \frac{1}{p^3+p^2q+pq^2+q^3},
\end{aligned}
$$

and

$$\Psi_3 = \frac{1}{p^3 + p^2 q + pq^2 + q^3}.$$

Proof Using Lemma 3.1, the (p,q)-Hölder–İşcan integral inequality, and the convexity of $|{}_aD^2_{p,q}f|^{\gamma_2}$ on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{qf(a)+pf(pb+(1-p)a)}{p+q} - \frac{1}{p^2(b-a)}\int_a^{p^2b+(1-p^2)a} f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\quad\le \frac{pq^2(b-a)^2}{p+q}\left\{\left(\int_0^1 (1-t)t(1-qt)^{\gamma_1}\,{}_a\mathrm{d}_{p,q}t\right)^{\frac{1}{\gamma_1}}\right. \\
&\qquad\times\left(\int_0^1 (1-t)t\left|{}_aD^2_{p,q}f\big(tb+(1-t)a\big)\right|^{\gamma_2}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_2}} \\
&\qquad\left.+\left(\int_0^1 t^2(1-qt)^{\gamma_1}\,{}_a\mathrm{d}_{p,q}t\right)^{\frac{1}{\gamma_1}}\left(\int_0^1 t^2\left|{}_aD^2_{p,q}f\big(tb+(1-t)a\big)\right|^{\gamma_2}\,{}_a\mathrm{d}_{p,q}t\right)^{\frac{1}{\gamma_2}}\right\} \\
&\quad\le \frac{pq^2(b-a)^2}{p+q}\left\{\left(\int_0^1 (1-t)t(1-qt)^{\gamma_1}\,{}_a\mathrm{d}_{p,q}t\right)^{\frac{1}{\gamma_1}}\right. \\
&\qquad\times\left[\int_0^1 (1-t)t\big((1-t)\left|{}_aD^2_{p,q}f(a)\right|^{\gamma_2} + t\left|{}_aD^2_{p,q}f(b)\right|^{\gamma_2}\big)\,{}_a\mathrm{d}_{p,q}x\right]^{\frac{1}{\gamma_2}} \\
&\qquad+\left(\int_0^1 t^2(1-qt)^{\gamma_1}\,{}_a\mathrm{d}_{p,q}t\right)^{\frac{1}{\gamma_1}} \\
&\qquad\left.\times\left[\int_0^1 t^2\big((1-t)\left|{}_aD^2_{p,q}f(a)\right|^{\gamma_2} + t\left|{}_aD^2_{p,q}f(b)\right|^{\gamma_2}\big)\,{}_a\mathrm{d}_{p,q}t\right]^{\frac{1}{\gamma_2}}\right\}.
\end{aligned}$$

We obtain the desired inequality by noting that

$$\begin{aligned}
&\int_0^1 (1-t)t(1-qt)^{\gamma_1}\,{}_a\mathrm{d}_{p,q}t = (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^2\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma_1}, \\
&\int_0^1 t^2(1-qt)^{\gamma_1}\,{}_a\mathrm{d}_{p,q}t = (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^3\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma_1}, \\
&\int_0^1 t(1-t)^2\,{}_a\mathrm{d}_{p,q}t = \frac{1}{p^3+p^2q+pq^2+q^3} - \frac{2}{p^2+pq+q^2} + \frac{1}{p+q}, \\
&\int_0^1 (1-t)t^2\,{}_a\mathrm{d}_{p,q}t = \frac{1}{p^2+pq+q^2} - \frac{1}{p^3+p^2q+pq^2+q^3},
\end{aligned}$$

and

$$\int_0^1 t^3\,{}_a\mathrm{d}_{p,q}t = \frac{1}{p^3+p^2q+pq^2+q^3}.$$

This ends the proof. □

Corollary 3.1 *If we set $p = 1$ and $q \to 1^-$ in Theorem 3.1, then we have*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\&\quad\le\frac{(b-a)^2}{2}\left\{\left(\frac{1}{(\gamma_1+2)(\gamma_1+3)}\right)^{\frac{1}{\gamma_1}}\left[\frac{1}{12}\left|f''(a)\right|^{\gamma_2}+\frac{1}{12}\left|f''(b)\right|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\right.\\&\qquad\left.+\left(\frac{2}{(\gamma_1+1)(\gamma_1+2)(\gamma_1+3)}\right)^{\frac{1}{\gamma_1}}\left[\frac{1}{12}\left|f''(a)\right|^{\gamma_2}+\frac{1}{4}\left|f''(b)\right|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\right\}.\end{aligned}\tag{3.2}$$

In Theorem 5.2 of [20], as $q \to 1^-$, the authors obtained the following result.

