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Nonlinear (m,∞)-isometries and (m,∞)-expansive (contractive) mappings on normed spaces

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Abstract

Let S be a self-mapping on a normed space $\mathcal{X}$. In this paper, we introduce three new classes of mappings satisfying the following conditions:

$$\max_{\substack{0\le k\le m\\ k\text{ even}}}\|S^kx-S^ky\| = \max_{\substack{0\le k\le m\\ k\text{ odd}}}\|S^kx-S^ky\|,$$

$$\max_{\substack{0\le k\le m\\ k\text{ even}}}\|S^kx-S^ky\| \le \max_{\substack{0\le k\le m\\ k\text{ odd}}}\|S^kx-S^ky\|,$$

$$\max_{\substack{0\le k\le m\\ k\text{ even}}}\|S^kx-S^ky\| \ge \max_{\substack{0\le k\le m\\ k\text{ odd}}}\|S^kx-S^ky\|,$$

for all $x,y\in\mathcal{X}$, where m is a positive integer. We prove some properties of these classes of mappings.

MSC: 54E40; 62H86

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1 Introduction

The notion of an m-isometry in the setting of Hilbert spaces was introduced by Agler [1]: a bounded linear operator T on a Hilbert space $\mathcal{H}$ is an m-isometry (integer $m\ge 1$) if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^k=0, \tag{1.1}$$

where T^* denotes the adjoint operator of T. It is clear that (1.1) is equivalent to

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^kx\|^2=0\quad (x\in\mathcal{H}). \tag{1.2}$$

A 1-isometry is an isometry and vice versa. We refer the reader to the trilogy [1–3] by Agler and Stankus for the fundamentals of the theory of m-isometries.

In the last years, a generalization of m-isometries to operators on general Banach spaces has been presented by several authors. Bayart [5] introduced the notion of (m,p)-isometries on general (real or complex) Banach spaces. An operator T on a Banach space $\mathcal{X}$ is called an (m,p)-isometry if there exist an integer $m \geq 1$ and $p \in [1,\infty)$ with

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^kx\|^p = 0 \quad (x \in \mathcal{H}). \tag{1.3}$$

In [11] the authors took off the restriction $p \geq 1$. They considered equation (1.3) for $p \in (0,\infty)$ and studied the role of the second parameter p and also discussed the case $p = \infty$.

Let $m \in \mathbb{N}$. An operator T acting on a Banach space $\mathcal{X}$ is called an (m,∞)-isometry (or (m,∞)-isometric operator) if

$$\max_{\substack{k\in\{0,1,\dots,m\}\\ k \text{ even}}}\|T^kx\| = \max_{\substack{k\in\{0,1,\dots,m\}\\ k \text{ odd}}}\|T^kx\|, \quad \forall x \in \mathcal{X}.$$

(See [8, 11].)

Let X and Y be metric spaces. A mapping $S : X \longrightarrow Y$ is called an isometry if it satisfies $d_Y(Sx,Sy) = d_X(x,y)$ for all $x,y \in X$, where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively.

In [6] the authors introduced the concept of (m,q)-isometry for maps on a metric space (X,d_X) as follows: a mapping $S : X \to X$ is called an (m,q)-isometry for integer $m \geq 1$ and real $q > 0$ if it satisfies

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}d_X\big(S^{m-k}x,S^{m-k}y\big)^q = 0, \quad \forall x,y \in X.$$

Very recently, in [4] the present author studied a class of mappings, called (m,∞)-isometries, acting on a metric space. A mapping S acting on a metric space (X,d_X) is called an (m,∞)-isometry for some positive integer m if for all $x,y \in X$,

$$\max_{\substack{0\le k\le m\\ k \text{ even}}} d_X\big(S^kx,S^ky\big) = \max_{\substack{0\le k\le m\\ k \text{ odd}}} d_X\big(S^kx,S^ky\big).$$

In [9] the author considers $A(m,p)$-isometries, where for an operator $A \in \mathcal{B}(\mathcal{X})$, $T \in \mathcal{B}(\mathcal{X})$ (the algebra of bounded linear operators) is $A(m,p)$-isometric if

$$\beta_m^{(p)}(T,A,x) := \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|AT^kx\|^p = 0 \quad (x \in \mathcal{X}). \tag{1.4}$$

Evidently, an $I(m,p)$-isometry is an (m,p)-isometry; if $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|AT^kx\|^p = 0 \quad\Longleftrightarrow\quad \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\||A|T^kx\|^p = 0 \quad (x \in \mathcal{X}).$$

If $\beta_m^{(p)}(T,A,x) \leq 0$ (resp., $\beta_m^{(p)}(T,A,x) \geq 0$) for $x \in \mathcal{X}$, then T is said to be (A,m,p)-expansive (resp., (A,m,p)-contractive). We refer the interested reader to [10, 13] for details.

A mapping S (not necessarily linear) on a normed space $\mathcal{X}$ [12] is an (m,p)-isometry for integer $m \geq 1$ and real $p > 0$ if for all $x, y \in \mathcal{X}$,

$$\beta_m^{(p)}(S,x,y) := \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\left\|S^k x - S^k y\right\|^p = 0. \tag{1.5}$$

When $m = 1$, (1.5) is equivalent to $\|Sx - Sy\| = \|x - y\|$ for $x, y \in \mathcal{X}$, and when $m = 2$, (1.5) is equivalent to

$$\left\|S^2 x - S^2 y\right\|^p - 2\|Sx - Sy\|^p + \|x - y\|^p = 0, \quad x, y \in \mathcal{X}.$$

After a short introduction and some connections with known results in this context, we present the main results of the paper as follows. In Sect. 2, we introduce and study some properties of (m,∞)-isometric mappings. Exactly, we give conditions under which a self-mapping S is an (m,∞)-isometry (Proposition 2.5, Corollary 2.7, Proposition 2.16). An (m,∞)-isometry becomes isometry (Theorems 2.10 and 2.20). An (m,∞)-isometric mapping becomes an $(m+1,\infty)$-isometric mapping. The product of two (m,∞)-isometries is an (m,∞)-isometry (Theorem 2.17), and a power of a $(2,\infty)$-isometry is again a $(2,\infty)$-isometry (Theorem 2.18). In Sect. 3, we present a parallel study of the classes of nonlinear (m,∞)-expansive and (m,∞)-contractive mappings.

2 Nonlinear (m,∞)-isometric mappings

This section is devoted to the study of some basic properties of the class of (m,∞)-isometric mappings (not necessary linear) on a normed space $\mathcal{X}$. Our inspiration comes from the papers [7, 11], and [14].

