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Global classical solutions to the elastodynamic equations with damping

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Abstract

In this paper, we show the global existence of classical solutions to the incompressible elastodynamics equations with a damping mechanism on the stress tensor in dimension three for sufficiently small initial data on periodic boxes, that is, with periodic boundary conditions. The approach is based on a time-weighted energy estimate, under the assumptions that the initial deformation tensor is a small perturbation around an equilibrium state and the initial data have some symmetry.

Keywords: Elastodynamics; Global classical solution; Time-weighted energy estimate; Damping mechanism

1 Introduction

The Oldroyd model for an incompressible viscoelastic fluid is governed by the following system of equations in $\mathbb{R}^3$:

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \mu \Delta u + \operatorname{div}(FF^{\mathrm{T}}), \\ F_t + u \cdot \nabla F = F \cdot \nabla u, \\ \operatorname{div} u = 0. \end{cases} \tag{1.1}$$

Here u denotes the fluid velocity, $F := (F_{ij})_{3\times 3}$ stands for the deformation tensor, p represents the fluids pressure, and $\mu > 0$ is a viscosity constant. The system (1.1) is also called as the viscoelastic Navier–Stokes equations. It have been studied by many authors (see [1–7] and the references cited therein) since the pioneering work of Renardy [8] and Baranger et al. [9]. There have been several interesting works on the initial value problem of (1.1), for instance, the short time existence of a smooth solution and the global existence of a smooth solution that is initially small have been established in various settings [10–12]. For large (rough) initial data, the global existence of weak solutions to (1.1) has been achieved by [13, 14] in dimension two. Recently, Jiang and Jiang [15] proved the global well-posedness of strong solutions for (1.1) in some classes of large data in dimension three.

The main difficulty in proving the global existence result of the viscoelastic Navier–Stokes equations lies in the equation of the stress tensor F in (1.1), which does not show any dissipative mechanism. In pursuing global weak solutions of (1.1), the authors of [11] proposed the following system as a way of approximating solutions of (1.1):

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \mu \Delta u + \operatorname{div}(FF^{\mathrm{T}}), \\ F_t + u \cdot \nabla F = \nu \Delta F + F \cdot \nabla u, \\ \operatorname{div} u = 0, \end{cases} \tag{1.2}$$

where $\nu > 0$ is a damping constant. We call (1.2) the viscoelastic Navier–Stokes equations with damping. It is not hard to establish the existence of a global in time weak solution of (1.2) by following the scheme of [16] on the incompressible Navier–Stokes equations. There are many studies about behaviors of solutions to (1.2), for example, partial regularity of weak solutions and forward self-similar solutions of (1.2) have been obtained in [17] and [18], respectively. Chemin and Masmoudi [19] established the global existence of small solutions to the Cauchy problem. Guillopé and Saut [20] also studied the initial-boundary value problem of this modified system (1.2). Finally, we would like to mention that the classical inviscid case of (1.2), i.e., $\mu = 0$, is a challenging problem to show the existence of global classical solutions. In this article, we are interested in studying the Cauchy problem only with the initial deformation that is a small displacement from equilibrium and the initial data have some symmetry.

Motivated by [15, 21], we investigate the global existence of the classical solutions to the following Cauchy problem in $\mathbb{R}^3$:

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \operatorname{div}(FF^{\mathrm{T}}), \\ F_t + u \cdot \nabla F - \nu \Delta F = F \cdot \nabla u, \\ \operatorname{div} u = 0, \\ u(0,x) = u_0(x), \qquad F(0,x) = F_0(x), \end{cases} \tag{1.3}$$

with periodic boundary conditions

$$x \in [-\pi, \pi]^3 = \mathbb{T}^3. \tag{1.4}$$

In what follows, we will make a fundamental simplification and assume that

$$\operatorname{div} F^{\mathrm{T}} = 0. \tag{1.5}$$

This means that the deformation F has divergence-free columns. It can be obtained by taking the divergence of the second equation in (1.3), which yields the following equation:

$$\partial_t \bigl(\operatorname{div} F^{\mathrm{T}}\bigr) + (u \cdot \nabla)\bigl(\operatorname{div} F^{\mathrm{T}}\bigr) = \nu \Delta \bigl(\operatorname{div} F^{\mathrm{T}}\bigr).$$

From the above transport equation, we can obtain that if $\operatorname{div} F^{\mathrm{T}}(x,0) = 0$, then $\operatorname{div} F^{\mathrm{T}} = 0$ for any $t > 0$.

And we assume that

$$\begin{aligned} &F_{i,j}(0,x) \text{ is even periodic with respect to } x_3, \quad i = j, \\ &F_{i,j}(0,x) \text{ is odd periodic with respect to } x_3, \quad i \neq j, \end{aligned} \tag{1.6}$$

moreover,

$$\int_{\mathbb{T}^3} u(0)\,\mathrm{d}x = 0, \qquad \int_{\mathbb{T}^3} F_{ij}(0)\delta_{ij}\,\mathrm{d}x = \alpha \neq 0. \tag{1.7}$$

Before stating our main result, we shall introduce some simplified notations in this article:

(1) Sobolev's spaces and norms:

$$L^p := L^p(\Omega) = W^{0,p}(\Omega), \qquad H^k := W^{k,2}(\Omega), \qquad \|\cdot\|_k := \|\cdot\|_{H^k},$$

$$a \lesssim b \text{ means that } a \leq cb \text{ for some positive constant } c,$$

where $1 < p \leq \infty$ and k are nonnegative integers;

(2) Estimates of the product of functions in Sobolev spaces (denoted as product estimates):

$$\|fg\|_j \lesssim \begin{cases} \|f\|_1\|g\|_1 & \text{for } j = 0; \\ \|f\|_j\|g\|_2 & \text{for } 0 \leq j \leq 2; \\ \|f\|_2\|g\|_j + \|f\|_j\|g\|_2 & \text{for } 3 \leq j \leq 5, \end{cases} \tag{1.8}$$

which can be easily verified by Hölder's inequality and the embedding inequality (see [22, Theorem 4.12]).