Proposition 3.1 *Let a continuous and integrable function $f:[a,b]\to\mathbb{R}$ be twice differentiable on (a,b). If $|f''|^{\gamma_2}$ is convex for $\gamma_2>1$ on $[a,b]$ and $\gamma_1^{-1}+\gamma_2^{-1}=1$, then we have*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\&\quad\le\frac{(b-a)^2}{2}\left(\frac{1}{(\gamma_1+1)(\gamma_1+2)}\right)^{\frac{1}{\gamma_1}}\left[\frac{|f''(a)|^{\gamma_2}+2|f''(b)|^{\gamma_2}}{6}\right]^{\frac{1}{\gamma_2}}.\end{aligned}\tag{3.3}$$

Remark 3.2 Inequality (3.2) is sharper than inequality (3.3). Indeed, since the function $h:[0,\infty)\to\mathbb{R}$, $h(\tau)=\tau^\kappa$, $\kappa\in(0,1]$, is concave, we can write

$$\frac{\mu^\kappa+\nu^\kappa}{2}=\frac{h(\mu)+h(\nu)}{2}\le h\left(\frac{\mu+\nu}{2}\right)=\left(\frac{\mu+\nu}{2}\right)^\kappa\tag{3.4}$$

for all $\mu,\nu\ge 0$. In inequality (3.4), choosing

$$\mu=\frac{1}{12}\left|f''(a)\right|^{\gamma_2}+\frac{1}{12}\left|f''(b)\right|^{\gamma_2},\qquad \nu=\frac{1}{12}\left|f''(a)\right|^{\gamma_2}+\frac{1}{4}\left|f''(b)\right|^{\gamma_2},$$

and $\kappa=\frac{1}{\gamma_2}$, $\gamma_2>1$, we have

$$\begin{aligned}&\frac{1}{2}\left[\frac{1}{12}\left|f''(a)\right|^{\gamma_2}+\frac{1}{12}\left|f''(b)\right|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}+\frac{1}{2}\left[\frac{1}{12}\left|f''(a)\right|^{\gamma_2}+\frac{1}{4}\left|f''(b)\right|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\\&\quad\le\left[\frac{|f''(a)|^{\gamma_2}+2|f''(b)|^{\gamma_2}}{12}\right]^{\frac{1}{\gamma_2}}.\end{aligned}$$

Thus we obtain the following result:

$$\begin{aligned}&\frac{(b-a)^2}{2}\left\{\left(\frac{1}{(\gamma_1+2)(\gamma_1+3)}\right)^{\frac{1}{\gamma_1}}\left[\frac{1}{12}\left|f''(a)\right|^{\gamma_2}+\frac{1}{12}\left|f''(b)\right|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\right.\\&\qquad\left.+\left(\frac{2}{(\gamma_1+1)(\gamma_1+2)(\gamma_1+3)}\right)^{\frac{1}{\gamma_1}}\left[\frac{1}{12}\left|f''(a)\right|^{\gamma_2}+\frac{1}{4}\left|f''(b)\right|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\right\}\\&\quad\le\frac{(b-a)^2}{2}\left(\frac{1}{(\gamma_1+1)(\gamma_1+2)}\right)^{\frac{1}{\gamma_1}}\left(\frac{1}{2^{\frac{1}{\gamma_1}}}\right)\left\{\left[\frac{1}{12}\left|f''(a)\right|^{\gamma_2}+\frac{1}{12}\left|f''(b)\right|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\right.\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{1}{12}\left|f''(a)\right|^{\gamma_2}+\frac{1}{4}\left|f''(b)\right|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\Bigg\} \\
&\le \frac{(b-a)^2}{2}\left(\frac{1}{(\gamma_1+1)(\gamma_1+2)}\right)^{\frac{1}{\gamma_1}}\left(\frac{1}{2^{\frac{1}{\gamma_1}}}\right)\left(\frac{2}{2^{\frac{1}{\gamma_2}}}\right)\left[\frac{|f''(a)|^{\gamma_2}+2|f''(b)|^{\gamma_2}}{6}\right]^{\frac{1}{\gamma_2}} \\
&= \frac{(b-a)^2}{2}\left(\frac{1}{(\gamma_1+1)(\gamma_1+2)}\right)^{\frac{1}{\gamma_1}}\left[\frac{|f''(a)|^{\gamma_2}+2|f''(b)|^{\gamma_2}}{6}\right]^{\frac{1}{\gamma_2}}.
\end{aligned}
$$

The upper bound for the right-hand side of the Hermite–Hadamard inequality for convex mappings obtained in inequality (3.2) is shaper than that of inequality (3.3), which can be illustrated by the following example.