Let $S : \mathcal{X} \longrightarrow \mathcal{X}$ be an (m,p)-isometric mapping. It obvious that

$$\begin{aligned} &\beta_m^{(p)}(S,x,y) = 0 \\ \iff\quad &\sum_{\substack{0\leq k\leq m\\ k \text{ even}}}\binom{m}{k}\left\|S^k x - S^k y\right\|^p = \sum_{\substack{0\leq k\leq m\\ k \text{ odd}}}\binom{m}{k}\left\|S^k x - S^k y\right\|^p \\ \iff\quad &\Biggl(\sum_{\substack{0\leq k\leq m\\ k \text{ even}}}\binom{m}{k}\left\|S^k x - S^k y\right\|^p\Biggr)^{\frac{1}{p}} = \Biggl(\sum_{\substack{0\leq k\leq m\\ k \text{ odd}}}\binom{m}{k}\left\|S^k x - S^k y\right\|^p\Biggr)^{\frac{1}{p}}. \end{aligned}$$

By taking the limit as $p \to \infty$ we arrive at the following definition of an (m,∞)-isometric nonlinear mapping.

Definition 2.1 A nonlinear mapping $S : \mathcal{X} \longrightarrow \mathcal{X}$ is said to be an (m,∞)-isometric mapping for some positive integer m if for all $x, y \in \mathcal{X}$,

$$\max_{\substack{0\leq k\leq m\\ k \text{ even}}}\left\|S^k x - S^k y\right\| = \max_{\substack{0\leq k\leq m\\ k \text{ odd}}}\left\|S^k x - S^k y\right\|. \tag{2.1}$$

Remark 2.2 (1) A self-mapping S on $\mathcal{X}$ is an $(1,\infty)$-isometry if for all $x,y \in \mathcal{X}$,

$$\|x-y\| = \|Sx-Sy\|.$$

(2) A self-mapping S on $\mathcal{X}$ is a $(2,\infty)$-isometry if for all $x,y \in \mathcal{X}$,

$$\|Sx-Sy\| = \max\left\{\left\|S^2x-S^2y\right\|, \|x-y\|\right\}.$$

(3) A self-mapping S on $\mathcal{X}$ is a $(3,\infty)$-isometry if for all $x,y \in \mathcal{X}$,

$$\max\left\{\|Sx-Sy\|, \left\|S^3x-S^3y\right\|\right\} = \max\left\{\left\|S^2x-S^2y\right\|, \|x-y\|\right\}.$$

Remark 2.3 The following remarks are obvious consequences of Definition 2.1.
(1) Every $(1,\infty)$-isometry is an isometry and vice versa.
(2) Every isometric mapping is an (m,∞)-isometric mapping for all $m \geq 1$.
Indeed, the classes of (m,∞)-isometries is a generalization of the class of isometries.
(3) If S is an (m,∞)-isometry that satisfies $S^2 = I$ (the identity map), then S is an isometry.

In the next example, we show that (m,∞)-isometries are in general neither continuous nor linear.

Example 2.4 Let $\mathcal{X} = \mathbb{R}$ with the usual norm $\|x\| = |x|$. Consider the map $S : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$Sx = \begin{cases} x+1, & x > -1, \\ -1, & x = -1, \\ x-1, & x < -1. \end{cases}$$

It is easy verify that S is a $(2,\infty)$-isometry, but S is neither continuous nor linear.

Proposition 2.5 *An mapping $S : \mathcal{X} \longrightarrow \mathcal{X}$ is an (m,∞)-isometric if and only if*

$$\max_{\substack{j \leq k \leq j+m \\ k \text{ even}}} \left\|S^kx-S^ky\right\| = \max_{\substack{j \leq k \leq j+m \\ k \text{ odd}}} \left\|S^kx-S^ky\right\|,$$

for all $x,y \in \mathcal{X}$ and $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of positive integers.

Proof The proof follows by substituting x by S^jx and y by S^jy into (2.1) for $j \in \mathbb{N}_0$, □

Proposition 2.6 ([11, Lemma 5.3]) *For all $k \in \mathbb{N}_0$, let $\pi(k) = k \bmod 2$ denote the parity of k. Let further $m \in \mathbb{N}$ with $m \geq 1$, and let $(a_k)_{k\in\mathbb{N}_0} \subset \mathbb{R}$. The following are equivalent.*
(1) *$(a_k)_{k\in\mathbb{N}_0}$ satisfies*

$$\max_{\substack{j \leq k \leq m+j \\ k \text{ even}}} a_k = \max_{\substack{j \leq k \leq m+j \\ k \text{ odd}}} a_k, \quad \forall j \in \mathbb{N}_0.$$

(2) $(a_k)_{k\in\mathbb{N}_0}$ *attains a maximum, and*

$$\max_{k\in\mathbb{N}_0}(a_k) = \max_{\substack{j\leq k\leq m-1+j \\ \pi(k)=\pi(m-1+j)}}(a_k), \quad \forall j\in\mathbb{N}_0.$$

Corollary 2.7 *Let $S:\mathcal{X}\longrightarrow\mathcal{X}$, and let $m\in\mathbb{N}$. Then S is an (m,∞)-isometric mapping if and only if*

$$\max_{k\in\mathbb{N}_0}\|S^kx-S^ky\| = \max_{\substack{j\leq k\leq m-1+j \\ \pi(k)=\pi(m-1+j)}}\|S^kx-S^ky\|$$

for all $x,y\in\mathcal{X}$ and $j\in\mathbb{N}_0$.

Proof The proof is essentially an application of Proposition 2.6. It suffices to consider $(a_k)_k := (\|S^kx-S^ky\|)_k$ for all $x,y\in\mathcal{X}$. □

Definition 2.8 A self-mapping S on a normed space $\mathcal{X}$ is called power bounded if

$$\sup_{n\in\mathbb{N}_0}\{\|S^nx\|\}<\infty, \quad \forall x\in\mathcal{X}.$$

Corollary 2.9 *Let $S:\mathcal{X}\longrightarrow\mathcal{X}$ be an (m,∞)-isometry. Then for all $n\in\mathbb{N}$ and $x\in\mathcal{X}$,*

$$\|S^nx-S^ny\|\leq\max_{0\leq k\leq m-1}\|S^kx-S^ky\|.$$

In particular, S is power bounded.

