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Positive radial solutions for a noncooperative resonant nuclear reactor model with sign-changing nonlinearities

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Abstract

This paper is concerned with the existence of positive radial solutions of the following resonant elliptic system:

$$\begin{cases} -\Delta u = uv + f(|x|, u), & 0 < R_1 < |x| < R_2, x \in \mathbb{R}^N, \\ -\Delta v = cg(u) - dv, & 0 < R_1 < |x| < R_2, x \in \mathbb{R}^N, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 = \frac{\partial v}{\partial \mathbf{n}}, & |x| = R_1, |x| = R_2, \end{cases}$$

where $\mathbb{R}^N$ ($N \geq 1$) is the usual Euclidean space, **n** indicates the outward unit normal vector, $f \in C([R_1, R_2] \times [0, \infty), \mathbb{R})$, $g \in C([0, \infty), [0, \infty))$, and c and d are positive constants. By employing the classical fixed point theory we establish several novel existence theorems. Our main findings enrich and complement those available in the literature.

MSC: 34B15

Keywords: Noncooperative models; Radial solutions; Resonance; Existence; Fixed point

1 Introduction

Let $N \geq 1$ be an integer, and let $\Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2, 0 < R_1 < R_2 < \infty\}$ be an annulus with boundary $\partial\Omega$. In this paper, we establish the existence of positive radial solutions to the elliptic system

$$\begin{cases} -\Delta u = uv + f(|x|, u), & x \in \Omega, \\ -\Delta v = cg(u) - dv, & x \in \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 = \frac{\partial v}{\partial \mathbf{n}}, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where **n** denotes the outward unit normal vector on $\partial\Omega$, and c and d are positive constants. For convenience, we write $q \gg 0$ for some function $q \in C[R_1, R_2]$ if it is strictly positive on $[R_1, R_2]$, and we denote by $\bar{q}$ and $\underline{q}$ the maximum and minimum of $q \gg 0$, respectively. Throughout the paper, we assume the following:

(H1) $f \in C([R_1, R_2] \times [0, \infty), \mathbb{R})$, and there is $\chi \gg 0$ such that

$$p(t)f(t,u) \geq -\chi(t)u, \quad (t,u) \in [R_1, R_2] \times [0, \infty),$$

where $p(t) = t^{N-1}$, $t \in [R_1, R_2]$.

(H2) $g \in C([0, \infty), [0, \infty))$.

Obviously, the nonlinear term f is allowed to change its sign. Since the Laplace operator $-\Delta$ is not invertible under the Neumann boundary conditions, elliptic system (1.1) is resonant.

Elliptic system (1.1) is closely related to the stationary version of the mathematical model of nuclear reactors

$$\begin{cases} u_t - \Delta u = uv - bu, & x \in \Omega_0, t > 0, \\ v_t - \Delta v = cu - dv, & x \in \Omega_0, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 = \frac{\partial v}{\partial \mathbf{n}}, & x \in \partial\Omega_0, t > 0, \\ u(x,0) = u_0(x) \geq 0, \qquad v(x,0) = v_0(x) \geq 0, & x \in \overline{\Omega}_0, \end{cases} \tag{1.2}$$

where $\Omega_0 \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega_0$ and represents a closed container, u and v are respectively the density of the neutron flux and temperature of the nuclear reactors. $b \in [0, \infty)$ and $c, d \in (0, \infty)$ are constants, and u_0 and v_0 are continuous functions on $\overline{\Omega}_0$. System (1.2) improves the original model

$$\begin{cases} u_t - D\Delta u = u(av - b), & (x,t) \in \Omega \times (0,T), \\ v_t = cu, & (x,t) \in \Omega \times (0,T), \end{cases} \tag{1.3}$$

put forward in [1] by adding the diffusion and linear feedback of the temperature, where the Neumann boundary condition

$$\frac{\partial u}{\partial \mathbf{n}} = 0 = \frac{\partial v}{\partial \mathbf{n}}, \quad x \in \partial\Omega_0, t > 0, \tag{1.4}$$

means that the neutron flux cannot cross the boundary of the closed container, and the boundary of the closed container is heat insulation.

