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Local Muckenhoupt class for variable exponents

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Abstract

This work extends the theory of Rychkov, who developed the theory of A_p^{loc} weights. It also extends the work by Cruz-Uribe SFO, Fiorenza, and Neugebauer. The class $A_{p(\cdot)}^{\text{loc}}$ is defined. The weighted inequality for the local Hardy–Littlewood maximal operator on Lebesgue spaces with variable exponents is proven. Cruz-Uribe SFO, Fiorenza, and Neugebauer considered the Muckenhoupt class for Lebesgue spaces with variable exponents. However, due to the setting of variable exponents, a new method for extending weights is needed. The proposed extension method differs from that by Rychkov. A passage to the vector-valued inequality is realized by means of the extrapolation technique. This technique is an adaptation of the work by Cruz-Uribe and Wang. Additionally, a theory of extrapolation adapted to our class of weights is also obtained.

MSC: 42B25; 42B35

Keywords: Variable exponent Lebesgue spaces; Muckenhoupt class; Extrapolation; Vector-valued maximal inequality

1 Introduction

This paper develops the theory of local Muckenhoupt weights in a variable exponent setting. It mixes the results obtained in [2, 3, 14]. Also see the textbook [6]. Due to the setting of variable exponents, we cannot directly use the ideas of Cruz-Uribe, Diening, and Hästö [2], Cruz-Uribe SFO, Fiorenza and Neugebauer [3], or Rychkov [14].

Herein, we use the following notation of variable exponents: Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ be a measurable function, and let w be a weight. In other words, $w : \mathbb{R}^n \rightarrow [1, \infty)$ is a measurable function that is positive almost everywhere. Then the weighted variable Lebesgue space $L^{p(\cdot)}(w)$ collects all measurable functions f such that

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} w(x) dx < \infty$$

for some $\lambda > 0$. For $f \in L^{p(\cdot)}(w)$, the norm is defined by

$$\|f\|_{L^{p(\cdot)}(w)} \equiv \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} w(x) dx \leq 1 \right\}.$$

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If $w \equiv 1$, then $\|\cdot\|_{L^{p(\cdot)}(1)} = \|\cdot\|_{p(\cdot)}$ and $L^{p(\cdot)}(1) = L^{p(\cdot)}(\mathbb{R}^n)$. Thus, we have the ordinary variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

The definition of $L^{p(\cdot)}(w)$ slightly differs from that in [3], where the authors considered the theory of Muckenhoupt weights for the Hardy–Littlewood maximal operator M for Lebesgue spaces with variable exponents. Recall that Rychkov established the theory of the local Muckenhoupt class [14]. Here and below, \mathcal{Q} denotes the set of all cubes whose edges are parallel to the coordinate axes. Herein we mix the notions considered in [3, 14] to define the local Muckenhoupt class with variable exponents.

Definition 1.1 Given an exponent $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$, a weight w belongs to $A_{p(\cdot)}^{\text{loc}}$ if $[w]_{A_{p(\cdot)}^{\text{loc}}} \equiv \sup_{Q \in \mathcal{Q}, |Q| \leq 1} |Q|^{-1} \|\chi_Q\|_{L^{p(\cdot)}(w)} \|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} < \infty$, where $\sigma \equiv w^{-\frac{1}{p(\cdot)-1}}$ and the supremum is taken over all cubes $Q \in \mathcal{Q}$ with $|Q| \leq 1$. Given a cube Q , analogously define $A_{p(\cdot)}^{\text{loc}}(Q)$ by restricting the cubes R to those contained in Q .

Remark that if $p(x) = 1$ for some $x \in \mathbb{R}^n$, then we define $\sigma(x) = 1$.

If $p(\cdot) \equiv p$ is a constant exponent, then $A_{p(\cdot)}^{\text{loc}}$ coincides with the class A_p^{loc} defined in [14]. Using a different method, we seek to establish that the local analog of the result in [2, 3] is available: Let f be a measurable function and let M^{loc} be the local maximal operator given by

$$M^{\text{loc}}f(x) \equiv \sup_{Q \in \mathcal{Q}, |Q| \leq 1} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| \, dy \quad (x \in \mathbb{R}^n).$$

Needless to say, this is an analog of the Hardy–Littlewood maximal operator given by

$$Mf(x) \equiv \sup_{Q \in \mathcal{Q}} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| \, dy \quad (x \in \mathbb{R}^n).$$

For the boundedness of M , we postulate the following two conditions on $p(\cdot)$:

- (1) The local log-Hölder continuity condition, which is given by

$$\text{LH}_0 : |p(x) - p(y)| \leq \frac{C}{-\log|x - y|}, \quad x, y \in \mathbb{R}^n, |x - y| \leq \frac{1}{2}. \tag{1.1}$$

- (2) The log-Hölder continuity condition at infinity. That is, there exists $p_\infty \in [0, \infty)$ such that

$$\text{LH}_\infty : |p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)}, \quad x \in \mathbb{R}^n. \tag{1.2}$$

Keeping these in mind, we state the main result of this paper.

Theorem 1.2 Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ satisfy conditions (1.1) and (1.2) and $1 < p_- \equiv \text{essinf}_{x \in \mathbb{R}^n} p(x) \leq p_+ \equiv \text{esssup}_{x \in \mathbb{R}^n} p(x) < \infty$. For any given $w \in A_{p(\cdot)}^{\text{loc}}$, there exists a constant $C > 0$ such that for all measurable functions f ,

$$\|M^{\text{loc}}f\|_{L^{p(\cdot)}(w)} \leq C \|f\|_{L^{p(\cdot)}(w)}.$$

It is easy to show that $w \in A_{p(\cdot)}^{\text{loc}}$ is necessary for the boundedness of M^{loc} , since $M^{\text{loc}}f(x) \geq \frac{1}{|Q|} \int_Q |f(y)| \, dx$ for all cubes Q with a volume less than or equal to 1 containing x .

Additionally, the matters are reduced to the estimate of the following maximal function. We consider the local maximal operator given by

$$M_{6^{-1}}^{\text{loc}}f(x) \equiv \sup_{Q \in \mathcal{Q}, |Q| \leq 6^{-n}} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| \, dy \quad (x \in \mathbb{R}^n),$$

for a measurable function f . In fact, if we denote the seven-fold composition of $M_{6^{-1}}^{\text{loc}}$ by $(M_{6^{-1}}^{\text{loc}})^7$, then there exists a constant $C > 0$ such that $M^{\text{loc}}f \leq C(M_{6^{-1}}^{\text{loc}})^7f$ for any measurable function f .

Before we go further, we offer some words on the technique of the proof. At first glance, the proof of Theorem 1.2 seems to be a reexamination of the original theorem [3, Theorem 1.5], which is recalled below.

Proposition 1.3 ([3, Theorem 1.5]) *Suppose that we have a variable exponent $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ such that $1 \leq p_- \leq p_+ < \infty$ and that $p(\cdot)$ satisfies (1.1) and (1.2). Let $w \in A_{p(\cdot)}$. Then, the Hardy–Littlewood maximal operator M satisfies the weak-type inequality*

$$\|t \chi_{\{x \in \mathbb{R}^n : Mf(x) > t\}} w\|_{p(\cdot)} \leq C \|fw\|_{p(\cdot)}, \quad t > 0$$

for all measurable functions f . If, in addition, $p_- > 1$, then

$$\|(Mf)w\|_{p(\cdot)} \leq C \|fw\|_{p(\cdot)}$$

for all measurable functions f .

However, as the example of $w(x) = \exp(|x|) \in A_{p(\cdot)}^{\text{loc}} \setminus A_{p(\cdot)}$ shows, inequality

$$\int_{\mathbb{R}^n} \frac{w(x)^{p(x)} \, dx}{(e + |x|)^K}, \int_{\mathbb{R}^n} \frac{\sigma(x) \, dx}{(e + |x|)^K} < \infty,$$

which is used in the proof of [3, Theorem 1.5], fails for local Muckenhoupt class with variable exponents. Hence, the proof of [3] cannot be used naively for the local Muckenhoupt class. This observation led to the technique of Rychkov, who detailed a method for creating global weights from given local weights.

Next, we consider why the technique employed by Rychkov [14] does not work directly. For simplicity, we work in \mathbb{R} . In [14], Rychkov considered a symmetric extension of weights. More precisely, given an interval I and a weight w on I , Rychkov defined a weight w_I on an interval J adjacent to I mirror-symmetrically with respect to the contact point in $I \cap J$. Here, we tried to repeat this procedure to define a weight w_I on \mathbb{R} . Hence, this method is not applicable because we cannot extend the variable exponents mirror-symmetrically. For example, if the weight w satisfies $w(t) = |t|^{-\frac{1}{3}}$ on $(-2, 2)$ and the exponent $p(\cdot)$ satisfies $p(t) = 2$ on $(-1, 1)$ and $p(t) = \frac{4}{3}$ on $(3, 5)$, then the weight $w_{(-2,2)}$ defined mirror-symmetrically from $w|_{(-2,2)}$ does not satisfy $\sigma = w_{(-2,2)}^{-\frac{1}{p(\cdot)-1}} \in L_{\text{loc}}^1(2, 6)$.

To overcome these issues, two devices are necessary. The first device is well known. We fix a dyadic grid $\mathcal{D}_{k,\mathbf{a}}$, $k \in \mathbb{Z}$ and $\mathbf{a} \in \{0, 1, 2\}^n$. More precisely, we let

$$\mathcal{D}_{k,\mathbf{a}}^0 \equiv \{2^{-k}[m + a/3, m + a/3 + 1) : m \in \mathbb{Z}\}$$

for $k \in \mathbb{Z}$ and $a = 0, 1, 2$, and consider

$$\mathcal{D}_{k,\mathbf{a}} \equiv \{Q_1 \times Q_2 \times \dots \times Q_n : Q_j \in \mathcal{D}_{k,a_j}^0, j = 1, 2, \dots, n\}$$

for $k \in \mathbb{Z}$ and $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \{0, 1, 2\}^n$. See [4, Lemma 4.8] for this construction. Herein, a dyadic grid is the family $\mathcal{D}_{\mathbf{a}} \equiv \bigcup_{k \in \mathbb{Z}} \mathcal{D}_{k,\mathbf{a}}$ for $\mathbf{a} \in \{0, 1, 2\}^n$. It is noteworthy that for any cube Q there exists $R \in \bigcup_{\mathbf{a} \in \{0,1,2\}^n} \mathcal{D}_{\mathbf{a}}$ such that $Q \subset R$ and $|R| \leq 6^n|Q|$. As in [11], we reduced the matters to the local maximal operator generated by a family \mathcal{D} given by

$$M_{\mathcal{D}}^{\text{loc}}f(x) \equiv \sup_{Q \in \mathcal{D}, |Q| \leq 1} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| \, dy \quad (x \in \mathbb{R}^n)$$

for a measurable function f and a dyadic grid $\mathcal{D} \in \{\mathcal{D}_{\mathbf{a}} : \mathbf{a} \in \{0, 1, 2\}^n\}$. In fact, we have

$$M_{\mathcal{D}}^{\text{loc}}f(x) \leq C \sum_{\mathbf{a} \in \{0,1,2\}^n} M_{\mathcal{D}_{\mathbf{a}}}^{\text{loc}}f(x) \quad (x \in \mathbb{R}^n).$$

Here and below, due to the similarity, we suppose $\mathbf{a} = (1, 1, \dots, 1)$. We abbreviate $\mathcal{D}_{(1,1,\dots,1)}$ to \mathcal{D} . The other values of \mathbf{a} can be handled similarly.