Example 3.1 Considering the mapping $f(x)=\frac{x^3}{6}$, $x>0$, we apply it to inequalities (3.3) and (3.2). Let the right-hand sides of inequalities (3.3) and (3.2), except a common factor $\frac{(b-a)^2}{2}$, be denoted by

$$
E_1(\gamma_1,\gamma_2)=\left(\frac{1}{(\gamma_1+1)(\gamma_1+2)}\right)^{\frac{1}{\gamma_1}}\left[\frac{a^{\gamma_2}+2b^{\gamma_2}}{6}\right]^{\frac{1}{\gamma_2}}
$$

and

$$
\begin{aligned}
E_2(\gamma_1,\gamma_2)=\Bigg\{&\left(\frac{1}{(\gamma_1+2)(\gamma_1+3)}\right)^{\frac{1}{\gamma_1}}\left[\frac{1}{12}a^{\gamma_2}+\frac{1}{12}b^{\gamma_2}\right]^{\frac{1}{\gamma_2}} \\
&+\left(\frac{2}{(\gamma_1+1)(\gamma_1+2)(\gamma_1+3)}\right)^{\frac{1}{\gamma_1}}\left[\frac{1}{12}a^{\gamma_2}+\frac{1}{4}b^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\Bigg\}.
\end{aligned}
$$

Next, let us compare $E_1(\gamma_1,\gamma_2)$ with $E_2(\gamma_1,\gamma_2)$. For $\gamma_2>2$ with $\gamma_1=\frac{\gamma_2}{\gamma_2-1}$, $a=2$, and $b=5$, from Fig. 1 we see that $E_2(\gamma_1,\gamma_2)$ is a sharper error bound than $E_1(\gamma_1,\gamma_2)$. Therefore it reveals that the result of Corollary 3.1 is sharper than that of Proposition 3.1.

The next result deals with the other case where $|{}_aD^2_{p,q}f|^{\gamma_2}$ is convex for $\gamma_2>1$.

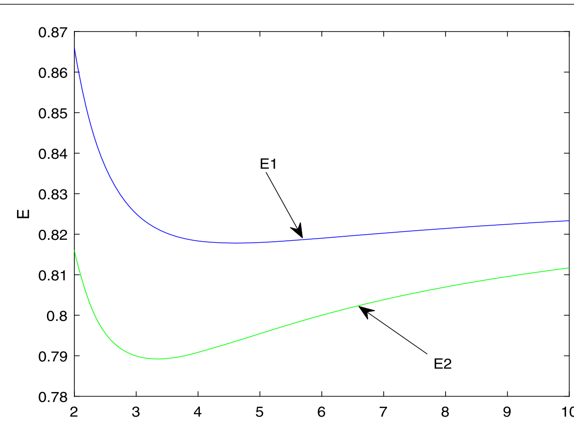

Figure 1 Error curves of E_1 and E_2 on the variable γ_2

Theorem 3.2 *Let $f:[a,b]\to\mathbb{R}$ be twice (p,q)-differentiable mapping with (a,b), and let ${}_aD^2_{p,q}f$ be continuous and integrable on $[a,b]$. If $|{}_aD^2_{p,q}f|^{\gamma_2}$ is convex for $\gamma_2>1$ on $[a,b]$ and $\gamma_1^{-1}+\gamma_2^{-1}=1$, then we have*

$$\begin{aligned}&\left|\frac{qf(a)+pf(pb+(1-p)a)}{p+q}-\frac{1}{p^2(b-a)}\int_a^{p^2b+(1-p^2)a}f(x)\,{}_a\mathrm{d}_{p,q}x\right|\\&\quad\le\frac{pq^2(b-a)^2}{p+q}\Big\{\mathcal{X}_1^{\frac{1}{\gamma_1}}\big[\Lambda_1\big|{}_aD^2_{p,q}f(a)\big|^{\gamma_2}+\Lambda_2\big|{}_aD^2_{p,q}f(b)\big|^{\gamma_2}\big]^{\frac{1}{\gamma_2}}\\&\qquad+\mathcal{X}_2^{\frac{1}{\gamma_1}}\big[\Lambda_2\big|{}_aD^2_{p,q}f(a)\big|^{\gamma_2}+\Lambda_3\big|{}_aD^2_{p,q}f(b)\big|^{\gamma_2}\big]^{\frac{1}{\gamma_2}}\Big\},\end{aligned}$$

where

$$\begin{aligned}\mathcal{X}_1&=(p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{\gamma_1+1}\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma_1},\\\mathcal{X}_2&=(p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{\gamma_1+2}\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma_1},\\\Lambda_1&=\frac{p+q-2}{p+q}+\frac{1}{p^2+pq+q^2},\\\Lambda_2&=\frac{1}{p+q}-\frac{1}{p^2+pq+q^2},\end{aligned}$$

and

$$\Lambda_3=\frac{1}{p^2+pq+q^2}.$$

Proof By Lemma 3.1, the (p,q)-Hölder–İşcan integral inequality, and the convexity of $|{}_aD^2_{p,q}f|^{\gamma_2}$ on $[a,b]$ we have