Proof From Corollary 2.7 we have

$$\max_{k\in\mathbb{N}_0}\|S^kx-S^ky\| = \max_{\substack{j\leq k\leq m-1+j \\ \pi(k)=\pi(m-1+j)}}\|S^kx-S^ky\|, \quad \forall x,y\in\mathcal{X},\forall j\in\mathbb{N}_0.$$

This gives that $\max_{k\in\mathbb{N}_0}\|S^kx-S^ky\|<\infty$. Further, we see that for all $n\in\mathbb{N}_0$,

$$\|S^nx-S^ny\|\leq\max_{k\in\mathbb{N}_0}\|S^kx-S^ky\|\leq\max_{0\leq k\leq m-1}\|S^kx-S^ky\|, \quad \forall x,y\in\mathcal{X}.$$

In particular,

$$\begin{aligned}\|S^nx\| &\leq \|S^nx-S^n0\|+\|S^n0\| \\ &\leq \max_{0\leq k\leq m-1}\|S^kx-S^ky\|+\|S^n0\|, \quad \forall x,y\in\mathcal{X}\end{aligned}$$

Therefore S is a power bounded mapping. □

In the following theorem, we show that if S is a self-mapping on a normed space $\mathcal{X}$ that is an (m,∞)-isometry, then there exists a metric d_∞ on $\mathcal{X}$ such that S is a $(1,\infty)$-isometry on $(\mathcal{X},d_\infty)$.

Theorem 2.10 *Let $S : \mathcal{X} \longrightarrow \mathcal{X}$ be an (m,∞)-isometry. Then there exists a metric d_∞ on $\mathcal{X}$ such that S is an isometry on $(\mathcal{X}, d_\infty)$. Moreover, d_∞ is given by*

$$d_\infty(x,y) = \max_{0\le k\le m-1} \| S^k x - S^k y \|, \quad \forall x,y \in \mathcal{X}.$$

Proof Since S is an (m,∞)-isometric mapping, we have by Corollary 2.7 that

$$\max_{k\in\mathbb{N}_0} \| S^k x - S^k y \| = \max_{0\le k\le m-1} \| S^k x - S^k y \|, \quad \forall x,y \in \mathcal{X}.$$

Define the map $d_\infty : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ by

$$d_\infty(x,y) := \max_{0\le k\le m-1} \| S^k x - S^k y \|, \quad \forall x,y \in \mathcal{X}.$$

It is easy to show that the map d_∞ define a metric on $\mathcal{X}$. On the other hand, since S is an (m,∞)-isometry, it follows that

$$\begin{aligned} d_\infty(x,y) &= \max_{0\le k\le m-1} \| S^k x - S^k y \|) \\ &= \max_{k\in\mathbb{N}_0} \| S^k x - S^k y \| \\ &= \max_{j\le k\le m-1+j} \| S^k x - S^k y \|, \quad \forall j \in \mathbb{N}_0. \end{aligned}$$

Consequently, $d_\infty(x,y) = d_\infty(Sx,Sy)$. So, S is an isometry on $(\mathcal{X}, d_\infty)$, and the proof is complete. □

Proposition 2.11 *Let $\mathcal{X}$ be a normed space, and let $S : \mathcal{X} \longrightarrow \mathcal{X}$ be a mapping (not necessarily linear). If S is an (m,∞)-isometry, then S is an $(m+1,\infty)$-isometry.*

Proof Since S is an (m,∞)-isometry, it follows that

$$\max_{k\in\mathbb{N}_0} \| S^k x - S^k y \| = \max_{\substack{j\le k\le m-1+j \\ \pi(k)=\pi(m-1+j)}} \| S^k x - S^k y \|,$$

for all $x,y \in \mathcal{X}$ and $j \in \mathbb{N}_0$. Hence, for for all $x,y \in \mathcal{X}$ and $j \in \mathbb{N}_0$, we have

$$\begin{aligned} \max_{k\in\mathbb{N}_0} \{ \| S^k x - S^k y \| \} &= \max_{\substack{j\le k\le m-1+l \\ \pi(k)=\pi(m-1+j)}} \{ \| S^k x - S^k y \| \} \\ &\le \max_{\substack{j\le k\le m+l \\ \pi(k)=\pi(m+j)}} \{ \| S^k x - S^k y \| \} \le \max_{k\in\mathbb{N}} \{ \| S^k x - S^k y \| \}. \end{aligned}$$

Consequently,

$$\max_{k\in\mathbb{N}_0} \{ \| S^k x - S^k y \| \} = \max_{\substack{j\le k\le m+l \\ \pi(k)=\pi(m+j)}} \{ \| S^k x - S^k y \| \}.$$

So, S is an $(m+1,\infty)$-isometry. □

Remark 2.12 In general, an (m,∞)-isometry is not necessary an $(m-1,\infty)$-isometry as shown in the following example.

Example 2.13 Let $\mathcal{X} = \mathbb{R}^2$ be equipped with the norm $\|(x,y)\| = |x| + |y|$. Define the map $S : \mathbb{R}^2 \to \mathbb{R}^2$ by $S(x,y) = (y+1, -x+y)$. A simple calculation shows that,

$$\begin{cases} S(x,y) = (y+1, -x+y), \\ S^2(x,y) = (-x+y+1, -x-1), \\ S^3(x,y) = (-x, -y-2), \\ S^4(x,y), (-y-1, x-y-2), \\ S^5(x,y) = (x-y-1, x-1). \end{cases}$$

From the above calculation we easily see that

$$\begin{aligned} &\max\bigl\{\bigl\|S(x,y)-S(u,v)\bigr\|, \bigl\|S^2(x,y)-S^2(u,v)\bigr\|, \bigl\|S^4(x,y)-S^4(u,v)\bigr\|\bigr\} \\ &\quad = \max\bigl\{\bigl\|S(x,y)-S(u,v)\bigr\|, \bigl\|S^3(x,y)-S^3(u,v)\bigr\|, \bigl\|S^5(x,y)-S^5(u,v)\bigr\|\bigr\} \end{aligned}$$

and

$$\begin{aligned} &\max\bigl\{\bigl\|(x,y)-(u,v)\bigr\|, \bigl\|S^2(x,y)-S^2(u,v)\bigr\|, \bigl\|S^4(x,y)-S^4(u,v)\bigr\|\bigr\} \\ &\quad \neq \max\bigl\{\bigl\|S(x,y)-S(u,v)\bigr\|, \bigl\|S^3(x,y)-S^3(u,v)\bigr\|\bigr\}. \end{aligned}$$

Consequently, S is a $(5,\infty)$-isometry but not a $(4,\infty)$-isometry.

Proposition 2.14 *Let $S : \mathcal{X} \longrightarrow \mathcal{X}$. If S^n is an isometry for odd integer n, then S is an (m,∞)-isometry for $m \geq 2n-1$.*

Proof In view of Proposition 2.11, it suffices to show that S is a $(2n-1,\infty)$-isometry.