Since we reduced the matters to a dyadic grid $\mathcal{D} = \mathcal{D}_{(1,1,\dots,1)}$, it is natural to define the class $A_{p(\cdot)}^{\text{loc}}(\mathcal{D})$.

Definition 1.4 Given an exponent $p(\cdot) : \mathbb{R}^n \rightarrow (1, \infty)$ with $p_- > 1$ and a weight w , we say that $w \in A_{p(\cdot)}^{\text{loc}}(\mathcal{D})$ if

$$[w]_{A_{p(\cdot)}^{\text{loc}}(\mathcal{D})} \equiv \sup_{Q \in \mathcal{D}, |Q| \leq 1} |Q|^{-1} \|\chi_Q\|_{L^{p(\cdot)}(w)} \|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} < \infty,$$

where $\sigma \equiv w^{-\frac{1}{p(\cdot)-1}}$ and the supremum is taken over all cubes $Q \in \mathcal{D}$ with $|Q| \leq 1$.

In an analogy to Theorem 1.2, we can prove the following theorem:

Theorem 1.5 Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ satisfy conditions (1.1) and (1.2). If $1 < p_- \leq p_+ < \infty$, then, for any $w \in A_{p(\cdot)}^{\text{loc}}(\mathcal{D})$, there exists a constant $C > 0$ such that

$$\|M_{\mathcal{D}}^{\text{loc}}f\|_{L^{p(\cdot)}(w)} \leq C \|f\|_{L^{p(\cdot)}(w)}$$

for any $f \in L^{p(\cdot)}(w)$.

The second device is a new local/global strategy. To prove Theorem 1.5 when dealing with local dyadic Muckenhoupt weights, we consider (global) dyadic Muckenhoupt weights.

Definition 1.6 Given an exponent $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ with $p_- > 1$ and a weight w , we say that $w \in A_{p(\cdot)}(\mathfrak{D})$ if

$$[w]_{A_{p(\cdot)}(\mathfrak{D})} \equiv \sup_{Q \in \mathfrak{D}} |Q|^{-1} \|\chi_Q\|_{L^{p(\cdot)}(w)} \|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} < \infty,$$

where $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$, $\sigma \equiv w^{-\frac{1}{p(\cdot)-1}}$ and the supremum is taken over all cubes $Q \in \mathfrak{D}$.

Here, we propose a new method to create a globally regular weight $w_Q \in A_{p(\cdot)}(\mathfrak{D})$, $Q \in \mathfrak{D}$ from a weight $w \in A_{p(\cdot)}^{\text{loc}}(\mathfrak{D})$ in Proposition 2.10. As we will see, this technique is valid only for the dyadic maximal operator (see Remark 2.11). In addition to the local/global strategy which differs from that in Rychkov [14], we use the localization principle due to Hästö [8, Theorem 2.4]. In analogy to Theorem 1.2, we can prove the following theorem:

Theorem 1.7 Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ satisfy conditions (1.1) and (1.2). If $1 < p_- \leq p_+ < \infty$, then given any $w \in A_{p(\cdot)}(\mathfrak{D})$, there exists a constant $C > 0$ such that

$$\|M_{\mathfrak{D}}f\|_{L^{p(\cdot)}(w)} \leq C\|f\|_{L^{p(\cdot)}(w)}$$

for any measurable function f .

As explained above, Theorem 1.2 will be proven once we prove Theorem 1.5, whose proof uses Theorem 1.7. We note that unlike the proof of Theorem 1.2, that for Theorem 1.7 is an analog of [3, Theorem 1.5]. However, we include its whole proof for the sake of completeness.

Finally, as an application of our results, we will prove the Rubio de Francia extrapolation theorem in our setting of weights, which in turn produces the weighted vector-valued maximal inequality. The theory of extrapolation is a powerful tool in harmonic analysis to extend many results starting from a weighted inequality. Cruz-Uribe and Wang [5] and Ho [9] extended the extrapolation theorem to weighted Lebesgue spaces with variable exponents. We will show the extrapolation theorem for $A_{p(\cdot)}^{\text{loc}}$ by applying the boundedness of the local maximal operator.

Theorem 1.8 Let $N : [1, \infty) \rightarrow [1, \infty)$ be an increasing function. Suppose that for some $p_0, 1 < p_0 < \infty$, and every $w_0 \in A_{p_0}^{\text{loc}}$, the inequality

$$\int_{\mathbb{R}^n} f(x)^{p_0} w_0(x) \, dx \leq N([w_0]_{A_{p_0}^{\text{loc}}}) \int_{\mathbb{R}^n} g(x)^{p_0} w_0(x) \, dx \tag{1.3}$$

holds for pairs of functions (f, g) contained in some family \mathcal{F} of nonnegative measurable functions. Let $p(\cdot)$ satisfy conditions (1.1) and (1.2) as well as $1 < p_- \leq p_+ < \infty$. Also let $w \in A_{p(\cdot)}^{\text{loc}}$. Then,

$$\|f\|_{L^{p(\cdot)}(w)} \leq C\|g\|_{L^{p(\cdot)}(w)}$$

for $(f, g) \in \mathcal{F}$.

Rychkov [14, Lemma 2.11] proved the weighted vector-valued inequality for M^{loc} and $w \in A_p^{\text{loc}}$ as an extension of the results in [1].

Proposition 1.9 *Let $1 < p < \infty$, $1 < q \leq \infty$, and $w \in A_p^{\text{loc}}$. Then for any sequence of measurable functions $\{f_j\}_{j \in \mathbb{N}}$, we have*

$$\left\| \left(\sum_{j=1}^{\infty} [M^{\text{loc}} f_j]^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \leq C \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)}.$$

Recall that Cruz-Urbe et al. extended the same result by Anderson and John [1] to variable Lebesgue spaces.

Proposition 1.10 *Suppose that $p(\cdot)$ satisfies conditions (1.1) and (1.2), as well as $1 < p_- \leq p_+ < \infty$, and let $w \in A_{p(\cdot)}$ and $1 < q \leq \infty$. Then for any sequence of measurable functions $\{f_j\}_{j \in \mathbb{N}}$, we have*

$$\left\| \left(\sum_{k=1}^{\infty} [M f_k]^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}(w)} \leq C \left\| \left(\sum_{k=1}^{\infty} |f_k|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}(w)}. \tag{1.4}$$

The following theorem is the weighted vector-valued inequality for the local variable weight:

Theorem 1.11 *Suppose that $p(\cdot)$ satisfies conditions (1.1) and (1.2), as well as $1 < p_- \leq p_+ < \infty$, and let $w \in A_{p(\cdot)}^{\text{loc}}$ and $1 < q \leq \infty$. Then for any sequence of measurable functions $\{f_j\}_{j \in \mathbb{N}}$, we have*

$$\left\| \left(\sum_{k=1}^{\infty} [M^{\text{loc}} f_k]^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}(w)} \leq C \left\| \left(\sum_{k=1}^{\infty} |f_k|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}(w)}. \tag{1.5}$$

Throughout the paper, we use the following notation. The relation $A \lesssim B$ means $A \leq CB$ for some constant $C > 0$, while $A \gtrsim B$ means $A \geq CB$ for some constant $C > 0$. The relation $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. For a weight w and measurable set E , we define $w(E) \equiv \int_E w(x) dx$.

The rest of this paper is organized as follows. Sect. 2 establishes various preliminaries and some notation. Sect. 3 proves Theorem 1.7, while Sect. 4 proves Theorem 1.5, which includes Theorem 1.2. Finally, as an application, Sect. 5 is devoted to the proof of the weighted vector-valued maximal inequality for $A_{p(\cdot)}^{\text{loc}}$.

2 Preliminaries

We collect some preliminary facts by investigating some elementary properties of the dyadic grids $\mathcal{D}_{k,1}^0$ and $\mathcal{D}_{k,(1,1,\dots,1)}$ and recalling the definition of variable Lebesgue spaces. Then we consider classes of weights.

2.1 Dyadic grids $\mathcal{D}_{k,1}^0$ and $\mathcal{D}_{k,(1,1,\dots,1)}$

Recall that the grid $\mathcal{D}_{k,1}^0$ is given by

$$\mathcal{D}_{k,1}^0 \equiv \{ [2^{-k}(m + 1/3), 2^{-k}(m + 4/3)) : m \in \mathbb{Z} \} \quad (k \in \mathbb{Z}).$$

Thus, $0 \in V_k \equiv [-2^{1-k}/3, 2^{-k}/3] \in \mathcal{D}_{k,1}^0$ for all $k \in \mathbb{Z}$. Concerning $\mathcal{D}_{k,(1,1,\dots,1)}$, we employ the following property.

Lemma 2.1 *Suppose that $Q \in \mathcal{D}_{(1,1,\dots,1)}$ does not contain 0. If we set*

$$U_k = \underbrace{V_k \times \dots \times V_k}_{n \text{ times}}, \quad k_Q = \max\{k \in \mathbb{Z} : Q \cap U_k \neq \emptyset\},$$

then $Q \subset U_{k_Q-1}$ and $Q \cap U_{k_Q+1} = \emptyset$. In particular, $\ell(Q) \sim 2^{-k_Q}$.

Proof By the property of “max”, it is clear that $Q \cap U_{k_Q+1} = \emptyset$. Let us prove $Q \subset U_{k_Q-1}$. First, we prove Q does not contain U_{k_Q} . To this end, it suffices prove $\ell(Q) \leq \ell(U_{k_Q})$. Since we can concentrate on the x_1 -direction, we may assume $n = 1$. Write

$$Q = [2^{-k}(m + 1/3), 2^{-k}(m + 4/3))$$

with $k, m \in \mathbb{Z}$. Since $0 \notin Q$, $m \neq -1$. Since U_{k_Q} and Q meet at a point, a geometric observation shows

$$\begin{aligned} 2^{-k} \left(m + \frac{1}{3} \right) < \frac{2^{-k_Q}}{3} \leq 2^{-k} \left(m + \frac{4}{3} \right) \quad \text{or} \\ 2^{-k} \left(m + \frac{1}{3} \right) \leq \frac{-2^{1-k_Q}}{3} < 2^{-k} \left(m + \frac{4}{3} \right). \end{aligned} \tag{2.1}$$

Equivalently,

$$3m + 1 < 2^{k-k_Q} \leq 3m + 4 \quad \text{or} \quad -(3m + 4) < 2^{k-k_Q} \leq -(3m + 1).$$

Thus, since $2^{k-k_Q} > 0$, $m \geq 0$ or $m \leq -2$, in which case $k \geq k_Q$.