Over the past few decades, existence and related properties of positive stationary solutions of (1.3) (and its more general forms) have been studied by many authors; see Kastenberg and Chambré [1], Pao [2, 3], Gu and Wang [4], Arioli [5], López-Gómez [6], and the references therein. Meanwhile, some authors have also focused on the existence of positive solutions of the one-dimensional analogue of (1.3). See, for instance, Wang and An [7–9], Li [10], Chen [11, 12], and references therein. However, as far as we know, most of papers mentioned are devoted to system (1.3) subject to Dirichlet boundary condition, which means that there is no neutron flux on the boundary of the container and the constant temperature on it, whereas the results associated with (1.4) are relatively rare. In addition, the existence results on positive solutions, obtained in [7–9, 11, 12], largely depend on the positivity of the nonlinearities, and only the nonresonant case has been treated. Based these reasons, our aim in the present paper is establishing the existence of positive radial solutions for elliptic system (1.1) at resonance.

To state our main results, we define

$$g_0 = \lim_{u\to 0+} \frac{g(u)}{u}, \qquad g_\infty = \lim_{u\to\infty} \frac{g(u)}{u};$$
$$f_0 = \lim_{u\to 0+} \frac{p(t)f(t,u)}{u}, \qquad f_\infty = \lim_{u\to\infty} \frac{p(t)f(t,u)}{u},$$

uniformly for $t \in [R_1, R_2]$.

Theorem 1.1 *Assume* (H1) *and* (H2). *If* $g_0 = 0, f_\infty = \infty$, *and*

$$\lim_{u\to 0+} \frac{p(t)f(t,u)}{u} = -\chi(t), \tag{1.5}$$

then (1.1) *has at least one positive radial solution.*

Theorem 1.2 *Assume* (H1) *and*

(H2)′ $g \in C([0,\infty),[0,\infty))$, *and* $\lim_{u\to+\infty} p(t)g(u) = 0$ *uniformly for* $t \in [R_1, R_2]$.

If $f_0 = \infty$ *and*

$$\lim_{u\to +\infty} \frac{p(t)f(t,u)}{u} = -\chi(t), \tag{1.6}$$

then (1.1) *admits at least one positive radial solution.*

Remark 1.1 (H1) implies that the nonlinearity f may be sign-changing, and hence it is more general than the corresponding conditions in the existing literature. For the first time, we establish the existence results of elliptic system (1.1) in the resonant case; related results for other problems with sigh-changing nonlinearities can be found in [13, 14] and references therein. To look for radially symmetric positive solutions, we impose a radial dependence of the coefficients involved in f, which is far from being the case in [15, 16] and most of the references therein; the results of these references can be adapted to deal with homogeneous Neumann boundary conditions, which we will do in some future work.

The rest of the paper is arranged as follows. In Sect. 2, we introduce some notations and preliminaries. In Sect. 3, we prove the main and some related results and give some remarks to demonstrate the feasibility of our main findings.

2 Preliminaries

As is well known, in finding a radial solution $(u,v) = (u(r), v(r))$, elliptic system (1.1) is equivalent to

$$\begin{cases} -u''(r) - \frac{N-1}{r}u'(r) = u(r)v(r) + f(r,u(r)), & R_1 < r < R_2, \\ -v''(r) - \frac{N-1}{r}v'(r) = cg(u(r)) - dv(r), & R_1 < r < R_2, \\ u'(R_1) = 0 = u'(R_2), \\ v'(R_1) = 0 = v'(R_2), \end{cases}$$

where $r = |x|$. Let $t = r$ and $p(t) = t^{N-1}$. Then we have $p(t) > 0$ on $[R_1, R_2]$, and the above system becomes

$$\begin{cases} (p(t)u')' + p(t)uv + p(t)f(t,u) = 0, & R_1 < t < R_2, \\ (p(t)v')' - dp(t)v + cp(t)g(u) = 0, & R_1 < t < R_2, \\ u'(R_1) = 0 = u'(R_2), \\ v'(R_1) = 0 = v'(R_2). \end{cases} \tag{2.1}$$

Hence, if we show that there is a positive solution to (2.1), then system (1.1) admits a positive radial solution. Here the positivity of a solution (u, v) of (2.1) means that $u, v \gg 0$.