Assume that S^n is an isometry. Then we have

$$\bigl\|S^{k+n}x - S^{k+n}y\bigr\| = \bigl\|S^k x - S^k y\bigr\| \quad \forall x,y \in \mathcal{X}, \forall k \in \mathbb{N}_0.$$

Since n is an odd integer, for $k \in \mathbb{N}_0$, we have that k is even if and only if $n+k$ is odd. Since S^n is an isometry, it follows that

$$\bigl\{\bigl\|S^k x - S^k y\bigr\|, k \in \{0,1,\dots,2n-1\}, k \text{ even}\bigr\} = \bigl\{\bigl\|S^k x - S^k y\bigr\|, k \in \{0,1,\dots,2n-1\}, k \text{ odd}\bigr\},$$

from which we deduce that S is a $(2n-1,\infty)$-isometry. □

Corollary 2.15 *Let $S : \mathcal{X} \longrightarrow \mathcal{X}$ be a mapping such that S^n is an isometry for an odd integer n. Then S^k is a $(2n-1,\infty)$-isometry for any integer $k \in \mathbb{N}$.*

Proof If $k = 1$, then the result follows from Proposition 2.14.

For $k > 1$, if S^n is an isometry, then $(S^k)^n$ is also an isometry, so by Proposition 2.14 we get that S^k is a $(2n-1,\infty)$-isometry. □

The following proposition generalizes [11, Proposition 5.8].

Proposition 2.16 *Let $S : \mathcal{X} \longrightarrow \mathcal{X}$, and let $m \in \mathbb{N}, m \geq 2$. Then the following properties hold.*

(1) *If $m \geq 3$ and S satisfy the conditions*
  (i) $\|S^m x - S^m y\| = \|S^{m-1}x - S^{m-1}y\|$ *and*
  (ii) $\|S^m x - S^m y\| \geq \|S^k x - S^k y\|$ *for $k = 0, \dots, m-2$ and all $x, y \in \mathcal{X}$,*

  then S is an (m, ∞)-isometry.

(2) *If $m = 2$, then S is an $(2, \infty)$-isometry if and only if*

$$\left\|S^2 x - S^2 y\right\| = \|Sx - Sy\| \quad \text{and} \quad \left\|S^2 x - S^2 y\right\| \geq \|x - y\|, \quad \forall x, y \in \mathcal{X}.$$

Proof (1) In view of conditions (i) and (ii), it is clear that

$$\max_{\substack{0 \leq k \leq m \\ k \text{ even}}} \left\|S^k x - S^k y\right\| = \max_{\substack{0 \leq k \leq m \\ k \text{ odd}}} \left\|S^k x - S^k y\right\|,$$

so that S is an (m, ∞)-isometry.

(2) Assume that S is an $(2, \infty)$-isometry. Then we have

$$\|Sx - Sy\| = \max\left\{\|x - y\|, \left\|S^2 x - S^2 y\right\|\right\},$$

and it follows that

$$\|Sx - Sy\| \geq \|x - y\| \quad \text{and} \quad \|Sx - Sy\| \geq \left\|S^2 x - S^2 y\right\|, \quad \forall x, y \in \mathcal{X}.$$

Replacing x by Sx and y by Sy, we get

$$\left\|S^2 x - S^2 y\right\| = \max\left\{\|Sx - Sy\|, \left\|S^3 x - S^3 y\right\|\right\}, \quad \forall x, y \in \mathcal{X},$$

and then

$$\left\|S^2 x - S^2 y\right\| \geq \|Sx - Sy\|, \quad \forall x, y \in \mathcal{X}.$$

So we have

$$\left\|S^2 x - S^2 y\right\|) = \|Sx - Sy\| \geq \|x - y\|, \quad \forall x, y \in \mathcal{X}.$$

The converse follows from statement (1). □

The authors in [6] proved that if $T, S : \mathcal{X} \longrightarrow \mathcal{X}$ are two linear maps such that $TS = ST$, T is an (m, p)-isometry, and S is an (n, p)-isometry, then TS is an $(m + n - 1, p)$-isometry. A similar result was proved in [4, Theorem 2.4]. In the following theorem, we show if T is an (m, ∞)-isometry and S is a $(2, \infty)$-isometry for which $TS = ST$, then TS is an (m, ∞)-isometry.

Theorem 2.17 *Let $T, S : \mathcal{X} \longrightarrow \mathcal{X}$ be two nonlinear mappings such that $ST = TS$. If T is an (m, ∞)-isometry and S is a $(2, \infty)$-isometry, then TS is an (m, ∞)-isometry.*

Proof Since S is a $(2,\infty)$-isometry, by statement (2) of Proposition 2.16 we have

$$\|S^2x - S^2y\| = \|Sx - Sy\| \geq \|x - y\| \quad \text{for all } x, y \in \mathcal{X}.$$

Now assume that T is a $(2,\infty)$-isometry. Then it follows that for all $x, y \in \mathcal{X}$,

$$\begin{aligned}
\|(TS)^2x - (TS)^2y\| &= \|T^2S^2x - T^2S^2y\| = \|TS^2x - TS^2y\| \\
&= \|S^2Tx - S^2Ty\| = \|TSx - TSy\| \\
&\geq \|Sx - Sy\| \\
&\geq \|x - y\|.
\end{aligned}$$

Consequently,

$$\|(TS)^2x - (TS)^2y\| = \|TSx - TSy\| \geq \|x - y\|, \quad \forall x, y \in \mathcal{X}.$$

This implies that TS is a $(2,\infty)$-isometry by statement (2) of Proposition 2.16.

We further suppose that $m > 2$. By the inequality

$$\|S^2x - S^2y\| = \|Sx - Sy\| \geq \|x - y\| \quad \text{for all } x, y \in \mathcal{X},$$

for all $k = 1, 2, \ldots,$ we have

$$\|S^kx - S^ky\| = \|Sx - Sy\| \geq \|x - y\| \quad \text{for all } x, y \in \mathcal{X}.$$

Thus for each $x, y \in \mathcal{X}$, we have

$$\begin{aligned}
\|(TS)^kx - (TS)^ky\| &= \|T^kS^kx - T^kS^ky\| \\
&= \|ST^kx - ST^ky\| \\
&\geq \|T^kx - T^ky\|.
\end{aligned}$$

Using this inequality, for all $x, y \in \mathcal{X}$, we have

$$\begin{aligned}
\max_{\substack{1\leq k\leq m \\ k \text{ even}}} \|(TS)^kx - (TS)^ky\| &= \max_{\substack{1\leq k\leq m \\ k \text{ even}}} \|T^kSx - T^kSy\| \\
&\geq \max_{\substack{1\leq k\leq m \\ k \text{ even}}} \|T^kx - T^ky\|.
\end{aligned}$$

We obtain

$$\max_{\substack{0\leq k\leq m \\ k \text{ even}}} \|(TS)^kx - (TS)^ky\| \geq \max_{\substack{0\leq k\leq m \\ k \text{ even}}} \|T^kx - T^ky\|.$$