Next, we will show that $Q \subset U_{k_Q-1}$. If $Q \subset U_{k_Q}$, this is clear. We assume otherwise. Then, the relations (2.1) hold. Since the left relation in (2.1) holds, we have

$$\frac{2^{-(k_Q-1)}}{3} - \left(2^{-k} \left(m + \frac{4}{3} \right) \right) = \frac{2^{-k_Q}}{3} + \left(\frac{2^{-k_Q}}{3} - 2^{-k} \left(m + \frac{1}{3} \right) - \frac{2^{-k}}{3} \right) > 0.$$

Similarly, since the right relation in (2.1) holds, we have

$$2^{-k} \left(m + \frac{1}{3} \right) - \frac{-2^{1-(k_Q-1)}}{3} > 0.$$

Therefore, we see that $Q \subset U_{k_Q-1}$. □

2.2 Weighted variable Lebesgue spaces

For any measurable subset $\Omega \subset \mathbb{R}^n$, denote

$$p_+(\Omega) \equiv \operatorname{esssup}_{x \in \Omega} p(x), \quad p_-(\Omega) \equiv \operatorname{essinf}_{x \in \Omega} p(x).$$

In particular, when $\Omega = \mathbb{R}^n$, we simply write p_+ and p_- , respectively.

Let $1 < p(\cdot) < \infty$. If $p(\cdot) \in LH_0$ then $p'(\cdot) \in LH_0$. Likewise, if $p(\cdot) \in LH_\infty$ then $p'(\cdot) \in LH_\infty$. Furthermore, $(p_\infty)' = (p')_\infty$.

In addition, recall generalized Hölder’s inequality.

Lemma 2.2 (Generalized Hölder’s inequality) *Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty]$ be a variable exponent. Then for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and all $g \in L^{p'(\cdot)}(\mathbb{R}^n)$,*

$$\|f \cdot g\|_{L^1(\mathbb{R}^n)} \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \tag{2.2}$$

where

$$r_p \equiv 1 + \frac{1}{p_-} - \frac{1}{p_+} = \frac{1}{p_-} + \frac{1}{(p')_-} \leq 2. \tag{2.3}$$

Let us recall some properties for the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 2.3 ([12, Lemma 2.2]) *Suppose that $p(\cdot)$ is a function satisfying (1.1) and (1.2).*

- (1) *For all cubes Q with $|Q| \leq 1$, we have $|Q|^{1/p_-(Q)} \lesssim |Q|^{1/p_+(Q)}$. In particular, we have $|Q|^{1/p_-(Q)} \sim |Q|^{1/p_+(Q)} \sim |Q|^{1/p(z)} \sim \|\chi_Q\|_{L^{p(\cdot)}}$.*
- (2) *For all cubes Q with $|Q| \geq 1$, we have $\|\chi_Q\|_{L^{p(\cdot)}} \sim |Q|^{1/p_\infty}$.*

Lemma 2.4 ([3, Lemma 2.2], [13, Lemma 2.17]) *Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ be such that $p_+ < \infty$. Then given any set Ω and any measurable function f ,*

- (1) *if $\|f\chi_\Omega\|_{p(\cdot)} \leq 1$, then $\|f\chi_\Omega\|_{p(\cdot)}^{p_+(\Omega)} \leq \int_\Omega |f(x)|^{p(x)} dx \leq \|f\chi_\Omega\|_{p(\cdot)}^{p_-(\Omega)}$ and*

$$\int_\Omega |f(x)|^{p(x)} dx \leq \|f\|_{p(\cdot)},$$

- (2) *if $\|f\chi_\Omega\|_{p(\cdot)} \geq 1$, then $\|f\chi_\Omega\|_{p(\cdot)}^{p_-(\Omega)} \leq \int_\Omega |f(x)|^{p(x)} dx \leq \|f\chi_\Omega\|_{p(\cdot)}^{p_+(\Omega)}$.*

Lemma 2.5 *Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ and f be a measurable function. Then $\|f\|_{p(\cdot)} \leq 1$ if and only if $\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \leq 1$.*

Remark 2.6 Let Q be a cube. In Lemmas 2.4 and 2.5, let $f = w^{\frac{1}{p(\cdot)}} \chi_Q$ to obtain the following equivalence:

$$\|\chi_Q\|_{L^{p(\cdot)}(w)} \leq 1 \iff \int_Q w(x) dx \leq 1. \tag{2.4}$$

A direct consequence of (2.4) is the following:

- (1) If $\|\chi_Q\|_{L^{p(\cdot)}(w)} \leq 1$, then

$$\|\chi_Q\|_{L^{p(\cdot)}(w)}^{p_+(Q)} \leq w(Q) \leq \|\chi_Q\|_{L^{p(\cdot)}(w)}^{p_-(Q)}. \tag{2.5}$$

- (2) If $\|\chi_Q\|_{L^{p(\cdot)}(w)} \geq 1$, then

$$\|\chi_Q\|_{L^{p(\cdot)}(w)}^{p_-(Q)} \leq w(Q) \leq \|\chi_Q\|_{L^{p(\cdot)}(w)}^{p_+(Q)}. \tag{2.6}$$

The following inequality is a key tool used in this paper. Although [3, Lemma 2.7] considers Borel measures, we consider Lebesgue measures. Here and below denote by $L^0(\mathbb{R}^n)$ the set of all measurable functions defined over \mathbb{R}^n .

Lemma 2.7 ([3, Lemma 2.7]) *Let μ be a weighted Lebesgue measure defined on a measurable set G . Given a set G and two exponents $s(\cdot)$ and $r(\cdot)$ such that*

$$|s(y) - r(y)| \lesssim \frac{1}{\log(e + |y|)} \quad (y \in G).$$

Then for all $t \in [1, \infty)$ and $f \in L^0(\mathbb{R}^n)$ with $0 \leq f \leq 1$,

$$\int_G f(y)^{s(y)} \, d\mu(y) \lesssim \int_G f(y)^{r(y)} \, d\mu(y) + \int_G \frac{d\mu(y)}{(e + |y|)^{tr_-(G)}}.$$

Finally, recall the localization principle due to Hästö.

Lemma 2.8 ([8]) *Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ satisfy conditions (1.1) and (1.2). Then*

$$\|f\|_{L^{p(\cdot)}} \sim \left(\sum_{Q \in \mathcal{D}_{0,(1,1,\dots,1)}} (\|f\chi_Q\|_{L^{p(\cdot)}})^{p_\infty} \right)^{\frac{1}{p_\infty}}$$

for all measurable functions f .

2.3 Weights

Here and below, we assume that $p(\cdot)$ takes value in $(1, \infty)$ and satisfies conditions (1.1) and (1.2). First, note that for $w \in A_{p(\cdot)}^{\text{loc}}$ we have, by the definition of $A_{p(\cdot)}^{\text{loc}}$,

$$\|([w]_{A_{p(\cdot)}^{\text{loc}}}|Q|)^{-1} \|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} \chi_Q\|_{L^{p(\cdot)}(w)} \leq 1,$$

or equivalently,

$$\int_Q \left(\frac{\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)}}{[w]_{A_{p(\cdot)}^{\text{loc}}}|Q|} \right)^{p(x)} w(x) \, dx \leq 1. \tag{2.7}$$

for $Q \in \mathcal{Q}$ with $|Q| \leq 1$.

Recall an equivalent definition of A_∞ . We refer to [7, Theorem 7.3.3] for its proof.

Lemma 2.9 ([7, Theorem 7.3.3]) *Given a weight w , the following are equivalent:*

- (1) $w \in A_\infty$.
- (2) *There exist constants $0 < C_1, C_2 < 1$ such that given any cube Q and any measurable set $E \subset Q$ with $|E| > C_1|Q|$, then $w(E) > C_2w(Q)$.*

If (2) holds, then it can be arranged so that C_1 and C_2 depend only on the A_∞ constant of w .

The next lemma is important in this paper. Rychkov extended a local weight mirror-symmetrically but a variable exponent cannot be set in this manner. Hence, we propose a different extension.

Proposition 2.10 *Suppose that $p_- > 1$. Let $w \in A_{p(\cdot)}^{\text{loc}}(\mathfrak{D})$. Let $I \in \mathfrak{D}$ be a cube with $|I| = 1$. Define*

$$\bar{w}(x) \equiv \begin{cases} (\|\chi_I\|_{L^{p(\cdot)}(w)})^{p(x)} & (x \in \mathbb{R}^n \setminus I), \\ w(x) & (x \in I). \end{cases}$$

Then $\bar{w} \in A_{p(\cdot)}(\mathfrak{D})$ and $[\bar{w}]_{A_{p(\cdot)}(\mathfrak{D})} \lesssim [w]_{A_{p(\cdot)}^{\text{loc}}(\mathfrak{D})}$.

Proof Arithmetic shows that $w \in A_{p(\cdot)}^{\text{loc}}(\mathfrak{D})$ if and only if $\sigma \in A_{p'(\cdot)}^{\text{loc}}(\mathfrak{D})$. We also note that $[w]_{A_{p(\cdot)}^{\text{loc}}(\mathfrak{D})} \gtrsim 1$ due to Hölder’s inequality (see Lemma 2.2). Write $\bar{\sigma} \equiv \bar{w}^{-\frac{1}{p(\cdot)-1}}$. Let $Q \in \mathfrak{D}$. We need to estimate

$$\frac{1}{|Q|} \|\chi_Q\|_{L^{p(\cdot)}(\bar{w})} \|\chi_Q\|_{L^{p'(\cdot)}(\bar{\sigma})}.$$

We distinguish three cases:

- Suppose $I \cap Q = \emptyset$. In this case, by virtue of Lemma 2.3,

$$\frac{1}{|Q|} \|\chi_Q\|_{L^{p(\cdot)}(\bar{w})} \|\chi_Q\|_{L^{p'(\cdot)}(\bar{\sigma})} = \frac{1}{|Q|} \|\chi_Q\|_{L^{p(\cdot)}} \|\chi_Q\|_{L^{p'(\cdot)}} \sim 1 \leq [w]_{A_{p(\cdot)}^{\text{loc}}(\mathfrak{D})}.$$

- Next, suppose $Q \subset I$. In this case, since $w \in A_{p(\cdot)}^{\text{loc}}(\mathfrak{D})$,

$$\frac{1}{|Q|} \|\chi_Q\|_{L^{p(\cdot)}(\bar{w})} \|\chi_Q\|_{L^{p'(\cdot)}(\bar{\sigma})} = \frac{1}{|Q|} \|\chi_Q\|_{L^{p(\cdot)}(w)} \|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} \leq [w]_{A_{p(\cdot)}^{\text{loc}}(\mathfrak{D})}.$$

- Finally, suppose $Q \supset I$. In this case, again by virtue of Lemma 2.3 and the fact that $w \in A_{p(\cdot)}^{\text{loc}}(\mathfrak{D})$,

$$\begin{aligned} & \frac{1}{|Q|} \|\chi_Q\|_{L^{p(\cdot)}(\bar{w})} \|\chi_Q\|_{L^{p'(\cdot)}(\bar{\sigma})} \\ & \sim \frac{1}{|Q|} (\|\chi_{Q \setminus I}\|_{L^{p(\cdot)}(\bar{w})} + \|\chi_I\|_{L^{p(\cdot)}(\bar{w})}) (\|\chi_{Q \setminus I}\|_{L^{p'(\cdot)}(\bar{\sigma})} + \|\chi_I\|_{L^{p'(\cdot)}(\bar{\sigma})}) \\ & \sim \frac{1}{|Q|} (|Q|^{\frac{1}{p_\infty}} \|\chi_I\|_{L^{p(\cdot)}(w)} + \|\chi_I\|_{L^{p(\cdot)}(w)}) \left(\frac{|Q|^{\frac{1}{p_\infty}}}{\|\chi_I\|_{L^{p(\cdot)}(w)}} + \|\chi_I\|_{L^{p'(\cdot)}(\sigma)} \right) \\ & \sim \frac{1}{|Q|} |Q|^{\frac{1}{p_\infty}} (|Q|^{\frac{1}{p_\infty}} + \|\chi_I\|_{L^{p(\cdot)}(w)} \|\chi_I\|_{L^{p'(\cdot)}(\sigma)}) \\ & \lesssim [w]_{A_{p(\cdot)}^{\text{loc}}(\mathfrak{D})}, \end{aligned}$$

□

as required.