Under the assumptions of (1.6) and (1.7), now we can state our main result in the following theorem.

Theorem 1.1 *Consider the* 3*D elastodynamic system* (1.3) *and* (1.4) *with initial data satisfying the conditions* (1.6) *and* (1.7). *Assume that* $(u_0, F_0) \in H^3(\mathbb{T}^3)$ *with* $\operatorname{div} u_0 = \operatorname{div} F_0^{\mathrm{T}} = 0$. *Then there exists a small constant* $\epsilon > 0$ *depending on* α *such that system* (1.3) *admits a global classical solution provided that*

$$\|u_0\|_3 + \|\nabla F_0\|_2 \leq \epsilon. \tag{1.9}$$

Without loss of generality, we assume that $\alpha = (2\pi)^3$, as our results do not change qualitatively as $\nu > 0$ is varied, so we set

$$\nu = 1.$$

Obviously, $(0, I)$ is an equilibrium-state solution of the system (1.3). Now, we denote the perturbation quantities by

$$u = u - 0, \qquad U = F - I,$$

where I denotes an identity matrix. By (1.7), we have

$$\int_{\mathbb{T}^3} u(0)\,\mathrm{d}x = \int_{\mathbb{T}^3} U(0)\,\mathrm{d}x = 0. \tag{1.10}$$

Then, (u, U) satisfies the perturbation equations:

$$\begin{cases} U_t + u \cdot \nabla U - \Delta U = \nabla u + U \cdot \nabla u, \\ u_t + u \cdot \nabla u + \nabla p = \operatorname{div}(U + UU^{\mathrm{T}}), \\ \operatorname{div} u = 0, \qquad \operatorname{div} U^{\mathrm{T}} = 0, \\ u(0,x) = u_0(x), \qquad U(0,x) = U_0(x). \end{cases} \tag{1.11}$$

And the properties of initial data (1.6) and (1.7) persist. Indeed,

$$\begin{aligned} &U_{i,j}(t,x) \text{ is even periodic with respect to } x_3, \quad i = j, \\ &U_{i,j}(t,x) \text{ is odd periodic with respect to } x_3, \quad i \neq j, \end{aligned} \tag{1.12}$$

and

$$\int_{\mathbb{T}^3} u \,\mathrm{d}x = \int_{\mathbb{T}_3} U \,\mathrm{d}x = 0. \tag{1.13}$$

Setting $U_j := Ue_j$ for $j = 1, 2, 3$, from the assumption $\operatorname{div} U^{\mathrm{T}} = 0$, we have

$$\operatorname{div}\bigl(UU^{\mathrm{T}}\bigr) = \sum_{k=1}^{3} (U_k \cdot \nabla) U_k. \tag{1.14}$$

For the system (1.11), now we define the following weighted energies which will enable us to achieve our desired estimates:

$$\begin{aligned} \mathcal{E}_0(t) &= \sup_{0 \le \tau \le t} \bigl(\|u(\tau)\|_3^2 + \|U(\tau)\|_3^2 \bigr) + \int_0^t \bigl(\|U(\tau)\|_4^2 + \|\nabla u(\tau)\|_2^2 \bigr) \,\mathrm{d}\tau, \\ \mathcal{E}_1(t) &= \sup_{0 \le \tau \le t} (1+\tau)^2 \bigl(\|u(\tau)\|_1^2 + \|U(\tau)\|_1^2 \bigr) \\ &\quad + \int_0^t (1+\tau)^2 \bigl(\|U(\tau)\|_2^2 + \|\nabla u(\tau)\|_0^2 \bigr) \,\mathrm{d}\tau. \end{aligned} \tag{1.15}$$

The energies above are defined on the domain $\mathbb{R}^+ \times \mathbb{T}^3$.

The rest of this paper is organized as follows. In Sect. 2, we will derive *a priori* estimates of the higher order energy $\mathcal{E}_0$ and lower order energy $\mathcal{E}_1$, and we only need to consider the highest-order norms in each energy estimate due to the condition (1.13) and the Poincaré inequality. And in Sect. 3, we will prove Theorem 1.1.

2 Energy estimate

First, we will deal with the higher-order energy $\mathcal{E}_0$. It shows that the highest-order norm $H^3(\mathbb{T}^3)$ of $u(t,\cdot)$ and $U(t,\cdot)$ can be bounded uniformly.

Lemma 2.1 *Under the condition* (1.10), *it holds that*

$$\mathcal{E}_0(t) \lesssim \mathcal{E}_0(0) + \mathcal{E}_0^{3/2}(t). \tag{2.1}$$

Proof We divide the proof into two parts. Instead of deriving the estimate of $\mathcal{E}_0(t)$ directly, we will first get the uniform bound of $\mathcal{E}_{0,1}$ which is defined by

$$\mathcal{E}_{0,1}(t) = \sup_{0\le\tau\le t}\left(\left\|u(\tau)\right\|_3^2 + \left\|U(\tau)\right\|_3^2\right) + \int_0^t \left\|U(\tau)\right\|_4^2\,\mathrm{d}\tau. \tag{2.2}$$