$$\begin{aligned}&\left|\frac{qf(a)+pf(pb+(1-p)a))}{p+q}-\frac{1}{p^2(b-a)}\int_a^{p^2b+(1-p^2)a}f(x)\,{}_a\mathrm{d}_{p,q}x\right|\\&\quad\le\frac{pq^2(b-a)^2}{p+q}\left\{\left(\int_0^1(1-t)t^{\gamma_1}(1-qt)^{\gamma_1}\,{}_a\mathrm{d}_{p,q}t\right)^{\frac{1}{\gamma_1}}\right.\\&\qquad\times\left(\int_0^1(1-t)\big|{}_aD^2_{p,q}f\big(tb+(1-t)a\big)\big|^{\gamma_2}\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma_2}}\\&\qquad\left.+\left(\int_0^1t^{\gamma_1+1}(1-qt)^{\gamma_1}\,{}_a\mathrm{d}_{p,q}t\right)^{\frac{1}{\gamma_1}}\left(\int_0^1t\big|{}_aD^2_{p,q}f\big(tb+(1-t)a\big)\big|^{\gamma_2}\,{}_a\mathrm{d}_{p,q}t\right)^{\frac{1}{\gamma_2}}\right\}\\&\quad\le\frac{pq^2(b-a)^2}{p+q}\left\{\left(\int_0^1(1-t)t^{\gamma_1}(1-qt)^{\gamma_1}\,{}_a\mathrm{d}_{p,q}t\right)^{\frac{1}{\gamma_1}}\right.\\&\qquad\times\left[\int_0^1(1-t)\big((1-t)\big|{}_aD^2_{p,q}f(a)\big|^{\gamma_2}+t\big|{}_aD^2_{p,q}f(b)\big|^{\gamma_2}\big)\,{}_a\mathrm{d}_{p,q}x\right]^{\frac{1}{\gamma_2}}\\&\qquad+\left(\int_0^1t^{\gamma_1+1}(1-qt)^{\gamma_1}\,{}_a\mathrm{d}_{p,q}t\right)^{\frac{1}{\gamma_1}}\end{aligned}$$

$$\times\left[\int_0^1 t\big((1-t)\big|{}_aD_{p,q}^2f(a)\big|^{\gamma_2}+t\big|{}_aD_{p,q}^2f(b)\big|^{\gamma_2}\big)\,{}_a\mathrm{d}_{p,q}t\right]^{\frac{1}{\gamma_2}}\Bigg\}.$$

We obtain the desired inequality by noting that

$$\int_0^1(1-t)t^{\gamma_1}(1-qt)^{\gamma_1}\,{}_a\mathrm{d}_{p,q}t=(p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{\gamma_1+1}\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma_1},$$

$$\int_0^1 t^{\gamma_1+1}(1-qt)^{\gamma_1}\,{}_a\mathrm{d}_{p,q}t=(p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{\gamma_1+2}\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma_1},$$

$$\int_0^1(1-t)^2\,{}_a\mathrm{d}_{p,q}t=\frac{p+q-2}{p+q}+\frac{1}{p^2+pq+q^2},$$

$$\int_0^1(1-t)t\,{}_a\mathrm{d}_{p,q}t=\frac{1}{p+q}-\frac{1}{p^2+pq+q^2},$$

and

$$\Lambda_3=\int_0^1 t^2\,{}_a\mathrm{d}_{p,q}t=\frac{1}{p^2+pq+q^2}.$$

This ends the proof. □

Corollary 3.2 *If we select $p=1$ and $q\to 1^-$ in Theorem 3.2, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{(b-a)^2}{2}\beta^{\frac{1}{\gamma_1}}(\gamma_1+1,\gamma_1+2)\left\{\left[\frac{1}{3}\big|f''(a)\big|^{\gamma_2}+\frac{1}{6}\big|f''(b)\big|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\right.$$
$$\left.+\left[\frac{1}{6}\big|f''(a)\big|^{\gamma_2}+\frac{1}{3}\big|f''(b)\big|^{\gamma_2}\right]^{\frac{1}{\gamma_2}}\right\},$$

where $\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$, $x>0$, $y>0$.

Theorem 3.3 *Let $f:[a,b]\to\mathbb{R}$ be twice (p,q)-differentiable mapping with (a,b), and let ${}_aD_{p,q}^2f$ be continuous and integrable on $[a,b]$. If $|{}_aD_{p,q}^2f|^{\gamma}$ is convex for $\gamma\ge 1$ on $[a,b]$, then we have*

$$\left|\frac{qf(a)+pf(pb+(1-p)a)}{p+q}-\frac{1}{p^2(b-a)}\int_a^{p^2b+(1-p^2)a}f(x)\,{}_a\mathrm{d}_{p,q}x\right|$$
$$\le\frac{pq^2(b-a)^2}{p+q}\Big\{\Theta_1^{1-\frac{1}{\gamma}}\big[\mathcal{H}_1\big|{}_aD_{p,q}^2f(a)\big|^{\gamma}+\mathcal{H}_2\big|{}_aD_{p,q}^2f(b)\big|^{\gamma}\big]^{\frac{1}{\gamma}}\tag{3.5}$$
$$+\Theta_2^{1-\frac{1}{\gamma}}\big[\mathcal{H}_2\big|{}_aD_{p,q}^2f(a)\big|^{\gamma}+\mathcal{H}_3\big|{}_aD_{p,q}^2f(b)\big|^{\gamma}\big]^{\frac{1}{\gamma}}\Big\},$$