On the other hand, it is obvious that for all $x, y \in \mathcal{X}$,

$$\max_{\substack{1\leq k\leq m \\ k \text{ even}}} \|T^kSx - T^kSy\| \leq \max_{\substack{1\leq k\leq m \\ k \text{ even}}} \|T^kx - T^ky\|.$$

Then we have

$$\max_{\substack{0\le k\le m\\ k \text{ even}}} \left\|(TS)^k x - (TS)^k y\right\| \le \max_{\substack{0\le k\le m\\ k \text{ even}}} \left\|T^k x - T^k y\right\| \quad \text{for all } x,y \in \mathcal{X}.$$

Using this inequality, we get

$$\max_{\substack{0\le k\le m\\ k \text{ even}}} \left\|(TS)^k x - (TS)^k y\right\| = \max_{\substack{0\le k\le m\\ k \text{ even}}} \left\|T^k x - T^k y\right\| \quad \text{for all } x,y \in \mathcal{X}.$$

In same way, we also have

$$\max_{\substack{0\le k\le m\\ k \text{ odd}}} \left\|(TS)^k x - (TS)^k y\right\| = \max_{\substack{0\le k\le m\\ k \text{ odd}}} \left\|T^k x - T^k y\right\| \quad \text{for all } x,y \in \mathcal{X}.$$

Since T is an (m,∞)-isometry, we deduce that

$$\max_{\substack{0\le k\le m\\ k \text{ even}}} \left\|(TS)^k x - (TS)^k y\right\| = \max_{\substack{0\le k\le m\\ k \text{ odd}}} \left\|(TS)^k x - (TS)^k y\right\|, \quad \forall x,y \in \mathcal{X}.$$

So, the desired conclusion is an immediate consequence of Definition 2.1. □

Patel [14] showed that if S is a 2-isometric operator on a Hilbert space, then S^2 is a 2-isometric operator. We now generalize this result to a $(2,\infty)$-isometric mappings.

Theorem 2.18 *A power of a $(2,\infty)$-isometric nonlinear mapping is again a $(2,\infty)$-isometric mapping.*

Proof Let $S : \mathcal{X} \longrightarrow \mathcal{X}$ be a $(2,\infty)$-isometric mapping. We need to prove that S^k is a $(2,\infty)$-isometric mapping for all positive integers k.

By statement (2) of Proposition 2.16 it suffices to show that

$$\left\|S^{2k}x - S^{2k}y\right\| = \left\|S^k x - S^k y\right\| \ge \|x-y\|, \quad \forall x,y \in \mathcal{X}.$$

Using mathematical induction on k we will show that

$$\left\|S^{2k}x - S^{2k}y\right\| = \left\|S^k x - S^k y\right\|, \quad \forall x,y \in \mathcal{X}.$$

For $k = 1$, it is true since S is an $(2,\infty)$-isometry. Assume that this equality is true for k and prove it for $k+1$. Indeed, we have

$$\begin{aligned} \left\|S^{2k+2}x - S^{2k+2}y\|\right\| &= \left\|S^{2k}S^2 x - S^{2k}S^2 y\right\| \\ &= \left\|S^k S^2 x - S^k S^2 y\right\| \\ &= \left\|S^{k+1}x - S^{k+1}y\right\|, \quad \forall x,y \in \mathcal{X}. \end{aligned}$$

Thus by induction we have proved that $\|S^{2k}x - S^{2k}y\| = \|S^k x - S^k y\|$ for all $x,y \in \mathcal{X}$, for all $k = 1,2,\ldots$.

It remains to show that for all $x, y \in \mathcal{X}$, $\|S^k x - S^k y\| \geq \|x - y\|$ for all $k = 1, 2, \ldots$.

Indeed, since $\|Sx - Sy\| \geq \|x - y\|$ for all $x, y \in \mathcal{X}$, by using the same inequality we have that for all $x, y \in \mathcal{X}$,

$$\begin{aligned}
\|S^k x - S^k y\| &= \|SS^{k-1}x - SS^{k-1}y\| \\
&\geq \|S^{k-1}x - S^{k-1}y\| = \|SS^{k-2}x - SS^{k-2}\| \\
&\geq \|S^{k-2}x - S^{k-2}y\| \\
&\geq \cdots \\
&\geq \|Sx - Sy\| \\
&\geq \|x - y\|.
\end{aligned}$$

By induction on k it follows that

$$\|S^{2k}x - S^{2k}y\| = \|S^k x - S^k y\| \geq \|x - y\|, \quad \forall x, y \in \mathcal{X}.$$

Therefore S^k also is a $(2, \infty)$-isometry. This completes the proof. □

Theorem 2.19 *Let $S : \mathcal{X} \longrightarrow \mathcal{X}$ be an invertible (m, ∞)-isometry. Then the following statements hold.*

(i) *S^{-1} is an (m, ∞)-isometry.*

(ii) *If m is even, then S is an $(m - 1, \infty)$-isometry.*

Proof (i) Since S is an (m, ∞)-isometry, from Definition 2.1 it follows that

$$\max_{\substack{0 \leq k \leq m \\ k \text{ even}}} \|S^k x - S^k y\| = \max_{\substack{0 \leq k \leq m \\ k \text{ odd}}} \|S^k x - S^k y\|, \quad \forall x, y \in \mathcal{X}.$$

Replacing x by $S^{-m}x$ and $S^{-m}y$, we obtain

$$\max_{\substack{0 \leq k \leq m \\ k \text{ even}}} \|S^{k-m}x - S^{m-k}y\| = \max_{\substack{0 \leq k \leq m \\ k \text{ odd}}} \|S^{k-m}x - S^{k-m}y\|, \quad \forall x, y \in \mathcal{X},$$

or, equivalently,

$$\max_{\substack{0 \leq k \leq m \\ k \text{ even}}} \|(S^{-1})^{m-k}x - (S^{-1})^{m-k}y\| = \max_{\substack{0 \leq k \leq m \\ k \text{ odd}}} \|(S^{-1})^{m-k}x - (S^{-1})^{m-k}y\|, \quad \forall x, y \in \mathcal{X},$$

which implies

$$\max_{\substack{0 \leq k \leq m \\ k \text{ even}}} \|(S^{-1})^k x - (S^{-1})^k y\| = \max_{\substack{0 \leq k \leq m \\ k \text{ odd}}} \|(S^{-1})^k x - S^{-1})^k y\|, \quad \forall x, y \in \mathcal{X}.$$

Consequently, S^{-1} is an (m, ∞)-isometry.