First, to get the estimate of $\mathcal{E}_{0,1}$, we apply the ∇^3 derivative on system (1.11). Then, we take the inner product with $\nabla^3 U$ in the first equation of (1.11) and also the inner product with $\nabla^3 u$ in the second equation of the same system. Adding them up, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|U\|_{\dot{H}^3}^2 + \|u\|_{\dot{H}^3}^2\right) + \|U\|_{\dot{H}^4}^2 = M_1 + M_2 + M_3 + M_4 + M_5, \tag{2.3}$$

where

$$\begin{aligned}
M_1 &= -\int_{\mathbb{T}^3} u\cdot\nabla\nabla^3 u\nabla^3 u + u\cdot\nabla\nabla^3 U\nabla^3 U\,\mathrm{d}x,\\
M_2 &= \int_{\mathbb{T}^3} \operatorname{div}\nabla^3 U\nabla^3 u + \nabla\nabla^3 u\nabla^3 U\,\mathrm{d}x,\\
M_3 &= \sum_{k=1}^{3} C_3^k \int_{\mathbb{T}^3} \nabla^k U\cdot\nabla\nabla^{3-k} u\nabla^3 U - \nabla^k u\cdot\nabla\nabla^{3-k} U\nabla^3 U\,\mathrm{d}x,\\
M_4 &= -\sum_{k=1}^{3} C_3^k \int_{\mathbb{T}^3} \nabla^k u\cdot\nabla\nabla^{3-k} u\nabla^3 u\,\mathrm{d}x,\\
M_5 &= \int_{\mathbb{T}^3} U\cdot\nabla\nabla^3 u\nabla^3 U + \nabla^3\operatorname{div}\left(UU^{\mathrm{T}}\right)\nabla^3 u\,\mathrm{d}x.
\end{aligned}$$

First, for the term M_1, using integration by parts and the divergence-free condition, we have

$$M_1 = 0. \tag{2.4}$$

For the M_2, by integration by parts, we get

$$M_2 = \int_{\mathbb{T}^3} \operatorname{div}\nabla^3 U\nabla^3 u - \nabla^3 u\operatorname{div}\nabla^3 U\,\mathrm{d}x. \tag{2.5}$$

By using the Hölder's inequality, we have

$$|M_2| \lesssim \left\|\operatorname{div}\nabla^3 U\right\|_0\left\|\nabla^3 u\right\|_0 \lesssim \|U\|_4\|u\|_3. \tag{2.6}$$

Then applying div to $(1.11)_1^{\mathrm{T}}$, where T represents the transpose of the matrix, we find that

$$-\Delta u = \operatorname{div}(U\cdot\nabla u - u\cdot\nabla U)^{\mathrm{T}}, \tag{2.7}$$

where we have used the condition $\operatorname{div} U^{\mathrm{T}} = 0$. From the regularity theory of elliptic equations [23, 24], thus we get

$$\begin{aligned}
\|u\|_3 &\lesssim \|U\cdot\nabla u\|_2 + \|u\cdot\nabla U\|_2,\\
&\lesssim \|U\|_3\|u\|_3.
\end{aligned} \tag{2.8}$$

Substituting the above inequality into (2.6), we obtain that

$$|M_2| \lesssim \|U\|_4^2 \|u\|_3. \tag{2.9}$$

Thus, from (2.9), we obtain

$$\begin{aligned} \int_0^t |M_2| \, \mathrm{d}\tau &\lesssim \sup_{0\le\tau\le t} \|u(\tau)\|_3 \int_0^t \|U(\tau)\|_4^2 \, \mathrm{d}\tau \\ &\lesssim \mathcal{E}_0^{3/2}(t). \end{aligned} \tag{2.10}$$

For the term M_3, by using the Hölder's inequality and product estimates, we obtain

$$\begin{aligned} |M_3| &\lesssim \left(\|\nabla U \nabla^3 u\|_0 + \|\nabla u \nabla^3 U\|_0 + \|\nabla^2 u \nabla^2 U\|_0 \right) \|\nabla^3 U\|_0 \\ &\lesssim \left(\|\nabla^3 u\|_0 \|\nabla U\|_2 + \|\nabla u\|_1 \|\nabla^3 U\|_1 + \|\nabla^2 u\|_1 \|\nabla^2 U\|_1 \right) \|\nabla^3 U\|_0 \\ &\lesssim \|u\|_3 \|U\|_4^2. \end{aligned}$$

Hence,

$$\begin{aligned} \int_0^t |M_3| \, \mathrm{d}\tau &\lesssim \sup_{0\le\tau\le t} \|u(\tau)\|_3 \int_0^t \|U(\tau)\|_4^2 \, \mathrm{d}\tau \\ &\lesssim \mathcal{E}_0^{3/2}(t). \end{aligned} \tag{2.11}$$

For the estimate of M_4, using the Hölder's inequality and product estimates, we have

$$\begin{aligned} |M_4| &\lesssim \|u\|_3 \left(\|\nabla u \cdot \nabla^3 u\|_0 + \|\nabla^2 u \cdot \nabla^2 u\|_0 \right) \\ &\lesssim \|u\|_3 \left(\|\nabla u\|_2 \|\nabla^3 u\|_0 + \|\nabla^2 u\|_1 \|\nabla^2 u\|_1 \right) \\ &\lesssim \|u\|_3 \|\nabla u\|_2^2, \end{aligned} \tag{2.12}$$

Hence,

$$\begin{aligned} &\int_0^t |M_4| \, \mathrm{d}\tau \\ &\quad \lesssim \sup_{0\le\tau\le t} \|u(\tau)\|_3 \int_0^t \|\nabla u(\tau)\|_2^2 \, \mathrm{d}\tau \\ &\quad \lesssim \mathcal{E}_0^{3/2}. \end{aligned} \tag{2.13}$$