where

$$\mathcal{H}_1=(p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^2\left(1-\frac{q^n}{p^{n+1}}\right)^2\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma},$$

$$\mathcal{H}_2 = (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^3\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma},$$
$$\mathcal{H}_3 = (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^4\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma},$$
$$\Theta_1 = \frac{1}{p+q} - \frac{1}{p^2+pq+q^2},$$

and

$$\Theta_2 = \frac{1}{p^2+pq+q^2}.$$

Proof First, we suppose that $\gamma = 1$. Using the convexity of $|{}_aD^2_{p,q}f|$ on $[a,b]$ and Lemma 3.1, we have

$$\begin{aligned}
&\left|\frac{qf(a)+pf(pb+(1-p)a))}{p+q} - \frac{1}{p^2(b-a)}\int_a^{p^2b+(1-p^2)a} f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\quad\le \frac{pq^2(b-a)^2}{p+q}\int_0^1 \left|t(1-qt)\right|\left|{}_aD^2_{p,q}f\bigl(tb+(1-t)a\bigr)\right|{}_0\mathrm{d}_{p,q}t \\
&\quad\le \frac{pq^2(b-a)^2}{p+q}\int_0^1 \left|t(1-qt)\right|\left[(1-t)\left|{}_aD^2_{p,q}f(a)\right| + t\left|{}_aD^2_{p,q}f(b)\right|\right]{}_0\mathrm{d}_{p,q}t \\
&\quad= \frac{p^2q^2(b-a)^2}{(p+q)(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)} \\
&\qquad\times\left[\frac{p^4+2p^3q+p^2q^2+pq^3-p^3-p^2q-q^4}{p+q}\left|{}_aD^2_{p,q}f(a)\right| + p^2\left|{}_aD^2_{p,q}f(b)\right|\right].
\end{aligned}$$

This ends the proof for the case of $\gamma = 1$. Second, we suppose that $\gamma > 1$. Using the improved (p,q)-power-mean integral inequality and the convexity of $|{}_aD^2_{p,q}f|^{\gamma}$ on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{qf(a)+pf(pb+(1-p)a)}{p+q} - \frac{1}{p^2(b-a)}\int_a^{p^2b+(1-p^2)a} f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\quad\le \frac{pq^2(b-a)^2}{p+q}\left\{\left(\int_0^1 (1-t)t\,{}_a\mathrm{d}_{p,q}t\right)^{1-\frac{1}{\gamma}}\right. \\
&\qquad\times\left(\int_0^1 (1-t)t(1-qt)^{\gamma}\left|{}_aD^2_{p,q}f\bigl(tb+(1-t)a\bigr)\right|^{\gamma}{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma}} \\
&\qquad\left.+\left(\int_0^1 t^2\,{}_a\mathrm{d}_{p,q}t\right)^{1-\frac{1}{\gamma}}\left(\int_0^1 t^2(1-qt)^{\gamma}\left|{}_aD^2_{p,q}f\bigl(tb+(1-t)a\bigr)\right|^{\gamma}{}_a\mathrm{d}_{p,q}t\right)^{\frac{1}{\gamma}}\right\} \\
&\quad\le \frac{pq^2(b-a)^2}{p+q}\left\{\left(\int_0^1 (1-t)t\,{}_a\mathrm{d}_{p,q}t\right)^{1-\frac{1}{\gamma}}\right. \\
&\qquad\times\left[\int_0^1 (1-t)t(1-qt)^{\gamma}\left((1-t)\left|{}_aD^2_{p,q}f(a)\right|^{\gamma} + t\left|{}_aD^2_{p,q}f(b)\right|^{\gamma}\right){}_a\mathrm{d}_{p,q}x\right]^{\frac{1}{\gamma}} \\
&\qquad+\left(\int_0^1 t^2\,{}_a\mathrm{d}_{p,q}t\right)^{1-\frac{1}{\gamma}}
\end{aligned}$$

$$\times \left[\int_0^1 t^2(1-qt)^\gamma \left((1-t)\left|{}_aD_{p,q}^2 f(a)\right|^\gamma + t\left|{}_aD_{p,q}^2 f(b)\right|^\gamma\right) {}_a\mathrm{d}_{p,q}t\right]^{\frac{1}{\gamma}}\Bigg\}.$$