Remark 2.11 Let $I \equiv (0, 1)$ and $J \equiv (-1, 0)$. Define $w(t) = t^{-\frac{1}{2}}$ on I and $w(t) = 1$ on J . Denote by $A_2(I)$ the A_2 class over I . Although $w \in A_2(I)$, w is not in $A_2(I \cup J \cup \{0\})$.

Lemma 2.12 *Suppose $w \in A_{p(\cdot)}(\mathfrak{D})$. Then for all $Q \in \mathfrak{D}$ and a measurable subset E of Q ,*

$$\frac{|E|}{|Q|} \leq 2[w]_{A_{p(\cdot)}(\mathfrak{D})} \frac{\|\chi_E\|_{L^{p(\cdot)}(w)}}{\|\chi_Q\|_{L^{p(\cdot)}(w)}}.$$

Here for the sake of convenience, we include the proof.

Proof By Hölder’s inequality (see Lemma 2.2), we have

$$|E| = \int_E dx \leq 2\|\chi_E\|_{L^{p(\cdot)}(w)}\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} \leq 2[w]_{A_{p(\cdot)}(\mathfrak{D})}|Q| \frac{\|\chi_E\|_{L^{p(\cdot)}(w)}}{\|\chi_Q\|_{L^{p(\cdot)}(w)}},$$

as required. □

Lemma 2.13 *Suppose $w \in A_{p(\cdot)}(\mathfrak{D})$. Then for all $Q \in \mathfrak{D}$,*

$$\|\chi_Q\|_{L^{p(\cdot)}(w)}^{p_-(Q)-p_+(Q)} \lesssim 1.$$

Proof We assume $\|\chi_Q\|_{L^{p(\cdot)}(w)} \leq 1$. Otherwise the inequality is trivial since $p_-(Q) \leq p_+(Q)$. We distinguish 3 cases. Let $Q_0 = [-4, 4]^n$.

- Suppose $|Q| \leq 1$ and $Q \cap Q_0 \neq \emptyset$. Then, $Q \subset 5Q_0$ and since $p(\cdot)$ satisfies the local log-Hölder continuity condition, $|Q|^{p_-(Q)-p_+(Q)} \lesssim 1$. Meanwhile, due to Lemma 2.5, since $\sigma(Q_0) \lesssim 1$, it follows that $\|\chi_{Q_0}\|_{L^{p'(\cdot)}(\sigma)} \lesssim 1$. Thus,

$$|Q| \lesssim \|\chi_Q\|_{L^{p(\cdot)}(w)}\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} \lesssim \|\chi_Q\|_{L^{p(\cdot)}(w)}\|\chi_{5Q_0}\|_{L^{p'(\cdot)}(\sigma)} \lesssim \|\chi_Q\|_{L^{p(\cdot)}(w)}.$$

- Suppose $Q \cap Q_0 = \emptyset$. Write x_Q for the left-lower corner of Q . Let $U_k = [-2^{-k}/3, 2^{1-k}/3]^n$ as before and write $k_Q = \max\{k \in \mathbb{Z} : U_k \cap Q \neq \emptyset\}$. Note that $k_Q \leq -3$. Then, since $Q \subset U_{k_Q-1}$ by Lemma 2.1, we have

$$\begin{aligned} |Q| &\lesssim \|\chi_Q\|_{L^{p(\cdot)}(w)}\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} \lesssim \|\chi_Q\|_{L^{p(\cdot)}(w)}\|\chi_{U_{k_Q-1}}\|_{L^{p'(\cdot)}(\sigma)} \\ &\lesssim |U_{k_Q-1}| \frac{\|\chi_Q\|_{L^{p(\cdot)}(w)}}{\|\chi_{U_{k_Q-1}}\|_{L^{p(\cdot)}(w)}}. \end{aligned}$$

We observe that

$$p_+(Q) - p_-(Q) = \sup_{y,z \in Q} |p(y) - p(z)| \leq 2 \sup_{y \in Q} |p(y) - p_\infty| \lesssim \frac{1}{\log(e + |x_Q|)}.$$

Thus, since $|U_{k_Q-1}| \sim |x_Q|^n$, we have

$$|U_{k_Q-1}|^{p_+(Q)-p_-(Q)} \sim |x_Q|^{\frac{Cn}{\log(e+|x_Q|)}} = \exp\left(\frac{Cn \log |x_Q|}{\log(e + |x_Q|)}\right) \lesssim 1$$

for some $C > 1$. Moreover, since $\|\chi_{U_{k_Q-1}}\|_{L^{p(\cdot)}(w)} \geq \|\chi_{U_0}\|_{L^{p(\cdot)}(w)} \gtrsim 1$, we obtain the desired result.

- Suppose $|Q| \geq 1$ and $Q \cap Q_0 \neq \emptyset$. Let $R \in \mathfrak{D}$ with $[-1, 1]^n \subset R$ and $Q \subset R$. Then $\|\chi_{[-1,1]^n}\|_{L^{p(\cdot)}(w)} \leq \|\chi_R\|_{L^{p(\cdot)}(w)} \lesssim \|\chi_Q\|_{L^{p(\cdot)}(w)} \leq 1$, thanks to Lemma 2.12. Therefore,

$$\|\chi_Q\|_{L^{p(\cdot)}(w)}^{p_-(Q)-p_+(Q)} \lesssim \|\chi_R\|_{L^{p(\cdot)}(w)}^{p_-(Q)-p_+(Q)} \sim 1. \quad \square$$

If we fix a cube $P \in \mathfrak{D}$ such that $0 \in P$ and if Q is any one of the adjacent cubes to P and has the same size, then $w(Q), \sigma(Q) \gtrsim 1$.

Corollary 2.14 *Let $w \in A_{p(\cdot)}(\mathfrak{D})$. Then*

$$\int_{\mathbb{R}^n} \frac{w(x) \, dx}{(e + |x|)^K}, \int_{\mathbb{R}^n} \frac{\sigma(x) \, dx}{(e + |x|)^K} < \infty$$

as long as $K \gg 1$.

Proof Let P_j be the j th parent of P . Then

$$\int_{\mathbb{R}^n} \frac{w(x) \, dx}{(e + |x|)^K} = \int_{P_1} \frac{w(x) \, dx}{(e + |x|)^K} + \sum_{j=1}^{\infty} \int_{P_{j+1} \setminus P_j} \frac{w(x) \, dx}{(e + |x|)^K} \lesssim \sum_{j=1}^{\infty} 2^{-jK} w(P_j).$$

By Lemmas 2.4 and 2.12,

$$w(P_j) \lesssim \max(\| \chi_{P_j} \|_{L^{p(\cdot)}(w)}^{p_-}, \| \chi_{P_j} \|_{L^{p(\cdot)}(w)}^{p_+}) \lesssim \max\left(\left(\frac{|P_j|}{|P|}\right)^{p_-}, \left(\frac{|P_j|}{|P|}\right)^{p_+}\right) \lesssim 2^{jnp_+}.$$

Thus, since K is sufficiently large, it follows that

$$\int_{\mathbb{R}^n} \frac{w(x) \, dx}{(e + |x|)^K} \lesssim \sum_{j=1}^{\infty} 2^{-j(K-np_+)} \sim 1.$$

The second inequality is proven similarly. □

Lemma 2.15 *Suppose $w \in A_{p(\cdot)}(\mathfrak{D})$. Let $Q \in \mathfrak{D}$ and E be a measurable subset of Q .*

- (1) *If $w(Q) \geq 1$, then $\| \chi_Q \|_{L^{p(\cdot)}(w)} \sim w(Q)^{\frac{1}{p_\infty}}$.*
- (2) *If $w(E) \geq 1$, then*

$$\frac{|E|}{|Q|} \lesssim \left(\frac{w(E)}{w(Q)}\right)^{\frac{1}{p_\infty}}.$$

- (3) *In general*

$$\frac{|E|}{|Q|} \lesssim \left(\frac{w(E)}{w(Q)}\right)^{\frac{1}{p_+}}.$$

Proof

- (1) Using Lemma 2.7 for the measure $w(x) \, dx$, we have

$$\int_Q \left(\frac{1}{w(Q)^{\frac{1}{p_\infty}}}\right)^{p(x)} w(x) \, dx \lesssim \int_Q \frac{w(x) \, dx}{w(Q)} + \int_Q \frac{w(x) \, dx}{(e + |x|)^K} \lesssim 1.$$

Thus, $w(Q) \gtrsim (\| \chi_Q \|_{L^{p(\cdot)}(w)})^{p_\infty}$. Likewise, due to Lemma 2.7,

$$\int_Q \left(\frac{1}{\| \chi_Q \|_{L^{p(\cdot)}(w)}}\right)^{p_\infty} w(x) \, dx \lesssim \int_Q \frac{w(x) \, dx}{(\| \chi_Q \|_{L^{p(\cdot)}(w)})^{p(x)}} + \int_Q \frac{w(x) \, dx}{(e + |x|)^K} \lesssim 1.$$

Thus, $w(Q) \lesssim (\| \chi_Q \|_{L^{p(\cdot)}(w)})^{p_\infty}$.

(2) As in (1), using Lemma 2.7 for the measure $w(x) dx$, we have

$$\int_E \left(\frac{1}{w(E)^{\frac{1}{p_\infty}}} \right)^{p(x)} w(x) dx \lesssim \int_E \frac{w(x) dx}{w(E)} + \int_E \frac{w(x) dx}{(e + |x|)^K} \lesssim 1.$$

Thus, since $w(Q)^{\frac{1}{p_\infty}} \sim \|\chi_Q\|_{L^{p(\cdot)}(w)}$,

$$\frac{|E|}{|Q|} \lesssim \frac{\|\chi_E\|_{L^{p(\cdot)}(w)}}{\|\chi_Q\|_{L^{p(\cdot)}(w)}} \lesssim \frac{w(E)^{\frac{1}{p_\infty}}}{w(Q)^{\frac{1}{p_\infty}}},$$

as required.

(3) If $w(E) \geq 1$, then this is clear from (2). Suppose $w(E) \leq 1$.