For the last term M_5, by integration by parts, we can obtain

$$M_5 = -\int_{\mathbb{T}^3} \operatorname{div} U \cdot \nabla^3 u \nabla^3 U + U \cdot \nabla^3 u \operatorname{div} \nabla^3 U \, \mathrm{d}x + \int_{\mathbb{T}^3} \nabla^3 \operatorname{div}\left(UU^{\mathrm{T}}\right) \nabla^3 u \, \mathrm{d}x. \tag{2.14}$$

By the Hölder's inequality and product estimates, we have

$$\begin{aligned} |M_5| &\lesssim \|\nabla^3 u\|_0 \left(\|\nabla U \nabla^3 U\|_0 + \|U \nabla^4 U\|_0 + \|\nabla^3 \operatorname{div}(UU^{\mathrm{T}})\|_0 \right) \\ &\lesssim \|\nabla^3 u\|_0 \left(\|\nabla U\|_2 \|\nabla^3 U\|_0 + \|\nabla^4 U\|_0 \|U\|_2 + \|UU^{\mathrm{T}}\|_4 \right) \end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\nabla^3 u\|_0 \left(\|U\|_4^2 + \|U\|_2 \|U^{\mathrm{T}}\|_4 + \|U\|_4 \|U^{\mathrm{T}}\|_2\right) \\
&\lesssim \|U\|_4^2 \|u\|_3,
\end{aligned}
\tag{2.15}
$$

thus, we can obtain that

$$
\begin{aligned}
\int_0^t \left|M_5(\tau)\right| \mathrm{d}\tau &\lesssim \sup_{0\le\tau\le t} \left\|u(\tau)\right\|_3 \int_0^t \left\|U(\tau)\right\|_4^2 \mathrm{d}\tau \\
&\lesssim \mathcal{E}_0^{3/2}.
\end{aligned}
\tag{2.16}
$$

Summing up the estimates for M_1–M_5, i.e., (2.4), (2.10), (2.11), (2.13), and (2.16), then integrating (2.3) with respect to time, we now get the estimate of $\mathcal{E}_{0,1}(t)$ which is defined in (2.2) as

$$
\mathcal{E}_{0,1} \lesssim \mathcal{E}_0(0) + \mathcal{E}_0^{3/2}.
\tag{2.17}
$$

Here, we have used the Poincaré inequality to consider the highest-order norm only.

Next, we work with the left term in $\mathcal{E}_0(t)$. Applying the ∇^2 derivative on the first equation of system (1.11), and taking the inner product with $\nabla^2\nabla u$, we get

$$
\|\nabla u\|_{\dot{H}^2}^2 = M_6 + M_7 + M_8,
\tag{2.18}
$$

where

$$
\begin{aligned}
M_6 &= \int_{\mathbb{T}^3} \nabla^2(u\cdot\nabla U - U\cdot\nabla u)\nabla^2\nabla u \,\mathrm{d}x, \\
M_7 &= -\int_{\mathbb{T}^3} \nabla^2 \Delta U \nabla^2\nabla u \,\mathrm{d}x, \\
M_8 &= \int_{\mathbb{T}^3} \nabla^2 U_t \nabla^2\nabla u \,\mathrm{d}x.
\end{aligned}
$$

As when getting the estimate of $\mathcal{E}_{0,1}$, we shall derive the estimate of the each term on the right-hand side of (2.18).

For M_6, by the Hölder's inequality and product estimates, we get

$$
\begin{aligned}
|M_6| &\lesssim \|\nabla^3 u\|_0 \left(\|\nabla^2(u\cdot\nabla U)\|_0 + \|\nabla^2(U\cdot\nabla u)\|_0\right) \\
&\lesssim \|\nabla u\|_2 \left(\|u\cdot\nabla U\|_2 + \|U\cdot\nabla u\|_2\right) \\
&\lesssim \|\nabla u\|_2 \left(\|u\|_2\|\nabla U\|_2 + \|U\|_2\|\nabla u\|_2\right) \\
&\lesssim \|\nabla u\|_2^2 \|U\|_4,
\end{aligned}
\tag{2.19}
$$

where we have used the Poincaré inequality in the last inequality.

Thus, we conclude

$$\begin{aligned}
&\int_0^t \big|M_6(\tau)\big|\,\mathrm{d}\tau \\
&\quad\lesssim \sup_{0\le\tau\le t}\big\|U(\tau)\big\|_4\int_0^t \big\|\nabla u(\tau)\big\|_2^2\,\mathrm{d}\tau \\
&\quad\lesssim \mathcal{E}_0^{3/2}.
\end{aligned}\tag{2.20}$$

The estimate for M_7 is almost the same, by using the Hölder's inequality, we can obtain that

$$\begin{aligned}
\int_0^t \big|M_7(\tau)\big|\,\mathrm{d}\tau &\lesssim \int_0^t \|U\|_4\|\nabla u\|_2\,\mathrm{d}\tau \\
&\lesssim \left(\int \big\|U(\tau)\big\|_4^2\,\mathrm{d}\tau\right)^{1/2}\left(\int_0^t \big\|\nabla u(\tau)\big\|_2^2\,\mathrm{d}\tau\right)^{1/2} \\
&\lesssim \mathcal{E}_{0,1}^{1/2}(t)\left(\int_0^t \big\|\nabla u(\tau)\big\|_2^2\,\mathrm{d}\tau\right)^{1/2}.
\end{aligned}\tag{2.21}$$

For the last term M_8, using integration by parts, we can write this term as

$$M_8 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\nabla^2 U\nabla\nabla^2 u\,\mathrm{d}x + \int_{\mathbb{T}^3}\nabla^2\operatorname{div} U\nabla^2 u_t\,\mathrm{d}x.\tag{2.22}$$