(ii) Since S is an (m, ∞)-isometry, it follows that

$$\max_{k \in \mathbb{N}_0} \|S^k x - S^k y\| < \infty$$

and, moreover,

$$\max_{k\in\mathbb{N}_0}\left\|S^kx-S^ky\right\| = \max_{\substack{j\le k\le m-1+j\\ \pi(k)=\pi(m-1+j)}}\left\|S^kx-S^ky\right\|,\quad \forall x,y\in\mathcal{X},\forall j\in\mathbb{N}_0.$$

Since S is invertible and $\pi(j-1)\neq\pi(m-2+j)$ for even m, we get that

$$\begin{aligned}\max_{\substack{j\le k\le m-1+j\\ \pi(k)=\pi(m-1+j)}}\left\|S^kx-S^ky\right\| &= \max_{\substack{j-1\le k\le m-2+j\\ \pi(k)=\pi(m-2+j)}}\left\|S^kx-S^ky\right\|\\ &= \max_{\substack{j\le k\le m-2+j\\ \pi(k)=\pi(m-2+j)}}\left\|S^kx-S^ky\right\|,\quad \forall x,y\in\mathcal{X},\forall j\in\mathbb{N}.\end{aligned}$$

This shows that

$$\max_{k\in\mathbb{N}_0}\left\|S^kx-S^ky\right\| = \max_{\substack{j\le k\le m-2+j\\ \pi(k)=\pi(m-2+j)}}\left\|S^kx-S^ky\right\|,\quad \forall x,y\in\mathcal{X},\forall j\in\mathbb{N}_0.$$

Hence the proof of the statement (ii) is complete. □

Theorem 2.20 *Let $S:\mathcal{X}\longrightarrow\mathcal{X}$ be a mapping such that S^2 is an isometry. Then the following conditions are equivalent.*
(1) *S is an isometry,*
(2) *S is an (m,∞)-isometry.*

Proof Since S^2 is an isometry, it follows that

$$\max_{k(odd)}\left\|S^kx-S^ky\right\| = \|Sx-Sy\|,\quad \forall x,y\in\mathcal{X},$$

and

$$\max_{k(even)}\left\|S^kx-S^ky\right\| = \|x-y\|,\quad \forall x,y\in\mathcal{X}.$$

This shows that (1) $\Longleftrightarrow$ (2). □

Similarly to the (m,q)-isometry (see [6, Proposition 2.18], we obtain the following theorem.

Theorem 2.21 *For $i=1,2,\dots,n$, let $(\mathcal{X}_i,\|\cdot\|_i)$ be a normed space, and let $S_i:\mathcal{X}_i\longrightarrow\mathcal{X}_i$, $m_i\ge 1$. Denote by $\mathcal{X}=\mathcal{X}_i\times\mathcal{X}_2\times\cdots\times\mathcal{X}_n$ the product space endowed with the product norm $\|(x_1,x_2,\dots,x_n)\|:=\max_{1\le i\le n}(\|x_i\|_i)$. Let $S:=S_1\times S_2\times\cdots\times S_n:\mathcal{X}\to\mathcal{X}$ be the mapping defined by*

$$S(x_1,\dots,x_n):=(S_1x_1,S_2x_2,\dots,S_nx_n).$$

If each S_i is an (m_i,∞)-isometry for $i=1,2,\dots,n$, then S is an (m,∞)-isometry, where $m=\max(m_1,\dots,m_n)$.

Proof Let $m = \max(m_1,\dots,m_n)$ and consider, for all $x,y \in X$,

$$\begin{aligned}\max_{\substack{0\le k\le m\\ k \text{ even}}} \|S^k x - S^k y\| &= \max_{\substack{0\le k\le m\\ k \text{ even}}} \Big(\max_{1\le i\le n} \|S_i^k x_i - S_i^k y_i\|_i\Big)\\ &= \max_{1\le i\le n} \Big(\max_{\substack{0\le k\le m\\ k \text{ even}}} \big\{\|S_i^k x_i - S_i^k y_i\|_i\big\}\Big).\end{aligned}$$

Since each S_i is an (m_i,∞)-isometric operator for $i = 1,2,\dots,n$, it follows that S_i is an (m,∞)-isometry for $i = 1,2,\dots,n$ by Proposition 2.11, from the above equality we have

$$\begin{aligned}\max_{\substack{0\le k\le m\\ k \text{ even}}} \|S^k x - S^k y\| &= \max_{1\le i\le n} \Big(\max_{\substack{0\le k\le m\\ k \text{ odd}}} \big\{\|S_i^k x_i - S_i^k y\|_i\big\}\Big)\\ &= \max_{\substack{0\le k\le m\\ k \text{ odd}}} \Big(\max_{1\le i\le n} \big\{\|S_i^k x_i - S_i^k y\|_i\big\}\Big).\end{aligned}$$

Thus we have

$$\max_{\substack{0\le k\le m\\ k \text{ even}}} \|S^k x - S^k y\| = \max_{\substack{0\le k\le m\\ k \text{ odd}}} \|S^k x - S^k y\|$$

for all $x,y \in \mathcal{X}$. Therefore S is an (m,∞)-isometric operator. □

3 Nonlinear (m,∞)-expansive and (m,∞)-contractive mappings

In this section, we introduce and study (m,∞)-expansive and (m,∞)-contractive nonlinear mappings on a normed space. We observe that

$$\begin{aligned}&\beta_m^{(p)}(S,x,y) \le 0\\ &\iff \sum_{\substack{0\le k\le m\\ k \text{ even}}} \binom{m}{k} \|S^k x - S^k y\|^p \le \sum_{\substack{0\le k\le m\\ k \text{ odd}}} \binom{m}{k} \|S^k x - S^k y\|^p\\ &\iff \Big(\sum_{\substack{0\le k\le m\\ k \text{ even}}} \binom{m}{k} \|S^k x - S^k y\|^p\Big)^{\frac{1}{p}} \le \Big(\sum_{\substack{0\le k\le m\\ k \text{ odd}}} \binom{m}{k} \|S^k x - S^k y\|^p\Big)^{\frac{1}{p}}.\end{aligned}$$

Taking the limit as $p \to \infty$, we arrive at the following definition of an (m,∞)-expansive mapping.

Definition 3.1 Let $m \in \mathbb{N}$. A mapping $S : \mathcal{X} \longrightarrow \mathcal{X}$ is said to be

(1) (m,∞)-expansive if

$$\max_{\substack{0\le k\le m\\ k \text{ even}}} \|S^k x - S^k y\| \le \max_{\substack{0\le k\le m\\ k \text{ odd}}} \|S^k x - S^k y\|), \quad \forall x,y \in \mathcal{X};$$

(2) (m,∞)-hyperexpansive if S is (k,∞)-expansive for $k = 1,\dots,m$;

(3) completely ∞-hyperexpansive if S is (k,∞)-expansive for all $k \in \mathbb{N}$.