- If $w(Q) \leq 1$, then by virtue of Lemma 2.5, $\|\chi_Q\|_{L^{p(\cdot)}(w)} \leq 1$ and $\|\chi_E\|_{L^{p(\cdot)}(w)} \leq 1$. Hence

$$\left(\|\chi_Q\|_{L^{p(\cdot)}(w)}\right)^{p_-(Q)} \geq w(Q), \quad \left(\|\chi_E\|_{L^{p(\cdot)}(w)}\right)^{p_+(Q)} \leq w(E), \tag{2.8}$$

thanks to Remark 2.6. Meanwhile, $\|\chi_Q\|_{L^{p(\cdot)}(w)}^{\frac{p_-(Q)}{p_+(Q)}-1} \lesssim 1$ due to Lemma 2.13. Thus, using (2.8), we have

$$\frac{|E|}{|Q|} \lesssim \frac{\|\chi_E\|_{L^{p(\cdot)}(w)}}{\|\chi_Q\|_{L^{p(\cdot)}(w)}} \lesssim \frac{w(E)^{\frac{1}{p_+(Q)}}}{w(Q)^{\frac{1}{p_+(Q)}} \|\chi_Q\|_{L^{p(\cdot)}(w)}^{1-\frac{p_-(Q)}{p_+(Q)}}} \lesssim \frac{w(E)^{\frac{1}{p_+}}}{w(Q)^{\frac{1}{p_+}}}.$$

- If $w(E) \leq 1 \leq w(Q)$, then

$$\left(\|\chi_Q\|_{L^{p(\cdot)}(w)}\right)^{p_+(Q)} \geq w(Q), \quad \left(\|\chi_E\|_{L^{p(\cdot)}(w)}\right)^{p_+(Q)} \leq w(E),$$

thanks to Remark 2.6. Thus,

$$\frac{|E|}{|Q|} \lesssim \frac{\|\chi_E\|_{L^{p(\cdot)}(w)}}{\|\chi_Q\|_{L^{p(\cdot)}(w)}} \leq \frac{w(E)^{\frac{1}{p_+(Q)}}}{w(Q)^{\frac{1}{p_+(Q)}}} \lesssim \frac{w(E)^{\frac{1}{p_+}}}{w(Q)^{\frac{1}{p_+}}}. \quad \square$$

Next, we consider some estimates related to the dyadic maximal operator. We define

$$m_Q(g) = \frac{1}{|Q|} \int_Q g(y) dy \quad (g \in L^1_{loc}(\mathbb{R}^n))$$

for a cube Q .

Lemma 2.16 (Sparse decomposition of f , [15, p. 250]) *Let $f \in L^\infty_c(\mathbb{R}^n) \setminus \{0\}$, $\lambda > 0$, $a \gg 2^n$, and let \mathfrak{D} be a dyadic grid. Then there exists a set of pairwise disjoint dyadic cubes $\{Q_j^k\}_{k \in \mathbb{Z}, j \in J_k}$ such that*

$$\Omega_k \equiv \{x \in \mathbb{R}^n : M_{\mathfrak{D}} f(x) > a^k\} = \bigcup_{j \in J_k} Q_j^k.$$

Further, these cubes have the property of $a^k < m_{Q_j^k}(|f|) \leq 2^n a^k$ for all $j \in J_k$. Furthermore, there exists a disjoint collection $\{E_j^k\}_{k \in \mathbb{Z}, j \in J_k}$, where each E_j^k is a subset of Q_j^k called a nutshell, such that $2|E_j^k| \geq |Q_j^k|$ and that

$$\Omega_k \setminus \Omega_{k+1} = \bigcup_{j \in J_k} E_j^k.$$

A direct consequence of Lemma 2.16 is that

$$M_{\mathfrak{D}} f \lesssim \sum_{k=-\infty}^{\infty} \sum_{j \in J_k} m_{Q_j^k}(|f|) \chi_{E_j^k}.$$

Given a weight W and a measurable function f , define

$$M_{W, \mathfrak{D}} f(x) = \sup_{Q \in \mathfrak{D}} \frac{\chi_Q(x)}{W(Q)} \int_Q |f(y)| W(y) dy.$$

The next lemma reflects the geometric property of \mathfrak{D} .

Lemma 2.17 ([10, Lemma 2.1]) *Given a weight W and $1 < p < \infty$, we have*

$$\int_{\mathbb{R}^n} M_{W, \mathfrak{D}} f(x)^p W(x) dx \leq \frac{p \cdot 2^p}{p-1} \int_{\mathbb{R}^n} |f(x)|^p W(x) dx.$$

We transform Lemma 2.17 as shown below.

Lemma 2.18 *Let $\{Q_j^k\}_{k \in \mathbb{Z}, j \in J_k}$ be a sparse collection with the nutshell $\{E_j^k\}_{k \in \mathbb{Z}, j \in J_k}$, and let $1 < r < \infty$. Also let $W \in A_{\infty}(\mathfrak{D})$. Then for all nonnegative $f \in L^0(\mathbb{R}^n)$,*

$$\sum_{k=-\infty}^{\infty} \sum_{j \in J_k} \left(\frac{1}{W(Q_j^k)} \int_{Q_j^k} f(y) W(y) dy \right)^r W(Q_j^k) \lesssim \int_{\mathbb{R}^n} f(x)^r W(x) dx.$$

Proof By Lemmas 2.9 and 2.17,

$$\begin{aligned} & \sum_{k=-\infty}^{\infty} \sum_{j \in J_k} \left(\frac{1}{W(Q_j^k)} \int_{Q_j^k} f(y) W(y) dy \right)^r W(Q_j^k) \\ & \lesssim \sum_{k=-\infty}^{\infty} \sum_{j \in J_k} \left(\frac{1}{W(Q_j^k)} \int_{Q_j^k} f(y) W(y) dy \right)^r W(E_j^k) \\ & = \sum_{k=-\infty}^{\infty} \sum_{j \in J_k} \int_{E_j^k} \left(\frac{1}{W(Q_j^k)} \int_{Q_j^k} f(y) W(y) dy \right)^r W(x) dx \\ & \leq \sum_{k=-\infty}^{\infty} \sum_{j \in J_k} \int_{E_j^k} M_{W, \mathfrak{D}} f(x)^r W(x) dx \\ & \lesssim \int_{\mathbb{R}^n} f(x)^r W(x) dx, \end{aligned}$$

as required. □

Lemma 2.19 *Let $w \in A_{p(\cdot)}(\mathfrak{D})$ and $Q \in \mathfrak{D}$.*

- (1) $\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)}^{p_-(Q)-p(x)} \lesssim 1$ for all $x \in Q$.
- (2) We have

$$\left(\frac{\sigma(Q)}{\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)}}\right)^{p_-(Q)} \leq \sigma(Q).$$

- (3) $\int_Q \sigma(Q)^{p_-(Q)} |Q|^{-p(x)} w(x) \, dx \lesssim \sigma(Q)$.

Proof Note that $p'_+ = (p')_-$ and $p'_- = (p')_+$, where $(p')_+$ and $(p')_-$ are the supremum and the infimum of $p'(\cdot)$, respectively.

- (1) Since $p(x) - p_-(Q) \lesssim (p')_+(Q) - (p')_-(Q)$ as in [3, p. 755], we can use Lemma 2.13.
- (2) If $\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} \geq 1$, then $\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} \geq \sigma(Q)^{\frac{1}{(p')_+(Q)}}$ by Remark 2.6. Hence

$$\left(\frac{\sigma(Q)}{\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)}}\right)^{p_-(Q)} \leq \left(\frac{\sigma(Q)}{\sigma(Q)^{\frac{1}{(p')_+(Q)}}}\right)^{p_-(Q)} = \left(\frac{\sigma(Q)}{\sigma(Q)^{\frac{1}{p'_-(Q)}}}\right)^{p_-(Q)} = \sigma(Q).$$

If $\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} \leq 1$, then, again by Remark 2.6,

$$\sigma(Q)^{\frac{1}{(p')_+(Q)}} \geq \|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} \geq \sigma(Q)^{\frac{1}{(p')_-(Q)}}.$$

Therefore

$$\begin{aligned} \left(\frac{\sigma(Q)}{\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)}}\right)^{p_-(Q)} &\leq (\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)}^{(p')_-(Q)-1})^{p_-(Q)} \\ &\leq (\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)}^{(p')_+(Q)-1})^{p_-(Q)} \\ &\leq (\sigma(Q)^{\frac{(p')_+(Q)-1}{(p')_+(Q)}})^{p_-(Q)} \\ &= (\sigma(Q)^{\frac{p'_-(Q)-1}{p'_-(Q)}})^{p_-(Q)} \\ &= \sigma(Q), \end{aligned}$$

as required.

- (3) By the definition of $A_{p(\cdot)}(\mathfrak{D})$, we have

$$\|([w]_{A_{p(\cdot)}(\mathfrak{D})}|Q|)^{-1} \|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} \chi_Q\|_{L^{p(\cdot)}(w)} \leq 1,$$

or equivalently,

$$\int_Q \left(\frac{\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)}}{[w]_{A_{p(\cdot)}(\mathfrak{D})}|Q|}\right)^{p(x)} w(x) \, dx \leq 1 \tag{2.9}$$

for $Q \in \mathfrak{D}$. From (1) and (2) we deduce

$$\int_Q \sigma(Q)^{p_-(Q)} |Q|^{-p(x)} w(x) \, dx$$

$$\begin{aligned}
 &= \left(\frac{\sigma(Q)}{\| \chi_Q \|_{L^{p'(\cdot)}(\sigma)}} \right)^{p-(Q)} \int_Q \| \chi_Q \|_{L^{p'(\cdot)}(\sigma)}^{p-(Q)-p(x)} \| \chi_Q \|_{L^{p'(\cdot)}(\sigma)}^{p(x)} |Q|^{-p(x)} w(x) \, dx \\
 &\lesssim \sigma(Q) \int_Q \| \chi_Q \|_{L^{p'(\cdot)}(\sigma)}^{p(x)} |Q|^{-p(x)} w(x) \, dx \lesssim \sigma(Q). \quad \square
 \end{aligned}$$

Lemma 2.20 *Suppose that $w \in A_{p(\cdot)}(\mathfrak{D})$. Let \mathcal{S} be a disjoint collection of cubes in dyadic grid \mathfrak{D} . Then*

$$\sum_{Q \in \mathcal{S}} \int_Q (e + |x|)^{-K} \sigma(Q)^{p-(Q)} |Q|^{-p(x)} w(x) \, dx \lesssim 1.$$

Proof Let $Q \in \mathcal{S}$. Then either $Q \supset P_l$ for some l or Q is included in some P_l . Since the first possibility can occur only in one cube, we only have to consider the second possibility. For such a cube Q , we let l be the smallest number such that $Q \subset P_l$. Then P_{l-1} and Q never intersect due to the minimality of l , since Q does not contain P_l . Thus, there exists uniquely an integer l such that $Q \subset P_l \setminus P_{l-1}$.