By $(1.11)_2$, we can obtain

$$\begin{aligned}
M_8 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\nabla^2 U\nabla\nabla^2 u\,\mathrm{d}x + \int_{\mathbb{T}^3}\operatorname{div}\nabla^2 U\nabla^2\big(\operatorname{div}\big(U+UU^{\mathrm{T}}\big)-\nabla p - u\cdot\nabla u\big)\,\mathrm{d}x \\
&=: K_1 + K_2.
\end{aligned}\tag{2.23}$$

First, by using the product estimates, we get

$$\int_0^t \big|K_1(\tau)\big|\,\mathrm{d}\tau \lesssim \|U\|_2\|u\|_3 \lesssim \mathcal{E}_{0,1}(t).\tag{2.24}$$

On the other hand, by the Hölder's inequality and product estimates, we get

$$\begin{aligned}
|K_2| &\lesssim \big\|\nabla^3 U\big\|_0\big(\|U\|_3 + \big\|UU^{\mathrm{T}}\big\|_3 + \big\|\nabla^3 p\big\|_0 + \big\|\nabla^2(u\cdot\nabla u)\big\|_0\big) \\
&\lesssim \|U\|_3\big(\|U\|_3 + \|U\|_3^2 + \|p\|_3 + \big\|\nabla^2 u\cdot\nabla u\big\|_0 + \big\|u\cdot\nabla^3 u\big\|_0\big) \\
&\lesssim \|U\|_3\big(\|U\|_3 + \|U\|_3^2 + \|p\|_3 + \|\nabla u\|_2\big\|\nabla^2 u\big\|_0 + \big\|\nabla^3 u\big\|_0\|u\|_2\big) \\
&\lesssim \|U\|_3\big(\|U\|_3 + \|U\|_3^2 + \|p\|_3 + \|\nabla u\|_2^2\big),
\end{aligned}\tag{2.25}$$

where we have used the Poincaré inequality in the last inequality.

Now, applying div to $(1.11)_2$, we get

$$\Delta p = \operatorname{div}\big(-u\cdot\nabla u + \operatorname{div}\big(U+UU^{\mathrm{T}}\big)\big),\tag{2.26}$$

thus, from the regularity theory of elliptic equations and product estimates, we get

$$\begin{aligned}\|p\|_3 &\lesssim \|u\cdot\nabla u\|_2 + \|U\|_3 + \left\|UU^{\mathrm{T}}\right\|_3\\ &\lesssim \|\nabla u\|_2\|u\|_2 + \|U\|_3 + \|U\|_3\left\|U^{\mathrm{T}}\right\|_2 + \|U\|_2\left\|U^{\mathrm{T}}\right\|_3\\ &\lesssim \|U\|_3 + \|U\|_3^2 + \|\nabla u\|_2^2, \end{aligned}\tag{2.27}$$

where we have used Poincaré inequality in the last inequality. Thus, we can get

$$|K_2| \lesssim \|U\|_3\big(\|U\|_3 + \|U\|_3^2 + \|\nabla u\|_2^2\big) \tag{2.28}$$

and so

$$\begin{aligned}\int_0^t |K_2|\,\mathrm{d}\tau &\lesssim \int_0^t \left\|U(\tau)\right\|_3^2\mathrm{d}\tau + \sup_{0\le\tau\le t}\left\|U(\tau)\right\|_3\int_0^t \left\|U(\tau)\right\|_3^2\mathrm{d}\tau\\ &\quad + \sup_{0\le\tau\le t}\left\|U(\tau)\right\|_3\int_0^t\left\|\nabla u(\tau)\right\|_2^2\mathrm{d}\tau\\ &\lesssim \mathcal{E}_{0,1}(t) + \mathcal{E}_0^{3/2}(t).\end{aligned}\tag{2.29}$$

By (2.24) and (2.29), we have

$$\int_0^t \left|M_8(\tau)\right|\mathrm{d}\tau \lesssim \mathcal{E}_{0,1}(t) + \mathcal{E}_0^{3/2}(t). \tag{2.30}$$

Integrating (2.18) with respect to time, using the estimates of (2.20), (2.21), (2.30), and Young's inequality, we obtain

$$\int_0^t \left\|\nabla u(\tau)\right\|_2^2\mathrm{d}\tau \lesssim \mathcal{E}_{0,1}(t) + \mathcal{E}_0^{3/2}(t). \tag{2.31}$$

Multiplying (2.17) by a suitably large number and adding (2.31), we then complete the proof of Lemma 2.1. □

Next, we want to give the estimate of the lower-order energy $\mathcal{E}_1(t)$ defined in (1.15). The result is given in the following lemma.

Lemma 2.2 *Under the condition* (1.10), *it holds that*

$$\mathcal{E}_1(t) \lesssim \mathcal{E}_1(0) + \mathcal{E}_0^{1/2}(t)\mathcal{E}_1^{1/2}(t) + \mathcal{E}_0^{1/2}(t)\mathcal{E}_1(t) + \mathcal{E}_1^{3/2}(t). \tag{2.32}$$

Proof Like the proof Lemma of 2.1, we divide this proof into two parts. Also, we will first get the estimate of $\mathcal{E}_{1,1}$ which is defined by the following:

$$\mathcal{E}_{1,1}(t) = \sup_{0\le\tau\le t}(1+\tau)^2\big(\left\|u(\tau)\right\|_1^2 + \left\|U(\tau)\right\|_1^2\big) + \int_0^t (1+\tau)^2\left\|U(\tau)\right\|_2^2\mathrm{d}\tau.$$

Now apply ∇ derivative on system (1.11), take the inner product with ∇U in the first equation of (1.11), and also take the inner product with ∇u in the second equation of