Let us denote by $K(t,s)$ the Green's function of

$$\begin{cases} (p(t)v')' - dp(t)v = 0, & R_1 < t < R_2, \\ v'(R_1) = 0 = v'(R_2). \end{cases}$$

Then it is easy to show that $K(t,s) > 0$ on $[R_1, R_2] \times [R_1, R_2]$ by an argument similar to the proof of [17, Lemmas 2.1 and 2.2], and therefore the linear problem

$$\begin{cases} (p(t)v')' - dp(t)v + cp(t)g(u) = 0, & R_1 < t < R_2, \\ v'(R_1) = 0 = v'(R_2) \end{cases}$$

can be equivalently written as

$$v(t) = c \cdot \int_{R_1}^{R_2} K(t,s)p(s)g\big(u(s)\big)\,ds =: c \cdot Tu(t). \tag{2.2}$$

Clearly, (H2) yields that $T : C[R_1, R_2] \to C[R_1, R_2]$ is a completely continuous operator. By (2.1) and (2.2) we get

$$\begin{cases} (p(t)u')' + cp(t)uTu + p(t)f(t,u) = 0, & R_1 < t < R_2, \\ u'(R_1) = 0 = u'(R_2), \end{cases} \tag{2.3}$$

which is a resonant problem. As this point, (2.3) can be transformed into the equivalent integral-differential equation

$$\begin{cases} (p(t)u')' - \chi(t)u + cp(t)uTu + (p(t)f(t,u) + \chi(t)u) = 0, & R_1 < t < R_2, \\ u'(R_1) = 0 = u'(R_2), \end{cases} \tag{2.4}$$

where the function χ is given as in (H1). In the following, we concentrate on the existence of positive solutions of (2.4). To this end, we denote by $G(t,s)$ the Green's function of the problem

$$\begin{cases} (p(t)u')' - \chi(t)u = 0, & R_1 < t < R_2, \\ u'(R_1) = 0 = u'(R_2). \end{cases}$$

Then by applying the same approach as in the proofs of [17, Lemmas 2.1 and 2.2] we can show that $G(t,s) > 0$ on $[R_1,R_2] \times [R_1,R_2]$ and (2.4) can be rewritten as the equivalent integral equation

$$\begin{aligned} u(t) &= c\int_{R_1}^{R_2} G(t,s)p(s)u(s)T\big(u(s)\big)\,ds + \int_{R_1}^{R_2} G(t,s)\big(p(s)f\big(s,u(s)\big) + \chi(s)u(s)\big)\,ds \\ &=: Au(t). \end{aligned}$$

Let E be the Banach space

$$E = \big\{u \in C[R_1,R_2] : u'(R_1) = 0 = u'(R_2)\big\}$$

equipped with the norm

$$\|u\| = \max_{t\in[R_1,R_2]} \big|u(t)\big|.$$

Denote by m_G and M_G the minimum and maximum of $G(t,s)$ on $[R_1,R_2] \times [R_1,R_2]$, respectively. Set $\sigma = \frac{m_G}{M_G}$ and

$$\mathcal{P} = \big\{u \in E : u(t) \ge \sigma\|u\|, t \in [R_1,R_2]\big\}.$$

Then $0 < \sigma < 1$, and $\mathcal{P}$ is a positive cone in E.

Lemma 2.1 *Assume* (H1) *and* (H2). *Then* $A(\mathcal{P}) \subseteq \mathcal{P}$, *and* $A : \mathcal{P} \to \mathcal{P}$ *is completely continuous.*

Proof Using (H1) and (H2), for any $u \in \mathcal{P}$, we get

$$\begin{aligned} Au(t) &= c\int_{R_1}^{R_2} G(t,s)p(s)u(s)T\big(u(s)\big)\,ds + \int_{R_1}^{R_2} G(t,s)\big(p(s)f\big(s,u(s)\big) + \chi(s)u(s)\big)\,ds \\ &\le M_G \cdot \int_{R_1}^{R_2} \big\{cp(s)u(s)T\big(u(s)\big) + \big(p(s)f\big(s,u(s)\big) + \chi(s)u(s)\big)\big\}\,ds, \quad \forall t \in [R_1,R_2], \end{aligned}$$

and therefore $\|Au\| \le M_G \cdot \int_{R_1}^{R_2}\{cp(s)u(s)T(u(s)) + (p(s)f(s,u(s)) + \chi(s)u(s))\}\,ds$. On the other hand,