Using Lemma 2.19(3) and Corollary 2.14, we estimate

$$\begin{aligned}
 &\sum_{Q \in \mathcal{S}} \int_Q (e + |x|)^{-K} \sigma(Q)^{p-(Q)} |Q|^{-p(x)} w(x) \, dx \\
 &= \sum_{l=1}^{\infty} \sum_{Q \in \mathcal{S}, Q \subset P_l \setminus P_{l-1}} \int_Q (e + |x|)^{-K} \sigma(Q)^{p-(Q)} |Q|^{-p(x)} w(x) \, dx \\
 &\lesssim \sum_{l=1}^{\infty} 2^{-Kl} \sum_{Q \in \mathcal{S}, Q \subset P_l \setminus P_{l-1}} \int_Q \sigma(Q)^{p-(Q)} |Q|^{-p(x)} w(x) \, dx \\
 &\lesssim \sum_{l=1}^{\infty} 2^{-Kl} \sum_{Q \in \mathcal{S}, Q \subset P_l \setminus P_{l-1}} \sigma(Q) \\
 &\lesssim \sum_{l=1}^{\infty} \sum_{Q \in \mathcal{S}, Q \subset P_l \setminus P_{l-1}} \int_Q \frac{\sigma(x) \, dx}{(e + |x|)^K} \\
 &\lesssim 1. \quad \square
 \end{aligned}$$

The next lemma is used in Sect. 3.2.

Lemma 2.21 *Suppose that $w \in A_{p(\cdot)}(\mathfrak{D})$. Let $\{Q_j^k\}_{k \in \mathbb{Z}, j \in J_k}$ be a sparse collection with the nutshell $\{E_j^k\}_{k \in \mathbb{Z}, j \in J_k}$. Set $\mathcal{H}_2 \equiv \{(k, j) : Q_j^k \cap P = \emptyset, \sigma(Q_j^k) \geq 1\}$. Then*

$$\sum_{(k,j) \in \mathcal{H}_2} \int_{E_j^k} \left(\frac{\| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)}}{|Q_j^k|} \right)^{p(x)} \frac{w(x) \, dx}{(e + |x|)^K} \lesssim 1.$$

Proof By (2.7), we obtain

$$\begin{aligned}
 &\sum_{(k,j) \in \mathcal{H}_2} \int_{E_j^k} \left(\frac{\| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)}}{|Q_j^k|} \right)^{p(x)} \frac{w(x) \, dx}{(e + |x|)^K} \\
 &\leq \sum_{(k,j) \in \mathcal{H}_2} \sup_{x \in Q_j^k} \frac{1}{(e + |x|)^K} \int_{Q_j^k} \left(\frac{\| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)}}{|Q_j^k|} \right)^{p(x)} w(x) \, dx \lesssim \sum_{(k,j) \in \mathcal{H}_2} \sup_{x \in Q_j^k} \frac{1}{(e + |x|)^K}.
 \end{aligned}$$

By Lemma 2.9, Corollary 2.14, and the fact that $\sigma(Q_j^k) \geq 1$ for any $(k, j) \in \mathcal{H}_2$, we obtain

$$\begin{aligned} & \sum_{(k,j) \in \mathcal{H}_2} \int_{E_j^k} \left(\frac{\| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)}}{|Q_j^k|} \right)^{p(x)} \frac{w(x) \, dx}{(e + |x|)^K} \\ & \lesssim \sum_{(k,j) \in \mathcal{H}_2} \sup_{x \in Q_j^k} \frac{1}{(e + |x|)^K} \sigma(Q_j^k) \\ & \lesssim \sum_{(k,j) \in \mathcal{H}_2} \int_{Q_j^k} \frac{\sigma(x)}{(e + |x|)^K} \, dx \\ & \lesssim \sum_{(k,j) \in \mathcal{H}_2} \int_{E_j^k} \frac{\sigma(x)}{(e + |x|)^K} \, dx \\ & \lesssim 1. \end{aligned} \quad \square$$

3 Proof of Theorem 1.7

Sect. 3 uses the notation in Lemma 2.21.

Suppose that $w \in A_{p(\cdot)}(\mathfrak{D})$ and $f \in L_c^\infty(\mathbb{R}^n)$ satisfy $\|f\|_{L^{p(\cdot)}(w)} < 1$. We may assume $f \geq 0$ a.e. by replacing f by $|f|$ if necessary. We write $f_1 \equiv f \chi_{\{f > \sigma\}}$ and $f_2 \equiv f - f_1$. Here, $\sigma \equiv w^{-\frac{1}{p(\cdot)-1}}$ is the dual weight. Then

$$\int_{\mathbb{R}^n} |f_j(x)|^{p(x)} w(x) \, dx < 1 \quad (j = 1, 2). \tag{3.1}$$

We only have to show that for $j = 1, 2$,

$$\int_{\mathbb{R}^n} M_{\mathfrak{D}} f_j(x)^{p(x)} w(x) \, dx \lesssim 1.$$

We form a sparse decomposition of f_1 and f_2 separately. The estimates of f_1 and f_2 will be established independently. So suppose that there exists a sparse family $\{Q_j^k\}_{k \in \mathbb{Z}, j \in J_k}$ with the nutshell $\{E_j^k\}_{k \in \mathbb{Z}, j \in J_k}$ such that

$$M_{\mathfrak{D}} f_l \lesssim \sum_{k=-\infty}^{\infty} \sum_{j \in J_k} m_{Q_j^k} (|f_l|) \chi_{E_j^k} \quad (l = 1, 2).$$

Since the E_j^k 's are disjoint, we have

$$\int_{\mathbb{R}^n} (M_{\mathfrak{D}} f_l(x))^{p(x)} w(x) \, dx \lesssim \sum_{k=-\infty}^{\infty} \sum_{j \in J_k} \int_{E_j^k} m_{Q_j^k} (|f_l|)^{p(x)} w(x) \, dx =: I_l \quad (l = 1, 2).$$

3.1 Estimate of $M_{\mathfrak{D}} f_1$

We use the sparse decomposition of $M_{\mathfrak{D}} f_1$:

$$M_{\mathfrak{D}} f_1 \lesssim \sum_{k=-\infty}^{\infty} \sum_{j \in J_k} m_{Q_j^k} (|f_1|) \chi_{E_j^k}.$$

Let $x \in \mathbb{R}^n$. Note that $f_1(x)\sigma^{-1}(x) \geq 1$ unless $f_1(x) = 0$. Since $p_-(Q_j^k) \geq 1$,

$$\begin{aligned} \int_{Q_j^k} (f_1(y)\sigma(y)^{-1})^{\frac{p(y)}{p_-(Q_j^k)}} \sigma(y) \, dy &\leq \int_{Q_j^k} (f_1(y)\sigma(y)^{-1})^{p(y)} \sigma(y) \, dy \\ &= \int_{Q_j^k} f_1(y)^{p(y)} w(y) \, dy \leq 1. \end{aligned}$$

Consequently,

$$\begin{aligned} \left(\int_{Q_j^k} (f_1(y)\sigma(y)^{-1}) \sigma(y) \, dy \right)^{p(x)} &\leq \left(\int_{Q_j^k} (f_1(y)\sigma(y)^{-1})^{\frac{p(y)}{p_-(Q_j^k)}} \sigma(y) \, dy \right)^{p(x)} \\ &\leq \left(\int_{Q_j^k} (f_1(y)\sigma(y)^{-1})^{\frac{p(y)}{p_-(Q_j^k)}} \sigma(y) \, dy \right)^{p_-(Q_j^k)}. \end{aligned}$$

Hence, by Lemma 2.19(3),

$$\begin{aligned} I_1 &\leq \sum_{k=-\infty}^{\infty} \sum_{j \in J_k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} (f_1(y)\sigma(y)^{-1})^{\frac{p(y)}{p_-(Q_j^k)}} \sigma(y) \, dy \right)^{p_-(Q_j^k)} \\ &\quad \times \int_{E_j^k} \sigma(Q_j^k)^{p_-(Q_j^k)} |Q_j^k|^{-p(x)} w(x) \, dx \\ &\lesssim \sum_{k=-\infty}^{\infty} \sum_{j \in J_k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} (f_1(y)\sigma(y)^{-1})^{\frac{p(y)}{p_-(Q_j^k)}} \sigma(y) \, dy \right)^{p_-(Q_j^k)} \sigma(Q_j^k) \\ &\leq \sum_{k=-\infty}^{\infty} \sum_{j \in J_k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} (f_1(y)\sigma(y)^{-1})^{\frac{p(y)}{p_-(Q_j^k)}} \sigma(y) \, dy \right)^{p_-} \sigma(Q_j^k). \end{aligned}$$

The last inequality used Hölder’s inequality. Since $\sigma \in A_\infty(\mathfrak{D})$, $\sigma(Q_j^k) \lesssim \sigma(E_j^k)$ by virtue of Lemma 2.9. Consequently, due to Lemma 2.18, we have

$$I_1 \lesssim \int_{\mathbb{R}^n} \left\{ [f_1(x)\sigma^{-1}(x)]^{\frac{p(x)}{p_-}} \right\}^{p_-} \sigma(x) \, dx \leq 1.$$

3.2 Estimate of Mf_2

Set

$$\begin{aligned} \mathcal{F} &\equiv \{(k, j) : Q_j^k \subset P\}, \\ \mathcal{G} &\equiv \{(k, j) : P \subset Q_j^k\}, \\ \mathcal{H} &\equiv \{(k, j) : P \cap Q_j^k = \emptyset\}. \end{aligned}$$

Accordingly,

$$I_{2,\mathcal{A}} \equiv \sum_{(k,j) \in \mathcal{A}} \int_{E_j^k} m_{Q_j^k}(|f_2|)^{p(x)} w(x) \, dx \quad (\mathcal{A} \in \{\mathcal{F}, \mathcal{G}, \mathcal{H}\}).$$

Estimate of $I_{2,\mathcal{F}}$ Since $f_2\sigma^{-1} \leq 1$, we have

$$\begin{aligned} I_{2,\mathcal{F}} &= \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y) \, dy \right)^{p(x)} w(x) \, dx \\ &\leq \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \sigma(Q_j^k)^{p(x)-p_-(Q_j^k)} \sigma(Q_j^k)^{p_-(Q_j^k)} |Q_j^k|^{-p(x)} w(x) \, dx \\ &\leq \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} (1 + \sigma(Q_j^k))^{p(x)-p_-(Q_j^k)} \sigma(Q_j^k)^{p_-(Q_j^k)} |Q_j^k|^{-p(x)} w(x) \, dx \\ &\leq (1 + \sigma(P))^{p_+-p_-} \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \sigma(Q_j^k)^{p_-(Q_j^k)} |Q_j^k|^{-p(x)} w(x) \, dx. \end{aligned}$$

From Lemmas 2.19(3) and 2.9, we obtain

$$\begin{aligned} I_{2,\mathcal{F}} &\lesssim (1 + \sigma(P))^{p_+-p_-} \sum_{(k,j) \in \mathcal{F}} \sigma(Q_j^k) \\ &\lesssim (1 + \sigma(P))^{p_+-p_-} \sum_{(k,j) \in \mathcal{F}} \sigma(E_j^k) \lesssim (1 + \sigma(P))^{p_+-p_-} \sigma(P) \lesssim 1. \end{aligned}$$