(1.11). Adding them up and multiplying by the time weight $(1+t)^2$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^2\big(\|U\|_{\dot{H}^1}^2+\|u\|_{\dot{H}^1}^2\big)+(1+t)^2\|U\|_{\dot{H}^2}^2=N_1+N_2+N_3+N_4, \tag{2.33}$$

where

$$\begin{aligned}
N_1&=(1+t)\big(\|U\|_{\dot{H}^1}^2+\|u\|_{\dot{H}^1}^2\big),\\
N_2&=(1+t)^2\int_{\mathbb{T}^3}\nabla(U\cdot\nabla u-u\cdot\nabla U+\nabla u)\nabla U\,\mathrm{d}x,\\
N_3&=(1+t)^2\int_{\mathbb{T}^3}\nabla\operatorname{div}\big(U+UU^{\mathrm{T}}\big)\nabla u\,\mathrm{d}x,\\
N_4&=-(1+t)^2\int_{\mathbb{T}^3}\nabla(u\cdot\nabla u)\nabla u\,\mathrm{d}x,
\end{aligned}$$

First, the term N_1 is equivalent to the following form:

$$N_1=(1+t)\big(\|\nabla U\|_0^2+\|\nabla u\|_0^2\big).$$

Thus,

$$|N_1|\lesssim(1+t)\big(\|U\|_4\|U\|_2+\|\nabla u\|_0\|\nabla u\|_2\big),$$

hence, by using the Hölder's inequality, we can get

$$\begin{aligned}
&\int_0^t|N_1|\,\mathrm{d}\tau\\
&\quad\lesssim\left(\int_0^t\big\|U(\tau)\big\|_4^2\,\mathrm{d}\tau\right)^{1/2}\cdot\left(\int_0^t(1+\tau)^2\big\|U(\tau)\big\|_2^2\,\mathrm{d}\tau\right)^{1/2}\\
&\qquad+\left(\int_0^t\big\|\nabla u(\tau)\big\|_2^2\,\mathrm{d}\tau\right)^{1/2}\cdot\left(\int_0^t(1+\tau)^2\big\|\nabla u(\tau)\big\|_0^2\,\mathrm{d}\tau\right)^{1/2}\\
&\quad\lesssim\mathcal{E}_0^{1/2}(t)\mathcal{E}_1^{1/2}(t).
\end{aligned}\tag{2.34}$$

For the term N_2, by using integrating by parts and Hölder's inequality, we can get

$$|N_2|\lesssim(1+t)^2\big(\|U\cdot\nabla u\|_0+\|u\cdot\nabla U\|_0+\|\nabla u\|_0\big)\big\|\nabla^2U\big\|_0. \tag{2.35}$$

Firstly, by (2.7) and the product estimates, we can obtain that

$$\|\nabla u\|_1\lesssim\|U\cdot\nabla u\|_1+\|u\cdot\nabla U\|_1\lesssim\|U\|_2\|u\|_2, \tag{2.36}$$

then putting (2.36) into (2.35), we obtain

$$\begin{aligned}
|N_2|&\lesssim(1+t)^2\big(\|\nabla u\|_0\|U\|_2+\|u\|_1\|\nabla U\|_1+\|U\|_2\|u\|_2\big)\|U\|_2\\
&\lesssim(1+t)^2\big(\|\nabla u\|_0\|U\|_2^2+\|u\|_2\|U\|_2^2\big)\\
&\lesssim(1+t)^2\|u\|_2\|U\|_2^2.
\end{aligned}\tag{2.37}$$

Hence, we get

$$\int_0^t \left|N_2(\tau)\right| \mathrm{d}\tau \lesssim \sup_{0\le\tau\le t} \left\|u(\tau)\right\|_2 \int_0^t (1+\tau)^2 \left\|U(\tau)\right\|_2^2 \mathrm{d}\tau$$
$$\lesssim \mathcal{E}_0^{1/2}(t)\mathcal{E}_1(t). \tag{2.38}$$

Also for the term N_3, by using the Hölder's inequality and product estimates, we obtain

$$|N_3| \lesssim (1+t)^2\|U\|_2^2\|u\|_2 + (1+t)^2\|U\|_2\|\nabla u\|_0. \tag{2.39}$$

By (2.36), we have

$$|N_3| \lesssim (1+t)^2\|U\|_2^2\|u\|_2.$$

Thus, we get

$$\int_0^t \left|N_3(\tau)\right| \mathrm{d}\tau \lesssim \sup_{0\le\tau\le t} \left\|u(\tau)\right\|_2 \int_0^t (1+\tau)^2 \left\|U(\tau)\right\|_2^2 \mathrm{d}\tau$$
$$\lesssim \mathcal{E}_0^{1/2}\mathcal{E}_1(t). \tag{2.40}$$

Next, we turn to estimating the last term N_4. By using integration by parts, we have

$$\begin{aligned} N_4 &= -(1+\tau)^2 \int_{\mathbb{T}^3} \nabla u\cdot\nabla u\nabla u + u\cdot\nabla\nabla u\nabla u \,\mathrm{d}x \\ &= -(1+\tau)^2 \int_{\mathbb{T}^3} \nabla u\cdot\nabla u\nabla u \,\mathrm{d}x, \end{aligned} \tag{2.41}$$

thus, by using the product estimates, we get

$$\begin{aligned} |N_4| &\lesssim (1+t)^2\|\nabla u\cdot\nabla u\|_0\|\nabla u\|_0 \\ &\lesssim (1+t)^2\|u\|_3\|\nabla u\|_0^2. \end{aligned} \tag{2.42}$$