We obtain the desired inequality by noting that

$$\int_0^1 (1-t)^2 t(1-qt)^\gamma \,{}_a\mathrm{d}_{p,q}t = (p-q)\sum_{n=0}^\infty \left(\frac{q^n}{p^{n+1}}\right)^2\left(1-\frac{q^n}{p^{n+1}}\right)^2\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^\gamma,$$

$$\int_0^1 t^2(1-t)(1-qt)^\gamma \,{}_a\mathrm{d}_{p,q}t = (p-q)\sum_{n=0}^\infty \left(\frac{q^n}{p^{n+1}}\right)^3\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^\gamma,$$

$$\int_0^1 t^3(1-qt)^\gamma \,{}_a\mathrm{d}_{p,q}t = (p-q)\sum_{n=0}^\infty \left(\frac{q^n}{p^{n+1}}\right)^4\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^\gamma,$$

$$\int_0^1 (1-t)t \,{}_a\mathrm{d}_{p,q}t = \frac{1}{p+q} - \frac{1}{p^2+pq+q^2},$$

and

$$\int_0^1 t^2 \,{}_a\mathrm{d}_{p,q}t = \frac{1}{p^2+pq+q^2}.$$

The proof of Theorem 3.3 is completed. □

Corollary 3.3 *If we take $p = 1$ and $q \to 1^-$ in Theorem 3.3, then we have*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\quad\le \frac{(b-a)^2}{2}\Bigg\{\left(\frac{1}{6}\right)^{1-\frac{1}{\gamma}}\left[\frac{1}{(\gamma+3)(\gamma+4)}\left|f''(a)\right|^\gamma \right.\\
&\qquad\left. + \frac{2}{(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(b)\right|^\gamma\right]^{\frac{1}{\gamma}} \\
&\qquad + \left(\frac{1}{3}\right)^{1-\frac{1}{\gamma}}\left[\frac{2}{(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(a)\right|^\gamma \right.\\
&\qquad\left. + \frac{6}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(b)\right|^\gamma\right]^{\frac{1}{\gamma}}\Bigg\}.
\end{aligned} \tag{3.6}$$

In Theorem 5.1 of [20], as $q \to 1^-$, the authors obtained the following result.

Proposition 3.2 *Let a continuous integrable function $f : [a, b] \to \mathbb{R}$ be twice differentiable on (a, b). If $|f''|^\gamma$ is convex for $\gamma \ge 1$ on $[a, b]$, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{2^{2-\frac{1}{\gamma}}}\left[\frac{(\gamma+1)|f''(a)|^\gamma + 2|f''(b)|^\gamma}{(\gamma+1)(\gamma+2)(\gamma+3)}\right]^{\frac{1}{\gamma}}. \tag{3.7}$$

Remark 3.3 Inequality (3.6) is shaper than inequality (3.7). Indeed, since the function $h : [0, \infty) \to \mathbb{R}$, $h(\tau) = \tau^\kappa$, $\kappa \in (0, 1]$, is concave, we can write

$$\frac{\mu^\kappa + \nu^\kappa}{2} = \frac{h(\mu)+h(\nu)}{2} \le h\left(\frac{\mu+\nu}{2}\right) = \left(\frac{\mu+\nu}{2}\right)^\kappa \tag{3.8}$$

for all $\mu, \nu \geq 0$. In inequality (3.8), choosing

$$\mu = \frac{1}{(\gamma+3)(\gamma+4)}\left|f''(a)\right|^{\gamma} + \frac{2}{(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(b)\right|^{\gamma},$$
$$\nu = \frac{2}{(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(a)\right|^{\gamma} + \frac{6}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(b)\right|^{\gamma},$$

and $\kappa = \frac{1}{\gamma}$, $\gamma \geq 1$, we have

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{(\gamma+3)(\gamma+4)}\left|f''(a)\right|^{\gamma} + \frac{2}{(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(b)\right|^{\gamma}\right]^{\frac{1}{\gamma}} \\
&\quad + \frac{1}{2}\left[\frac{2}{(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(a)\right|^{\gamma} + \frac{6}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(b)\right|^{\gamma}\right]^{\frac{1}{\gamma}} \\
&\quad \leq \left[\frac{(\gamma+1)|f''(a)|^{\gamma} + 2|f''(b)|^{\gamma}}{2(\gamma+1)(\gamma+2)(\gamma+3)}\right]^{\frac{1}{\gamma}}.
\end{aligned}$$

Thus we obtain the following result:

$$\begin{aligned}
&\frac{(b-a)^2}{2}\left\{\left(\frac{1}{6}\right)^{1-\frac{1}{\gamma}}\left[\frac{1}{(\gamma+3)(\gamma+4)}\left|f''(a)\right|^{\gamma} + \frac{2}{(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(b)\right|^{\gamma}\right]^{\frac{1}{\gamma}}\right. \\
&\quad + \left(\frac{1}{3}\right)^{1-\frac{1}{\gamma}}\left[\frac{2}{(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(a)\right|^{\gamma}\right. \\
&\quad \left.\left. + \frac{6}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(b)\right|^{\gamma}\right]^{\frac{1}{\gamma}}\right\} \\
&\quad \leq \frac{(b-a)^2}{2^{2-\frac{1}{\gamma}}}\left(\frac{1}{2^{1-\frac{1}{\gamma}}}\right)\left\{\left[\frac{1}{(\gamma+3)(\gamma+4)}\left|f''(a)\right|^{\gamma} + \frac{2}{(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(b)\right|^{\gamma}\right]^{\frac{1}{\gamma}}\right. \\
&\quad \left. + \left[\frac{2}{(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(a)\right|^{\gamma} + \frac{6}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)}\left|f''(b)\right|^{\gamma}\right]^{\frac{1}{\gamma}}\right\} \\
&\quad \leq \frac{(b-a)^2}{2^{2-\frac{1}{\gamma}}}\left(\frac{1}{2^{1-\frac{1}{\gamma}}}\right)\left(\frac{2}{2^{\frac{1}{\gamma}}}\right)\left[\frac{(\gamma+1)|f''(a)|^{\gamma} + 2|f''(b)|^{\gamma}}{(\gamma+1)(\gamma+2)(\gamma+3)}\right]^{\frac{1}{\gamma}} \\
&\quad = \frac{(b-a)^2}{2^{2-\frac{1}{\gamma}}}\left[\frac{(\gamma+1)|f''(a)|^{\gamma} + 2|f''(b)|^{\gamma}}{(\gamma+1)(\gamma+2)(\gamma+3)}\right]^{\frac{1}{\gamma}}.
\end{aligned}$$