Similarly,

$$\beta_m^{(p)}(S,x,y) \ge 0$$

$$\iff \sum_{\substack{0\le k\le m\\ k \text{ even}}} \binom{m}{k} \|S^k x - S^k y\|^p \ge \sum_{\substack{0\le k\le m\\ k \text{ odd}}} \binom{m}{k} \|S^k x - S^k y\|^p$$

$$\iff \Bigg(\sum_{\substack{0\le k\le m\\ k \text{ even}}} \binom{m}{k} \|S^k x - S^k y\|^p\Bigg)^{\frac{1}{p}} \ge \Bigg(\sum_{\substack{0\le k\le m\\ k \text{ odd}}} \binom{m}{k} \|S^k x - S^k y\|^p\Bigg)^{\frac{1}{p}}.$$

Taking the limit as $p \to \infty$, we arrive at the following definition of an (m,∞)-contractive mapping.

Definition 3.2 Let $m \in \mathbb{N}$. A mapping $S : \mathcal{X} \longrightarrow \mathcal{X}$ is said to be

(1) (m,∞)-contractive if

$$\max_{\substack{0\le k\le m\\ k \text{ even}}} \|S^k x - S^k y\| \ge \max_{\substack{0\le k\le m\\ k \text{ odd}}} \|S^k x - S^k y\|, \quad \forall x,y \in \mathcal{X};$$

(2) (m,∞)-hypercontractive if S is (k,∞)-contractive for $k = 1,\dots,m$;

(3) completely ∞-hypercontractive if S is (k,∞)-contractive for all $k \in \mathbb{N}$.

Remark 3.3 We make the following observations.

(1) Every $(1,\infty)$-expansive mapping is expansive, that is, $\|Sx - Sy\| \ge \|x - y\|$ for all $x,y \in \mathcal{X}$.

(2) Every $(1,\infty)$-contractive mapping is contractive, that is, $\|Sx - Sy\| \le \|x - y\|$ for all $x,y \in \mathcal{X}$.

(3) S is $(2,\infty)$-expansive if for all $x,y \in \mathcal{X}$,

$$\|Sx - Sy\| \ge \max\{\|x - y\|, \|S^2 x - S^2 y\|\}.$$

(4) S is $(2,\infty)$-contractive if for all $x,y \in \mathcal{X}$,

$$\|Sx - Sy\| \le \max\{\|x - y\|, \|S^2 x - S^2 y\|\}.$$

Remark 3.4 Observe that every (m,∞)-isometry is an (m,∞)-expansive and (m,∞)-contractive mapping.

Theorem 3.5 *Let $S : \mathcal{X} \longrightarrow \mathcal{X}$. The we have the following properties*:

(1) *T is (m,∞)-expansive if and only if*

$$\max_{\substack{j\le k\le j+m\\ k \text{ even}}} \|S^k x - S^k y\| \le \max_{\substack{j\le k\le j+m\\ k \text{ odd}}} \|S^k x - S^k y\|, \quad \forall x,y \in X, \forall j \in \mathbb{N}_0.$$

(2) *S is (m,∞)-contractive if and only if*

$$\max_{\substack{j\le k\le j+m\\ k \text{ even}}} \|S^k x - S^k y\| \ge \max_{\substack{j\le k\le j+m\\ k \text{ odd}}} \|S^k x - S^k y\|), \quad \forall x,y \in \mathcal{X}, \forall j \in \mathbb{N}_0.$$

Proof Let $j \in \mathbb{N}_0$. The desired characterizations follow by substituting $S^j x$ for x and $S^j y$ for y in statement (1) of Definition 3.1 and statement (2) of Definition 3.2. □

Proposition 3.6 *Let $S : \mathcal{X} \longrightarrow \mathcal{X}$ be a mapping such that S^2 is an isometry. Then the following are equivalent.*
(1) *S is (m,∞)-expansive,*
(2) *S is expansive,*
(3) *S is an isometry,*
(4) *S is contractive,*
(5) *S is (m,∞)-contractive.*

Proof Since S^2 is an isometry, it follows that

$$\max_{k(\mathrm{odd})} \|S^k x - S^k y\| = \|Sx - Sy\|, \quad \forall x, y \in \mathcal{X},$$

and

$$\max_{k(even)} \|S^k x - S^k y\| = \|x - y\|, \quad \forall x, y \in \mathcal{X},$$

and this shows that (1) $\Longleftrightarrow$ (2) and (4) $\Longleftrightarrow$ (5). The equivalence of (2), (3), and (4) follows on replacing x by Sx and y by Sy. □

Theorem 3.7 *Let $S : \mathcal{X} \longrightarrow \mathcal{X}$ be a is invertible mapping. Then we have*:
(1) *If S is (m,∞)-expansive, then*
 (i) *S^{-1} is (m,∞)-expansive for even m, and*
 (ii) *S^{-1} is (m,∞)-contractive for odd m.*
(2) *If S is (m,∞)-contractive, then*
 (i) *S^{-1} is (m,∞)-contractive for even m, and*
 (ii) *S^{-1} is (m,∞)-expansive for odd m.*

Proof (1) Since S is an invertible (m,∞)-expansive mapping, we have

$$\max_{\substack{0\leq k\leq m\\ k \text{ even}}} \|S^k x - S^k y\| \leq \max_{\substack{0\leq k\leq m\\ k \text{ odd}}} \|S^k x - S^k y\| \tag{3.1}$$

for all $x, y \in \mathcal{X}$. Replacing x by $S^{-m}x$ and y by $S^{-m}y$ in this inequality, we get

$$\max_{\substack{0\leq k\leq m\\ k \text{ even}}} \|(S^{-1})^{m-k} x - (S^{-1})^{m-k} y\| \leq \max_{\substack{0\leq k\leq m\\ k \text{ odd}}} \|(S^{-1})^{m-k} x - (S^{-1})^{m-k} y|\|$$

for all $x, y \in \mathcal{X}$. From this it immediately follows that

$$\begin{cases} \max_{\substack{0\leq j\leq m\\ j \text{ even}}} \|(S^{-1})^j x - (S^{-1})^j y\| \leq \max_{\substack{0\leq j\leq m\\ j \text{ odd}}} \|(S^{-1})^j x - (S^{-1})^j y\| \quad \text{for even } m, \\ \text{and} \\ \max_{\substack{0\leq j\leq m\\ j \text{ odd}}} \|(S^{-1})^j x - (S^{-1})^j y\| \geq \max_{\substack{0\leq j\leq m\\ j \text{ odd}}} \|(S^{-1})^j x - (S^{-1})^j y\| \quad \text{for odd } m, \end{cases}$$

$$\forall x, y \in \mathcal{X},$$

proving the first statement.