Estimate of $I_{2,\mathcal{G}}$ We note that $w(Q_j^k) \geq w(P) \geq 1$ and $\sigma(Q_j^k) \geq \sigma(P) \geq 1$. Consequently, from Lemma 2.15(1)–(2),

$$\frac{1}{|Q_j^k|} \lesssim \frac{1}{|P|} \sigma(P)^{\frac{1}{p_\infty}} \sigma(Q_j^k)^{-\frac{1}{p_\infty}} \lesssim \sigma(Q_j^k)^{-\frac{1}{p_\infty}} \lesssim \frac{1}{\|\chi_{Q_j^k}\|_{L^{p'(\cdot)}(\sigma)}}.$$

Thus, by Hölder’s inequality and (3.1),

$$m_{Q_j^k}(f_2) \lesssim \frac{1}{\|\chi_{Q_j^k}\|_{L^{p'(\cdot)}(\sigma)}} \int_{Q_j^k} f_2(y) \, dy \lesssim \frac{\|f_2\|_{L^{p(\cdot)}(w)} \|\chi_{Q_j^k}\|_{L^{p'(\cdot)}(\sigma)}}{\|\chi_{Q_j^k}\|_{L^{p'(\cdot)}(\sigma)}} \lesssim 1.$$

Using Lemma 2.7 for the measure $w(x) \, dx$, we have

$$\begin{aligned} I_{2,\mathcal{G}} &\lesssim \sum_{(k,j) \in \mathcal{G}} \int_{E_j^k} (m_{Q_j^k}(f_2))^{p_\infty} w(x) \, dx + \sum_{(k,j) \in \mathcal{G}} \int_{E_j^k} \frac{w(x) \, dx}{(e + |x|)^K} \\ &\lesssim \sum_{(k,j) \in \mathcal{G}} \int_{E_j^k} (m_{Q_j^k}(f_2))^{p_\infty} w(x) \, dx + 1. \end{aligned}$$

To complete the estimate of $I_{2,\mathcal{G}}$, we only have to show that the first sum is bounded by a constant. We calculate

$$\begin{aligned} &\sum_{(k,j) \in \mathcal{G}} \int_{E_j^k} (m_{Q_j^k}(f_2))^{p_\infty} w(x) \, dx \\ &= \sum_{(k,j) \in \mathcal{G}} w(E_j^k) \left(\frac{\sigma(Q_j^k)}{|Q_j^k|} \right)^{p_\infty} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y) \, dy \right)^{p_\infty}. \end{aligned}$$

We note that

$$\sigma(Q_j^k)^{p_\infty-1} = \sigma(Q_j^k)^{\frac{p_\infty}{p_\infty}} \sim (\|\chi_{Q_j^k}\|_{L^{p(\cdot)}(\sigma)})^{p_\infty} \lesssim \left(\frac{|Q_j^k|}{\|\chi_{Q_j^k}\|_{L^{p(\cdot)}(w)}}\right)^{p_\infty} \sim \frac{|Q_j^k|^{p_\infty}}{w(Q_j^k)}$$

thanks to Lemma 2.15(1) and the definition of $A_{p(\cdot)}(\mathfrak{D})$. Thus,

$$\begin{aligned} & \sum_{(k,j) \in \mathcal{G}} \int_{E_j^k} (m_{Q_j^k}(f_2))^{p_\infty} w(x) \, dx \\ &= \sum_{(k,j) \in \mathcal{G}} w(E_j^k) \left(\frac{\sigma(Q_j^k)}{|Q_j^k|}\right)^{p_\infty} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y) \, dy\right)^{p_\infty} \\ &\lesssim \sum_{(k,j) \in \mathcal{G}} \sigma(Q_j^k) \frac{w(E_j^k)}{w(Q_j^k)} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y) \, dy\right)^{p_\infty} \\ &\lesssim \sum_{(k,j) \in \mathcal{G}} \sigma(E_j^k) \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y) \, dy\right)^{p_\infty} \\ &\lesssim \int_{\mathbb{R}^n} (f_2(x) \sigma(x)^{-1})^{p_\infty} \sigma(x) \, dx, \end{aligned}$$

owing to Lemma 2.18. Since $0 \leq f_2 \sigma^{-1} \leq 1$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} (f_2(x) \sigma(x)^{-1})^{p_\infty} \sigma(x) \, dx &\lesssim \int_{\mathbb{R}^n} (f_2(x) \sigma(x)^{-1})^{p(x)} \sigma(x) \, dx + \int_{\mathbb{R}^n} \frac{\sigma(x)}{(e + |x|)^K} \, dx \\ &\lesssim \int_{\mathbb{R}^n} f_2(x)^{p(x)} w(x) \, dx + 1 \lesssim 1, \end{aligned}$$

thanks to Lemma 2.7 applied for the weighted measure $\sigma(x) \, dx$. Consequently, $I_{2,\mathcal{G}} \lesssim 1$.

Estimate of $I_{3,\mathcal{G}}$ Set

$$\mathcal{H}_1 \equiv \{(k, j) \in \mathcal{H} : \sigma(Q_j^k) \leq 1\}, \quad \mathcal{H}_2 \equiv \{(k, j) \in \mathcal{H} : \sigma(Q_j^k) > 1\}.$$

Accordingly, we consider

$$I_{2,\mathcal{H}_l} \equiv \sum_{(k,j) \in \mathcal{H}_l} \int_{E_j^k} m_{Q_j^k}(|f_2|)^{p(x)} w(x) \, dx \quad (l = 1, 2).$$

Let $(k, j) \in \mathcal{H}_1$. Let $x_+ \in \overline{Q_j^k}$ satisfy $p(x_+) = p_+(Q_j^k)$. Then

$$|p_+(Q_j^k) - p(x)| \leq |p(x) - p_\infty| + |p(x_+) - p_\infty| \lesssim \frac{1}{\log(e + |x|)} \quad (x \in Q_j^k).$$

Also see [3, (5.14)]. Consequently, from Lemma 2.7 applied for the measure $w(x) dx$, we deduce

$$\begin{aligned} I_{2, \mathcal{H}_1} &\lesssim \sum_{(k,j) \in \mathcal{H}_1} \int_{E_j^k} m_{Q_j^k}(f_2)^{p_+(Q_j^k)} w(x) dx + \sum_{(k,j) \in \mathcal{H}_1} \int_{E_j^k} \frac{w(x) dx}{(e + |x|)^K} \\ &\lesssim \sum_{(k,j) \in \mathcal{H}_1} \int_{E_j^k} m_{Q_j^k}(f_2)^{p_+(Q_j^k)} w(x) dx + 1. \end{aligned}$$

Note that

$$\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y) dy \leq 1$$

from the definition of f_2 . We calculate

$$\begin{aligned} I_{2, \mathcal{H}_1} &\lesssim \sum_{(k,j) \in \mathcal{H}_1} \int_{E_j^k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y) dy \right)^{p_+(Q_j^k)} \left(\frac{\sigma(Q_j^k)}{|Q_j^k|} \right)^{p_+(Q_j^k)} w(x) dx + 1 \\ &\leq \sum_{(k,j) \in \mathcal{H}_1} \int_{E_j^k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y) dy \right)^{p_\infty} \left(\frac{\sigma(Q_j^k)}{|Q_j^k|} \right)^{p_+(Q_j^k)} w(x) dx \\ &\quad + \sum_{(k,j) \in \mathcal{H}_1} \int_{E_j^k} (e + |x|)^{-K} \left(\frac{\sigma(Q_j^k)}{|Q_j^k|} \right)^{p_+(Q_j^k)} w(x) dx + 1. \end{aligned}$$

Since $\sigma(Q_j^k) \leq 1$ and $p(x) \leq p_+(Q_j^k)$ for $x \in Q_j^k$, we have

$$\begin{aligned} I_{2, \mathcal{H}_1} &\lesssim \sum_{(k,j) \in \mathcal{H}_1} \int_{E_j^k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y) dy \right)^{p_\infty} \left(\frac{\sigma(Q_j^k)}{|Q_j^k|} \right)^{p_+(Q_j^k)} w(x) dx \\ &\quad + \sum_{(k,j) \in \mathcal{H}_1} \int_{E_j^k} (e + |x|)^{-K} \frac{\sigma(Q_j^k)^{p_-(Q_j^k)}}{|Q_j^k|^{p(x)}} w(x) dx + 1 \\ &\lesssim \sum_{(k,j) \in \mathcal{H}_1} \int_{E_j^k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y) dy \right)^{p_\infty} \left(\frac{\sigma(Q_j^k)}{|Q_j^k|} \right)^{p_+(Q_j^k)} w(x) dx + 1, \end{aligned}$$

by virtue of Lemma 2.20. Consequently, we only have to show

$$\sum_{(k,j) \in \mathcal{H}_1} \int_{E_j^k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y) dy \right)^{p_\infty} \left(\frac{\sigma(Q_j^k)}{|Q_j^k|} \right)^{p_+(Q_j^k)} w(x) dx \lesssim 1.$$

In fact, from Lemma 2.19(3), we get

$$\begin{aligned} &\sum_{(k,j) \in \mathcal{H}_1} \int_{E_j^k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y) dy \right)^{p_\infty} \frac{\sigma(Q_j^k)^{p_-(Q_j^k)}}{|Q_j^k|^{p(x)}} w(x) dx \\ &\leq \sum_{(k,j) \in \mathcal{H}_1} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y) dy \right)^{p_\infty} \sigma(Q_j^k). \end{aligned}$$

Thus, using Lemmas 2.18 and 2.20, we have

$$\begin{aligned} & \sum_{(k,j) \in \mathcal{H}_1} \int_{E_j^k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y) \, dy \right)^{p_\infty} \frac{\sigma(Q_j^k)^{p_\infty - (Q_j^k)}}{|Q_j^k|^{p(x)}} w(x) \, dx \\ & \lesssim \sum_{(k,j) \in \mathcal{H}_1} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y) \, dy \right)^{p_\infty} \sigma(E_j^k) \\ & \lesssim \int_{\mathbb{R}^n} (f_2(y) \sigma(y)^{-1})^{p_\infty} \sigma(x) \, dx \\ & \lesssim \int_{\mathbb{R}^n} (f_2(y) \sigma(y)^{-1})^{p(x)} \sigma(x) \, dx + \int_{\mathbb{R}^n} \frac{\sigma(x) \, dx}{(e + |x|)^M} \lesssim 1. \end{aligned}$$

We consider \mathcal{H}_2 . By Hölder’s inequality,

$$\int_{Q_j^k} f_2(y) \, dy \lesssim \|f_2\|_{L^{p(\cdot)}(w)} \| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)} \leq \| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)}.$$