The upper bound for the right-hand side of the Hermite–Hadamard inequality for convex mappings obtained in inequality (3.6) is better than that of inequality (3.7), as the the following example shows.

Example 3.2 Considering the mapping $f(x) = e^x$, $x > 0$, we apply it to inequalities (3.7) and (3.6). Let the right-hand sides of inequalities (3.7) and (3.6), except a common factor $\frac{(b-a)^2}{2}$, be denoted by

$$E_3(\gamma) = \frac{1}{2^{1-\frac{1}{\gamma}}}\left[\frac{(\gamma+1)e^{a\gamma} + 2e^{b\gamma}}{(\gamma+1)(\gamma+2)(\gamma+3)}\right]^{\frac{1}{\gamma}}$$

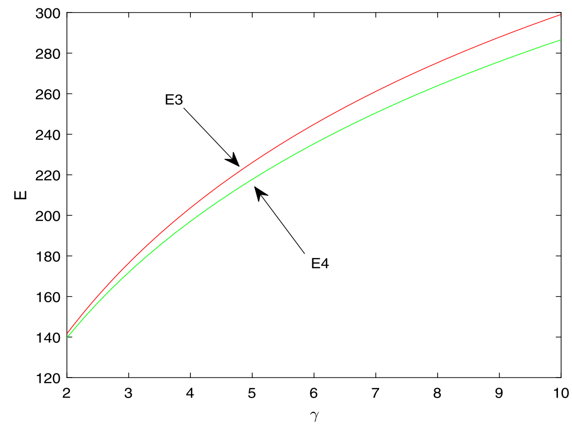

Figure 2 Error curves of E_3 and E_4 on the variable γ

and

$$E_4(\gamma) = \left\{\left(\frac{1}{6}\right)^{1-\frac{1}{\gamma}}\left[\frac{1}{(\gamma+3)(\gamma+4)}e^{a\gamma} + \frac{2}{(\gamma+2)(\gamma+3)(\gamma+4)}e^{b\gamma}\right]^{\frac{1}{\gamma}} + \left(\frac{1}{3}\right)^{1-\frac{1}{\gamma}}\left[\frac{2}{(\gamma+2)(\gamma+3)(\gamma+4)}e^{a\gamma} + \frac{6}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)}e^{b\gamma}\right]^{\frac{1}{\gamma}}\right\}.$$

Let us compare $E_3(\gamma)$ with $E_4(\gamma)$. For $\gamma > 2$, $a = 3$, and $b = 7$, from Fig. 2 we see that $E_4(\gamma)$ is a shaper error bound than $E_3(\gamma)$. Therefore it reveals that the result of Corollary 3.3 is shaper than that of Proposition 3.2.

Finally, we get the following result dealing with the other case where $|{}_aD^2_{p,q}f|^\gamma$ is convex for $\gamma \geq 1$.

Theorem 3.4 *Let $f : [a,b] \to \mathbb{R}$ be a twice (p,q)-differentiable mapping on (a,b), and let ${}_aD^2_{p,q}f$ be continuous and integrable on $[a,b]$. If $|{}_aD^2_{p,q}f|^\gamma$ is convex for $\gamma \geq 1$ on $[a,b]$, then we have*

$$\begin{aligned} &\left|\frac{qf(a)+pf(pb+(1-p)a)}{p+q} - \frac{1}{p^2(b-a)}\int_a^{p^2b+(1-p^2)a} f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\ &\quad\leq \frac{pq^2(b-a)^2}{p+q}\left\{\left(\frac{p+q-1}{p+q}\right)^{1-\frac{1}{\gamma}}\left[\mathcal{G}_1\left|{}_aD^2_{p,q}f(a)\right|^\gamma + \mathcal{G}_2\left|{}_aD^2_{p,q}f(b)\right|^\gamma\right]^{\frac{1}{\gamma}} \right. \\ &\quad\quad \left. + \left(\frac{1}{p+q}\right)^{1-\frac{1}{\gamma}}\left[\mathcal{G}_2\left|{}_aD^2_{p,q}f(a)\right|^\gamma + \mathcal{G}_3\left|{}_aD^2_{p,q}f(b)\right|^\gamma\right]^{\frac{1}{\gamma}}\right\}, \end{aligned} \tag{3.9}$$

where

$$\mathcal{G}_1 = (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{\gamma+1}\left(1-\frac{q^n}{p^{n+1}}\right)^2\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^\gamma,$$

$$\mathcal{G}_2 = (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{\gamma+2}\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^\gamma,$$

and

$$\mathcal{G}_3 = (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{\gamma+3}\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma}.$$