$$\begin{aligned} Au(t) &= c\int_{R_1}^{R_2} G(t,s)p(s)u(s)T\big(u(s)\big)\,ds + \int_{R_1}^{R_2} G(t,s)\big(p(s)f\big(s,u(s)\big) + \chi(s)u(s)\big)\,ds \\ &\ge m_G \cdot \int_{R_1}^{R_2} \big\{cp(s)u(s)T\big(u(s)\big) + \big(p(s)f\big(s,u(s)\big) + \chi(s)u(s)\big)\big\}\,ds \\ &= \sigma \cdot M_G \int_{R_1}^{R_2} \big\{cp(s)u(s)T\big(u(s)\big) + \big(p(s)f\big(s,u(s)\big) + \chi(s)u(s)\big)\big\}\,ds. \end{aligned}$$

Combining the above two inequalities, we obtain $Au(t) \ge \sigma\|Au\|$. Hence $A(\mathcal{P}) \subseteq \mathcal{P}$. Finally, using (H1)–(H2), in a standard way, we can easily show that $A : \mathcal{P} \to \mathcal{P}$ is completely continuous. □

The main tool adopted in the paper is the following:

Lemma 2.2 ([18]) *Let E be a Banach space, and let $\mathcal{P} \subseteq E$ be a cone. Let Ω_1 and Ω_2 be open bounded subsets of E satisfying $0 \in \Omega_1$ and $\bar{\Omega}_1 \subseteq \Omega_2$, and let $T : \mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$ be a completely continuous operator such that*

(i) $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$,

or

(ii) $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

We conclude this section by giving some notations to be used later. Set

$$l = R_2 - R_1 \tag{2.5}$$

and

$$m = \min_{(t,s)\in[R_1,R_2]\times[R_1,R_2]} K(t,s),$$

$$M = \max_{(t,s)\in[R_1,R_2]\times[R_1,R_2]} K(t,s),$$

where $K(t,s)$ is as before. Define

$$p_0 = \int_{R_1}^{R_2} p(t)\,dt. \tag{2.6}$$

Then it is not difficult to see that $p_0 > 0$.

3 Proof of main results

Proof of Theorem 1.1 For positive constants $r < R$, set

$$\Omega_1 = \{u \in E : \|u\| < r\}, \qquad \Omega_2 = \{u \in E : \|u\| < R\}.$$

Then Ω_1 and Ω_2 are open bounded subsets of E with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subseteq \Omega_2$.

By (1.5) there exists $r_1 > 0$ such that for any $0 < u \leq r_1$,

$$p(t)f(t,u) \leq \epsilon u - \chi(t)u,$$

where $\epsilon > 0$ is a constant small enough so that $\epsilon l M_G \leq \frac{1}{2}$, and M_G is defined as in Sect. 2. Thus for $u \in \mathcal{P}$ with $\|u\| \leq r_1$,

$$p(t)f(t,u) + \chi(t)u \leq \epsilon u, \quad t \in [R_1, R_2].$$

From $g_0 = 0$ it follows there exists a positive constant

$$r_2 \ll 1 \tag{3.1}$$

such that $g(u) \leq \varepsilon u$ for any $0 < u \leq r_2$, and therefore for $u \in \mathcal{P}$ satisfying $\|u\| \leq r_2$, simple estimation shows that

$$\begin{aligned} c \cdot Tu(t) &= c \cdot \int_{R_1}^{R_2} K(t,s)p(s)g\big(u(s)\big)\, ds \\ &\leq \varepsilon cM\|u\| \int_{R_1}^{R_2} p(s)\, ds \\ &\leq \varepsilon cMp_0\|u\|, \end{aligned}$$

where ε is a sufficiently small positive constant such that $\varepsilon cMM_G p_0^2 \leq \frac{1}{2}$, and p_0 is given by (2.6). Let $r = \min\{r_1, r_2\}$. Then for $u \in \mathcal{P}$ with $\|u\| = r$, we get

$$\begin{aligned} Au(t) &= c\int_{R_1}^{R_2} G(t,s)p(s)u(s)T\big(u(s)\big)\, ds + \int_{R_1}^{R_2} G(t,s)\big(p(s)f\big(s,u(s)\big) + \chi(s)u(s)\big)\, ds \\ &\leq \varepsilon cMM_G p_0^2\|u\| + \epsilon lM_G\|u\| \\ &\leq \|u\|, \end{aligned}$$

which implies $\|Tu\| \leq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_1$.