(2) This statement is proved in the same way as statement (1). □

Corollary 3.8 *Let $S : \mathcal{X} \longrightarrow \mathcal{X}$ be an invertible mapping. We have*:

(1) *If S is $(2,\infty)$-expansive, then S is a $(1,\infty)$-isometry.*

(2) *If S is $(2,\infty)$-contractive, then S is a $(1,\infty)$-isometry.*

Proof (1) If S is $(2,\infty)$-expansive, then we have

$$\|Sx - Sy\| \geq \max\{\|x-y\|, \|S^2x - S^2y\|\} \geq \|x-y\|, \quad \forall x,y \in \mathcal{X}.$$

By Theorem 3.7, S^{-1} is $(2,\infty)$-expansive, so

$$\|S^{-1}x - s^{-1}y\| \geq \|x-y\|, \quad \forall x,y \in \mathcal{X}.$$

This means that $\|x-y\| \geq \|Sx - Sy\|$ for all $x,y \in \mathcal{X}$. Therefore

$$\|Sx - Sy\| = \|x-y\| \quad \text{for all } x,y \in \mathcal{X}.$$

(2) This statement is proved in the same way as statement (1). □

Proposition 3.9 *Let $S : \mathcal{X} \longrightarrow \mathcal{X}$ be a $(2,\infty)$-expansive mapping and an (m,∞)-isometry, then S is a $(2,\infty)$-isometry.*

Proof Since S is a $(2,\infty)$-expansive mapping and an (m,∞)-isometry, it follows that

$$\|Sx - Sy\| \geq \max\{\|x-y\|, \|S^2x - S^2y\|\}, \quad \forall x,y \in \mathcal{X}, \tag{3.2}$$

and

$$\max_{\substack{0\leq k\leq m \\ k \text{ even}}} \|S^kx - S^ky\| = \max_{\substack{0\leq k\leq m \\ k \text{ odd}}} \|S^kx - S^ky\|, \quad \forall x,y \in \mathcal{X}. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$\|Sx - Sy\| = \max\{\|x-y\|, \|S^2x - S^2y\|\}, \quad \forall x,y \in \mathcal{X}.$$

So, S is a $(2,\infty)$-isometry. □

Theorem 3.10 *For $i = 1,2,\ldots,n$, let $(\mathcal{X}_i, \|\cdot\|_i)$ be a normed space, and let $S_i : \mathcal{X}_i \to \mathcal{X}_i$, $m_i \geq 1$. Denote by $\mathcal{X} = \mathcal{X}_i \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_n$ the product space endowed with the product norm $\|(x_1,x_2,\ldots,x_n)\| := \max_{1\leq i\leq n}(\|x_i\|)$. Let $S := S_1 \times S_2 \times \cdots \times S_n : \mathcal{X} \to \mathcal{X}$ be the mapping defined by*

$$S(x_1,\ldots,x_n) := (S_1x_1, S_2x_2, \ldots, S_nx_n).$$

Then we have:

(1) *If each S_i is (m_i,∞)-hyperexpansive for $i = 1,2,\ldots,n$, then S is (m,∞)-expansive, where $m = \min(m_1,\ldots,m_n)$.*

(2) *If each S_i is (m_i,∞)-hypercontractive for $i = 1,2,\ldots,n$, then S is (m,∞)-contractive, where $m = \min(m_1,\ldots,m_n)$.*

(3) *If each S_i is completely ∞-hyperexpansive for $i = 1, 2, \dots, n$, then so is S.*

(4) *If each S_i is completely ∞-hypercontractive for $i = 1, 2, \dots, n$, then so is S.*

Proof (1) Let $m = \min(m_1, \dots, m_n)$ and consider, for all $x, y \in X$,

$$\begin{aligned}\max_{\substack{0\leq k\leq m\\ k \text{ even}}} \left\|S^k x - S^k y\right\| &= \max_{\substack{0\leq k\leq m\\ k \text{ even}}} \left(\max_{1\leq i\leq n} \left\| \left(S_i^k x_i - S_i^k y_i\right\|_i \right\}\right) \\ &= \max_{1\leq i\leq n} \left(\max_{\substack{0\leq k\leq m\\ k \text{ even}}} \left\|S_i^k x_i - S_i^k y_i\right\|_i\right).\end{aligned}$$

Since S_i is (m_i, ∞)-hyperexpansive for $i = 1, 2, \dots, n$, it follows that S_i is (m, ∞)-expansive for $i = 1, 2, \dots, n$, and hence

$$\begin{aligned}\max_{\substack{0\leq k\leq m\\ k \text{ even}}} \left\|S^k x - S^k y\right\| &\leq \max_{1\leq i\leq n} \left(\max_{\substack{0\leq k\leq m\\ k \text{ odd}}} \left\{\left\|S_i^k x_i - S_i^k y_i\right\|_i\right\}\right) \\ &= \max_{\substack{0\leq k\leq m\\ k \text{ odd}}} \left(\max_{1\leq i\leq n} \left\{\left\|S_i^k x_i - S_i^k y_i\right\|_i\right\}\right).\end{aligned}$$

Thus we have

$$\max_{\substack{0\leq k\leq m\\ k \text{ even}}} \|S^k x - S^k y) \leq \max_{\substack{0\leq k\leq m\\ k \text{ odd}}} \left\|S^k x - S^k y\right\|.$$

Consequently, S is an (m, ∞)-expansive mapping.

(2) This statement follows from statement (1) by reversing the above inequality.

(3) Suppose that S_i is completely ∞-hyperexpansive for each $i = 1, 2, \dots, n$, and hence each S_i is (k, ∞)-expansive for any $k \in \mathbb{N}$. As a consequence of this observation, we deduce the following inequality for all $x, y \in X$:

$$\begin{aligned}\max_{\substack{0\leq j\leq k\\ j \text{ even}}} \left\|S^j x - S^j y\right\| &= \max_{\substack{0\leq j\leq k\\ j \text{ even}}} \left(\max_{1\leq i\leq n} \left\|S_i^j x_i - S_i^j y_j\right\|_i\right) \\ &= \max_{1\leq i\leq n} \left(\max_{\substack{0\leq j\leq k\\ j \text{ even}}} \left\|S_i^j x_i - S_i^j y_j\right\|_i\right) \\ &\leq \max_{\substack{0\leq j\leq k\\ j \text{ odd}}} \left\|S^j x - S^j y\right\|, \quad \forall k \in \mathbb{N},\end{aligned}$$

from which statement (3) follows.

(4) This statement is proved in the same way as statement (3). □

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