Proof Using Lemma 3.1, the improved (p,q)-power-mean integral inequality, and the convexity of $|{}_aD_{p,q}^2 f|^\gamma$ on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{qf(a)+pf(pb+(1-p)a)}{p+q}-\frac{1}{p^2(b-a)}\int_a^{p^2b+(1-p^2)a} f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\quad\leq \frac{pq^2(b-a)^2}{p+q}\left\{\left(\int_0^1 (1-t)\,{}_a\mathrm{d}_{p,q}t\right)^{1-\frac{1}{\gamma}}\right. \\
&\qquad\times\left(\int_0^1 (1-t)t^\gamma(1-qt)^\gamma\left|{}_aD_{p,q}^2 f\left(tb+(1-t)a\right)\right|^\gamma\,{}_a\mathrm{d}_{p,q}x\right)^{\frac{1}{\gamma}} \\
&\qquad\left.+\left(\int_0^1 t\,{}_a\mathrm{d}_{p,q}t\right)^{1-\frac{1}{\gamma}}\left(\int_0^1 t^{\gamma+1}(1-qt)^\gamma\left|{}_aD_{p,q}^2 f\left(tb+(1-t)a\right)\right|^\gamma\,{}_a\mathrm{d}_{p,q}t\right)^{\frac{1}{\gamma}}\right\} \\
&\quad\leq \frac{pq^2(b-a)^2}{p+q}\left\{\left(\frac{p+q-1}{p+q}\right)^{1-\frac{1}{\gamma}}\right. \\
&\qquad\times\left[\int_0^1 (1-t)t^\gamma(1-qt)^\gamma\left((1-t)\left|{}_aD_{p,q}^2 f(a)\right|^\gamma+t\left|{}_aD_{p,q}^2 f(b)\right|^\gamma\right)\,{}_a\mathrm{d}_{p,q}x\right]^{\frac{1}{\gamma}} \\
&\qquad\left.+\left(\frac{1}{p+q}\right)^{1-\frac{1}{\gamma}}\left[\int_0^1 t^{\gamma+1}(1-qt)^\gamma\left((1-t)\left|{}_aD_{p,q}^2 f(a)\right|^\gamma+t\left|{}_aD_{p,q}^2 f(b)\right|^\gamma\right)\,{}_a\mathrm{d}_{p,q}t\right]^{\frac{1}{\gamma}}\right\}.
\end{aligned}$$

We obtain the desired inequality by noting that

$$\int_0^1 (1-t)^2 t^\gamma(1-qt)^\gamma\,{}_a\mathrm{d}_{p,q}t = (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{\gamma+1}\left(1-\frac{q^n}{p^{n+1}}\right)^2\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma},$$

$$\int_0^1 t^{\gamma+1}(1-t)(1-qt)^\gamma\,{}_a\mathrm{d}_{p,q}t = (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{\gamma+2}\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma},$$

and

$$\int_0^1 t^{\gamma+2}(1-qt)^\gamma\,{}_a\mathrm{d}_{p,q}t = (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{\gamma+3}\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{\gamma}.$$

The proof is completed. □

Corollary 3.4 *If we choose $p=1$ and $q\to 1^-$ in Theorem 3.4, then we have*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\quad\leq \frac{(b-a)^2}{2^{2-\frac{1}{\gamma}}}\left\{\left[\beta(\gamma+1,\gamma+3)\left|f''(a)\right|^\gamma+\beta(\gamma+2,\gamma+2)\left|f''(b)\right|^\gamma\right]^{\frac{1}{\gamma}}\right. \\
&\qquad\left.+\left[\beta(\gamma+2,\gamma+2)\left|f''(a)\right|^\gamma+\beta(\gamma+3,\gamma+1)\left|f''(b)\right|^\gamma\right]^{\frac{1}{\gamma}}\right\}.
\end{aligned}$$

4 Conclusion

We extend some important integral inequalities of analysis to (p,q)-calculus, which include the Hermite–Hadamard, Hölder, and power-mean integral inequalities. As applications, for mappings with convex absolute values of the second derivatives, we derive certain analogue of (p,q)-Hermite–Hadamard inequalities based on the established (p,q)-integral identity. By an interesting comparison it turns out that the results obtained in this paper are shaper than the existing results. With these ideas and techniques developed in this work, the interested readers can be inspired to explore this fascinating field of (p,q)-integral inequalities.

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