On the other hand, $f_\infty = \infty$ yields that there exists $\tilde{R} > 0$ such that

$$p(t)f(t,u) \geq \eta u, \quad u \geq \tilde{R},$$

where $\eta > 0$ is a constant large enough with $\sigma lm_G(\eta + \underline{\chi}) \geq 1$. Fixing $R > \max\{r, \frac{\tilde{R}}{\sigma}\}$ and letting $u \in \mathcal{P}$ with $\|u\| = R$, we have

$$u(t) \geq \sigma\|u\| = \sigma R > \tilde{R},$$

and therefore

$$p(t)f(t,u) + \chi(t)u \geq \eta u + \chi(t)u \geq \sigma(\eta + \underline{\chi})\|u\|, \quad t \in [R_1, R_2].$$

Therefore we can deduce from (H2) that for $u \in \mathcal{P}$ with $\|u\| = R$,

$$\begin{aligned} Au(t) &= c\int_{R_1}^{R_2} G(t,s)p(s)u(s)T\big(u(s)\big)\, ds + \int_{R_1}^{R_2} G(t,s)\big(p(s)f\big(s,u(s)\big) + \chi(s)u(s)\big)\, ds \\ &\geq \sigma lm_G(\eta + \underline{\chi})\|u\| \\ &\geq \|u\|, \end{aligned}$$

which shows that $\|Tu\| \geq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_2$.

By Lemma 2.2(i) A possesses a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$, which is just a positive solution of (2.4). Accordingly, it follows from (2.2) that the original elliptic system (1.1) admits at least one positive radial solution. □

Proof of Theorem 1.2 To apply Lemma 2.2, we adopt the same strategy and notations as before. First, we show that for $r > 0$ sufficiently small,

$$\|Au\| \geq \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.2}$$

Indeed, by $f_0 = \infty$ there exists $\tilde{r} > 0$ such that

$$p(t)f(t,u) \geq \beta u, \quad 0 < u \leq \tilde{r},$$

where $\beta > 0$ is a constant large enough with $\sigma lm_G(\beta + \underline{\chi}) \geq 1$. Thus, for $0 < r \leq \tilde{r}$, if $u \in \mathcal{P}$ and $\|u\| = r$, then

$$p(t)f(t,u) + \chi(t)u \geq \beta u + \chi(t)u \geq \sigma(\beta + \underline{\chi})\|u\|, \quad t \in [R_1, R_2],$$

which, together with (H2)′, implies

$$\begin{aligned} Au(t) &= c\int_{R_1}^{R_2} G(t,s)p(s)u(s)T\big(u(s)\big)\,ds \\ &\quad + \int_{R_1}^{R_2} G(t,s)\big(p(s)f\big(s,u(s)\big) + \chi(s)u(s)\big)\,ds \\ &\geq \sigma lm_G(\beta + \underline{\chi})\|u\| \\ &\geq \|u\|. \end{aligned}$$

Hence (3.2) holds.

Next, we prove that for $R > 0$ large enough,

$$\|Au\| \leq \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.3}$$

From (1.6) it follows that there exists $\tilde{R} > 0$ such that

$$p(t)f(t,u) \leq \epsilon u - \chi(t)u$$

for $u \geq \tilde{R}$, where $\epsilon > 0$ satisfies $\epsilon lM_G \leq \frac{1}{2}$. Let $\tilde{R}_1 > \max\{\tilde{r}, \frac{\tilde{R}}{\sigma}\}$. Then for $u \in \mathcal{P}$ with $\|u\| \geq \tilde{R}_1$, we get

$$u(t) \geq \sigma\|u\| \geq \sigma\tilde{R}_1 > \tilde{R},$$

and thus

$$p(t)f(t,u) + \chi(t)u \leq \mu u \leq \mu\|u\|, \quad t \in [R_1, R_2].$$

On the other hand, (H2)′ implies that there exists $\tilde{R}_2 > 0$ such that $p(t)g(u) \leq \varepsilon$ for any $u \geq \tilde{R}_2$. Therefore, for $u \in \mathcal{P}$ with $\|u\| \geq \tilde{R}_2$, we have