Hence,

$$\begin{aligned} &\int_0^t \left|N_4(\tau)\right| \mathrm{d}\tau \\ &\quad\lesssim \sup_{0\le\tau\le t} \left\|u(\tau)\right\|_3 \int_0^t (1+\tau)^2 \left\|\nabla u(\tau)\right\|_0^2 \mathrm{d}\tau \\ &\quad\lesssim \mathcal{E}_0^{1/2}\mathcal{E}_1(t). \end{aligned} \tag{2.43}$$

Now, summing up the estimates for N_1–N_4, i.e., (2.34), (2.38), (2.40), and (2.43), and integrating (2.33) with respect to time, we get the estimate of $\mathcal{E}_{1,1}$,

$$\mathcal{E}_{1,1}(t) \lesssim \mathcal{E}_1(0) + \mathcal{E}_0^{1/2}(t)\mathcal{E}_1^{1/2}(t) + \mathcal{E}_0^{1/2}(t)\mathcal{E}_1(t). \tag{2.44}$$

Here, we have used the Poincaré inequality to consider the highest-order norm only.

By an identical argument as in the proof of Lemma 2.1, multiplying the first equation of (1.11) by ∇u and taking the inner product, then multiplying by the time weight $(1+t)^2$, we get

$$(1+t)^2\|\nabla u\|_0^2 = N_5 + N_6 + N_7, \tag{2.45}$$

where

$$\begin{aligned}
N_5 &= (1+t)^2 \int_{\mathbb{T}^3} (u\cdot\nabla U - U\cdot\nabla u)\nabla u\,\mathrm{d}x,\\
N_6 &= -(1+t)^2 \int_{\mathbb{T}^3} \Delta U \nabla u\,\mathrm{d}x,\\
N_7 &= (1+t)^2 \int_{\mathbb{T}^3} U_t \nabla u\,\mathrm{d}x.
\end{aligned}$$

Using the product estimates, we obtain

$$|N_5| \lesssim (1+t)^2\|u\|_2\|U\|_2\|\nabla u\|_0,$$

hence, we conclude that

$$\begin{aligned}
&\int_0^t \bigl|N_5(\tau)\bigr|\,\mathrm{d}\tau\\
&\quad\lesssim \sup_{0\le\tau\le t}\bigl\|u(\tau)\bigr\|_2 \int_0^t (1+\tau)^2\bigl\|U(\tau)\bigr\|_2\bigl\|\nabla u(\tau)\bigr\|_0\,\mathrm{d}\tau\\
&\quad\lesssim \sup_{0\le\tau\le t}\bigl\|u(\tau)\bigr\|_2 \biggl(\int_0^t (1+\tau)^2\bigl\|U(\tau)\bigr\|_2^2\,\mathrm{d}\tau\biggr)^{1/2}\biggl(\int_0^t \bigl\|(1+\tau)^2\nabla u(\tau)\bigr\|_0^2\,\mathrm{d}\tau\biggr)^{1/2}\\
&\quad\lesssim \mathcal{E}_0^{1/2}\mathcal{E}_1(t).
\end{aligned} \tag{2.46}$$

Similarly, we have

$$|N_6| \lesssim (1+t)^2\|U\|_2\|\nabla u\|_0,$$

thus,

$$\begin{aligned}
\int_0^t \bigl|N_6(\tau)\bigr|\,\mathrm{d}\tau &\lesssim \int_0^t (1+t)^2\|U\|_2\|\nabla u\|_0\,\mathrm{d}\tau\\
&\lesssim \biggl(\int_0^t (1+\tau)^2\bigl\|U(\tau)\bigr\|_2^2\,\mathrm{d}\tau\biggr)^{1/2}\biggl(\int_0^t (1+\tau)^2\bigl\|\nabla u(\tau)\bigr\|_0^2\,\mathrm{d}\tau\biggr)^{1/2}\\
&\lesssim \mathcal{E}_{1,1}^{1/2}(t)\biggl(\int_0^t (1+\tau)^2\bigl\|\nabla u(\tau)\bigr\|_0^2\,\mathrm{d}\tau\biggr)^{1/2}.
\end{aligned} \tag{2.47}$$

For the last term N_7, we first rewrite it using integration by parts and have

$$\begin{aligned}N_7 &= (1+t)^2\int_{\mathbb{T}^3} U_t\nabla u\,\mathrm{d}x\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\biggl[(1+t)^2\int_{\mathbb{T}^3} U\nabla u\,\mathrm{d}x\biggr] - 2(1+t)\int_{\mathbb{T}^3} U\nabla u\,\mathrm{d}x\\ &\quad + (1+t)^2\int_{\mathbb{T}^3}\operatorname{div} U u_t := J_1 + J_2 + J_3,\end{aligned} \tag{2.48}$$

thus, by using the product estimates, we get

$$\begin{aligned}&\int_0^t \bigl|J_1(\tau)\bigr| + \bigl|J_2(\tau)\bigr|\,\mathrm{d}\tau\\ &\quad\lesssim \sup_{0\le\tau\le t}(1+\tau)^2\bigl\|U(\tau)\bigr\|_0\bigl\|\nabla u(\tau)\bigr\|_0 + \int_0^t (1+\tau)\bigl\|U(\tau)\bigr\|_0\bigl\|\nabla u(\tau)\bigr\|_0\,\mathrm{d}\tau\\ &\quad\lesssim \mathcal{E}_{1,1}(t) + \biggl(\int_0^t (1+\tau)^2\bigl\|U(\tau)\bigr\|_0^2\,\mathrm{d}\tau\biggr)^{1/2}\biggl(\int_0^t \bigl\|\nabla u(\tau)\bigr\|_0^2\,\mathrm{d}\tau\biggr)^{1/2}\\ &\quad\lesssim \mathcal{E}_{1,1}(t) + \mathcal{E}_0^{1/2}\mathcal{E}_1^{1/2}(t).\end{aligned} \tag{2.49}$$

On the other hand, by using $(1.11)_2$, we get

$$J_3 = (1+t)^2\int_{\mathbb{T}^3}\operatorname{div} U\bigl[\operatorname{div}\bigl(U + UU^{\mathrm{T}}\bigr) - \nabla p - u\cdot\nabla u\bigr]\,\mathrm{d}x. \tag{2.50}$$

Thus, by the product estimates, we get

$$|J_3| \lesssim (1+t)^2\|U\|_1\bigl(\|U\|_1 + \|U\|_2^2 + \|\nabla p\|_0 + \|\nabla u\|_0\|u\|_2\bigr). \tag{2.51}$$

Now, by (2.26), we get

$$\begin{aligned}\|p\|_2 &\lesssim \|\nabla u\nabla u\|_0 + \|U\|_2 + \|U\|_2^2\\ &\lesssim \|\nabla u\|_0\|u\|_3 + \|U\|_2 + \|U\|_2^2,\end{aligned} \tag{2.52}$$

thus,

$$|J_3| \lesssim (1+t)^2\|U\|_1\bigl(\|U\|_2 + \|U\|_2^2 + \|\nabla u\|_0\|u\|_3\bigr). \tag{2.53}$$

Hence, we obtain

$$\begin{aligned}&\int_0^t \bigl|J_3(\tau)\bigr|\,\mathrm{d}\tau\\ &\quad\lesssim \int_0^t (1+\tau)^2\bigl\|U(\tau)\bigr\|_1\bigl\|U(\tau)\bigr\|_2\,\mathrm{d}\tau + \int_0^t (1+\tau)^2\bigl\|U(\tau)\bigr\|_1\bigl\|U(\tau)\bigr\|_2^2\,\mathrm{d}\tau\\ &\qquad + \int_0^t (1+\tau)^2\bigl\|U(\tau)\bigr\|_1\bigl\|\nabla u(\tau)\bigr\|_0\bigl\|u(\tau)\bigr\|_3\,\mathrm{d}\tau\end{aligned}$$

$$\lesssim \int_0^t (1+\tau)^2 \|U(\tau)\|_2^2 \,\mathrm{d}\tau + \sup_{0\le\tau\le t} \|U(\tau)\|_1 \int_0^\tau (1+\tau)^2 \|U(\tau)\|_2^2 \,\mathrm{d}\tau + \sup_{0\le\tau\le t} \|u(\tau)\|_3 \left(\int_0^t \|(1+\tau)^2 U(\tau)\|_1^2 \,\mathrm{d}\tau\right)^{1/2} \left(\int_0^t (1+\tau)^2 \|\nabla u\|_0^2 \,\mathrm{d}\tau\right)^{1/2},$$

thus, we get

$$\int_0^t |J_3(\tau)| \,\mathrm{d}\tau \lesssim \mathcal{E}_{1,1}(t) + \mathcal{E}_1^{3/2}(t) + \mathcal{E}_0^{1/2}\mathcal{E}_1(t). \tag{2.54}$$

By (2.49) and (2.54), we get

$$\int_0^t |N_7(\tau)| \,\mathrm{d}\tau \lesssim \mathcal{E}_{1,1}(t) + \mathcal{E}_0^{1/2}\mathcal{E}_1^{1/2}(t) + \mathcal{E}_1^{3/2}(t) + \mathcal{E}_0^{1/2}\mathcal{E}_1(t). \tag{2.55}$$

Integrating (2.45) with respect to time, using (2.46), (2.47), (2.55), and Young's inequality, we get

$$\int_0^t (1+\tau)^2 \|\nabla u(\tau)\|_0^2 \,\mathrm{d}\tau \lesssim \mathcal{E}_{1,1}(t) + \mathcal{E}_1^{3/2}(t) + \mathcal{E}_0^{1/2}(t)\mathcal{E}_1^{1/2}(t) + \mathcal{E}_0^{1/2}\mathcal{E}_1(t). \tag{2.56}$$

Now, multiplying (2.44) by a suitably large number and adding (2.56), using the Young's inequality, we complete the proof of Lemma 2.2. □

3 Proof of Theorem 1.1

Now, we will combine the above *a priori* estimates of all the energies defined in (1.15) together, and give the proof of Theorem 1.1. First, we define the total energy as follows:

$$\mathcal{E}(t) = \mathcal{E}_0(t) + \mathcal{E}_1(t).$$

Multiplying (2.1) and (2.32) in the above two lemmas by a different suitable number and summing them up, we can get the following inequality:

$$\mathcal{E}(t) \le C_1\mathcal{E}(0) + C_1\mathcal{E}^{3/2}(t), \tag{3.1}$$

for some positive constant C_1.

Under the setting of initial data (1.9), there exists a positive constant C_2 such that the initial total energy satisfies

$$\mathcal{E}(0) \le C_2\epsilon. \tag{3.2}$$

According to the standard local well-posedness theory which can be obtained by classical arguments, there exists a positive time T such that for $C_3 = C_1C_2$,

$$\mathcal{E}(t) \le 2C_3\epsilon, \quad \forall t \in [0, T]. \tag{3.3}$$

Let T^* be the largest possible time of T satisfying (3.3), it is then left to show that $T^* = \infty$. Noticing the estimate (3.1), we can use a standard continuation argument to show that $T^* = \infty$ provided that ϵ is small enough. We omit the details here. Hence, we finish the proof of Theorem 1.1.

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Authors' contributions
This work was carried out in collaboration between both authors. XL designed the study and guided the research. ML performed the analysis and wrote the first draft of the manuscript. XL and ML managed the analysis of the study. Both authors read and approved the final manuscript.

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