$$\begin{aligned} c \cdot Tu(t) &= c \cdot \int_{R_1}^{R_2} K(t,s)p(s)g\big(u(s)\big)\,ds \\ &\leq \varepsilon cM \int_{R_1}^{R_2} ds \\ &\leq \varepsilon cMl, \end{aligned}$$

where $\varepsilon > 0$ is a constant satisfying $\varepsilon clMM_G p_0 \leq \frac{1}{2}$. Let $R = \max\{\tilde{R}_1, \tilde{R}_2\}$. Then for $u \in \mathcal{P}$ with $\|u\| = R$, we easily verify that

$$\begin{aligned} Au(t) &= c\int_{R_1}^{R_2} G(t,s)p(s)u(s)T\big(u(s)\big)\,ds + \int_{R_1}^{R_2} G(t,s)\big(p(s)f\big(s,u(s)\big) + \chi(s)u(s)\big)\,ds \\ &\leq \varepsilon clMM_G p_0\|u\| + \epsilon lM_G\|u\| \\ &\leq \|u\|, \end{aligned} \tag{3.4}$$

which yields (3.3).

Consequently, Lemma 2.2(ii) ensures that A has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$, and thus system (1.1) admits a positive radial solution. □

Remark 3.1 To illustrate the results of Theorem 1.1, we choose

$$\chi(t) = p(t) = t^{N-1}, \quad t \in [R_1, R_2].$$

Let $g(u) = u^\alpha$, $u \in [0,\infty)$, and

$$f(t,u) = \begin{cases} -u, & u \in [0,1], \\ -(u-2)^2, & u \in (1,2], \\ (u-2)^2, & u \in (2,+\infty), \end{cases}$$

where $\alpha > 1$ is a constant. Then it is not hard to verify that the assumptions in Theorem 1.1 are all satisfied. Therefore elliptic system (1.1) admits at least one positive radial solution.

Remark 3.2 To estimate (3.4), we assume (H2)′ in Theorem 1.2. Nevertheless, we believe that system (1.1) may admit positive radial solutions under (H2) and some suitable conditions on the nonlinearity g, which will be treated in the forthcoming paper. Clearly, Theorems 1.1 and 1.2 apply to models that cannot be dealt with by the results in the existing literature, and thus our main results are novel.

In the rest of the section, we consider the elliptic system

$$\begin{cases} -\Delta u = uv + f(|x|,u), & x \in \Omega, \\ -\Delta v = cg(u) - dv, & x \in \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0, \quad \frac{\partial v}{\partial \mathbf{n}} + \alpha v = 0, & x \in \partial\Omega, \end{cases} \tag{3.5}$$

where Ω is the annulus introduced in Sect. 1. Note that the boundary condition in (3.5) means that the nuclear reactors exchange heat energy with the outside and neutron flux cannot cross the boundary of the container, which is the case closer to the reality. In this case the positive constant α is called the heat transfer coefficient. Obviously, system (3.5) corresponds to the nuclear reactor model

$$\begin{cases} u_t - \Delta u = uv - bu, & x \in \Omega_0, t > 0, \\ v_t - \Delta v = cu - dv, & x \in \Omega_0, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = 0, \quad \frac{\partial v}{\partial \mathbf{n}} + \alpha v = 0, & x \in \partial\Omega_0, t > 0, \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, & x \in \overline{\Omega}_0. \end{cases}$$

For radial solutions, elliptic system (3.5) is equivalent to

$$\begin{cases} (p(t)u')' + p(t)uv + p(t)f(t,u) = 0, & R_1 < t < R_2, \\ (p(t)v')' - dp(t)v + cp(t)g(u) = 0, & R_1 < t < R_2, \\ u'(R_1) = 0 = u'(R_2), \\ v'(R_1) + \alpha v(R_1) = 0, \quad v'(R_2) + \alpha v(R_2) = 0. \end{cases}$$

Applying Lemma 2.2, by an argument similar to that of Sects. 2 and 3 we can show that the results of Theorems 1.1 and 1.2 are still valid for elliptic system (3.5).

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
RC carried out the analysis and proof the main results and was a major contributor in writing the manuscript. JL and GZ participated in checking the proofs. XK checked the English grammar and typing errors in the text. All authors read and approved the final manuscript.

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