Consequently, using Lemma 2.7 for the measure $w(x) \, dx$ and Lemma 2.21, we have

$$\begin{aligned} & \sum_{(k,j) \in \mathcal{H}_2} \int_{E_j^k} \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} f_2(y) \, dy \right)^{p(x)} w(x) \, dx \\ & = \sum_{(k,j) \in \mathcal{H}_2} \int_{E_j^k} \left(\frac{1}{\| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)} \int_{Q_j^k} f_2(y) \, dy \right)^{p(x)} \left(\frac{\| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)}}{|Q_j^k|} \right)^{p(x)} w(x) \, dx \\ & \lesssim \sum_{(k,j) \in \mathcal{H}_2} \int_{E_j^k} \left(\frac{1}{\| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)} \int_{Q_j^k} f_2(y) \, dy \right)^{p_\infty} \left(\frac{\| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)}}{|Q_j^k|} \right)^{p(x)} w(x) \, dx \\ & \quad + \sum_{(k,j) \in \mathcal{H}_2} \int_{E_j^k} \left(\frac{\| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)}}{|Q_j^k|} \right)^{p(x)} \frac{w(x) \, dx}{(e + |x|)^K} \\ & \lesssim \sum_{(k,j) \in \mathcal{H}_2} \int_{E_j^k} \left(\frac{1}{\| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)} \int_{Q_j^k} f_2(y) \, dy \right)^{p_\infty} \left(\frac{\| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)}}{|Q_j^k|} \right)^{p(x)} w(x) \, dx + 1 \\ & \equiv J_1 + 1. \end{aligned}$$

Thanks to Lemma 2.15 applied to the dual exponent,

$$\left(\frac{\sigma(Q_j^k)}{\| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)}} \right)^{p_\infty} \lesssim (\sigma(Q_j^k))^{1 - \frac{1}{p_\infty}})^{p_\infty} = \sigma(Q_j^k).$$

Thus we obtain

$$\begin{aligned} I_1 & = \sum_{(k,j) \in \mathcal{H}_2} \int_{E_j^k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \, dy \right)^{p_\infty} \left(\frac{\sigma(Q_j^k)}{\| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)}} \right)^{p_\infty} \\ & \quad \times \left(\frac{\| \chi_{Q_j^k} \|_{L^{p'(\cdot)}(\sigma)}}{|Q_j^k|} \right)^{p(x)} w(x) \, dx \end{aligned}$$

$$\begin{aligned} &\leq \sum_{(k,j) \in \mathcal{H}_2} \int_{E_j^k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \, dy \right)^{p_\infty} \left(\frac{\|\chi_{Q_j^k}\|_{L^{p'(\cdot)}(\sigma)}}{|Q_j^k|} \right)^{p(x)} \sigma(Q_j^k) w(x) \, dx \\ &\lesssim \sum_{(k,j) \in \mathcal{H}_2} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \, dy \right)^{p_\infty} \sigma(Q_j^k) \\ &\lesssim 1, \end{aligned}$$

where in the third inequality, we used (2.7) and Lemmas 2.7 and 2.9. Together, we obtain the desired result.

3.3 Equivalent condition on weights

Finally, we consider the condition on which Q in the definition of $w \in A_{p(\cdot)}^{\text{loc}}$. We generalize $w \in A_{p(\cdot)}^{\text{loc}}$ as follows:

Definition 3.1 Given an exponent $p(\cdot) : \mathbb{R}^n \rightarrow (1, \infty)$ with $p_- > 1$, a positive number $R \geq 1$ and a weight w , we say that $w \in A_{p(\cdot)}^{\text{loc},R}$ if

$$[w]_{A_{p(\cdot)}^{\text{loc},R}} \equiv \sup_{|Q| \leq R^n} |Q|^{-1} \|\chi_Q\|_{L^{p(\cdot)}(w)} \|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} < \infty,$$

where $\sigma \equiv w^{-\frac{1}{p(\cdot)-1}}$ as before and the supremum is taken over all cubes $Q \in \mathcal{Q}$ with volume R^n .

Accordingly, we consider the local maximal operator given by

$$M^{\text{loc},R} f(x) \equiv \sup_{Q \in \mathcal{Q}, |Q| \leq R^n} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| \, dy \quad (x \in \mathbb{R}^n)$$

for a measurable function f and $R \geq 1$.

Similar to Theorem 1.2, we can prove the following theorem:

Theorem 3.2 *Suppose that a variable exponent $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ satisfies conditions (1.1) and (1.2) and $1 < p_- \leq p_+ < \infty$. Let $R \geq 1$. Then given any $w \in A_{p(\cdot)}^{\text{loc},R}$,*

$$\|M^{\text{loc},R} f\|_{L^{p(\cdot)}(w)} \leq C \|f\|_{L^{p(\cdot)}(w)}.$$

We remark that the class $w \in A_{p(\cdot)}^{\text{loc},R}$ with $R \geq 1$ is independent of $R \geq 1$.

Proposition 3.3 *Suppose that a variable exponent $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ satisfies conditions (1.1) and (1.2) and $1 < p_- \leq p_+ < \infty$. The class $w \in A_{p(\cdot)}^{\text{loc},R}$ with $R \geq 1$ is independent of $R \geq 1$.*

Proof Let $w \in A_{p(\cdot)}^{\text{loc}}$. If $m \equiv [2R + 1]$, then $M^{\text{loc},R} f(x) \leq C_m (M^{\text{loc}})^m f(x)$ for any measurable function f , where $(M^{\text{loc}})^m$ denotes the m -fold composition of M^{loc} . Consequently, $M^{\text{loc},R}$ is bounded on $L^{p(\cdot)}(w)$. Thus, $w \in A_{p(\cdot)}^{\text{loc},R}$. □

4 Proof of Theorem 1.5

Thanks to Lemma 2.8, we have

$$\|M_{\mathfrak{D}}^{\text{loc}} f\|_{L^{p(\cdot)}(w)} \sim \left(\sum_{Q \in \mathcal{D}_{0,(1,1,\dots,1)}} (\|M_{\mathfrak{D}}^{\text{loc}} f\|_{L^{p(\cdot)}(w)} \chi_Q)^{p_{\infty}} \right)^{\frac{1}{p_{\infty}}}.$$

Since $(M_{\mathfrak{D}}^{\text{loc}} f)\chi_Q = M_{\mathfrak{D}}^{\text{loc}}[f\chi_Q] = (M_{\mathfrak{D}}[f\chi_Q])\chi_Q$ for any $Q \in \mathcal{D}_{0,(1,1,\dots,1)}$, we can use Theorem 1.7 to get

$$\|M_{\mathfrak{D}}^{\text{loc}} f\|_{L^{p(\cdot)}(w)} \lesssim \left(\sum_{Q \in \mathcal{D}_{0,(1,1,\dots,1)}} (\|f\chi_Q\|_{L^{p(\cdot)}(w)})^{p_{\infty}} \right)^{\frac{1}{p_{\infty}}}.$$

Using Lemma 2.8, we obtain the desired result.

5 Application – the weighted vector-valued maximal inequality

Finally, as an application, we consider the weighted vector-valued inequality for M^{loc} . If Theorem 1.8 is proven, Theorem 1.11 follows immediately from Proposition 1.9. So, we concentrate on Theorem 1.8 using an extrapolation for $A_{p(\cdot)}^{\text{loc}}$. We prepare two lemmas.

Lemma 5.1 *Let $w_0, w_1 \in A_1^{\text{loc}}$ and $1 < p < \infty$. Then, $w = w_0 w_1^{1-p} \in A_p^{\text{loc}}$.*

Proof The proof is analogous to the corresponding assertion for A_1 and A_p . For convenience, here we supply the proof. Fix a cube Q with $|Q| \leq 1$. Then

$$\begin{aligned} & \frac{1}{|Q|} \int_Q w_0(x) w_1(x)^{1-p} \, dx \left(\frac{1}{|Q|} \int_Q w_0(x)^{1-p'} w_1(x) \, dx \right)^{p-1} \\ & \lesssim \frac{1}{|Q|} \int_Q w_0(x) \left(\frac{1}{|Q|} \int_Q w_1(y) \, dy \right)^{1-p} \, dx \\ & \quad \times \left(\frac{1}{|Q|} \int_Q w_1(x) \, dx \right)^{p-1} \left(\frac{1}{|Q|} \int_Q w_0(y) \, dy \right)^{-1} \\ & = 1. \end{aligned}$$

Thus, $[w]_{A_p^{\text{loc}}} \lesssim 1$ and $w \in A_p^{\text{loc}}$. □

Let us conclude the proof of Theorem 1.8.

Let $w \in A_{p(\cdot)}^{\text{loc}}$ and $(f, g) \in \mathcal{F}$ with $\|f\|_{L^{p(\cdot)}(w)} < \infty$. Assume that $\|f\|_{L^{p(\cdot)}(w)} > 0$ and $0 < \|g\|_{L^{p(\cdot)}(w)} < \infty$, also set

$$h_1 \equiv \frac{f}{\|f\|_{L^{p(\cdot)}(w)}} + \frac{g}{\|g\|_{L^{p(\cdot)}(w)}}.$$

Then, $h_1 \in L^{p(\cdot)}(w)$ and $\|h_1\|_{L^{p(\cdot)}(w)} \leq 2$. We define the operator \mathcal{R} as

$$\mathcal{R}h(x) \equiv \sum_{k=0}^{\infty} \frac{(M^{\text{loc}})^k h(x)}{2^k \|M^{\text{loc}}\|_{\mathcal{B}(L^{p(\cdot)}(w))}^k} \quad (x \in \mathbb{R}^n)$$

for $h \in L^{p(\cdot)}(w)$. Then, we can show that

- (i) for all $x \in \mathbb{R}^n$, $|h(x)| \leq \mathcal{R}h(x)$,
- (ii) $\|\mathcal{R}h\|_{L^{p(\cdot)}(w)} \leq 2\|h\|_{L^{p(\cdot)}(w)}$,
- (iii) $\mathcal{R}h \in A_1^{\text{loc}}$ with $[\mathcal{R}h]_{A_1^{\text{loc}}} \leq 2\|M^{\text{loc}}\|_{\mathcal{B}(L^{p(\cdot)}(w))}$.

Define $M'h \equiv w^{-1} \cdot M^{\text{loc}}(hw)$. Note that if $\sigma = w^{-\frac{1}{p(\cdot)-1}} \in A_{p(\cdot)}^{\text{loc}}$, then M^{loc} is bounded on $L^{p(\cdot)}(\sigma)$. Hence, M' is bounded on $L^{p(\cdot)}(w)$. In fact,

$$\begin{aligned} \|M'h\|_{L^{p(\cdot)}(w)} &= \|w^{-1} \cdot M^{\text{loc}}(hw)w^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}} \\ &= \|M^{\text{loc}}(hw) \cdot \sigma^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}} \\ &= \|M^{\text{loc}}(hw)\|_{L^{p(\cdot)}(\sigma)} \lesssim \|hw\|_{L^{p(\cdot)}(\sigma)} = \|h\|_{L^{p(\cdot)}(w)}. \end{aligned}$$

Moreover, define

$$\mathcal{R}'h(x) \equiv \sum_{k=0}^{\infty} \frac{(M')^k h(x)}{2^k \|M'\|_{\mathcal{B}(L^{p(\cdot)}(w))}^k} \quad (x \in \mathbb{R}^n)$$

for $h \in L^{p(\cdot)}$. Then, we also have

- (i) for all $x \in \mathbb{R}^n$, $|h(x)| \leq \mathcal{R}'h(x)$,
- (ii) $\|\mathcal{R}'h\|_{L^{p(\cdot)}(w)} \leq 2\|h\|_{L^{p(\cdot)}(w)}$,
- (iii) $(\mathcal{R}'h)w \in A_1^{\text{loc}}$ with $[(\mathcal{R}'h)w]_{A_1^{\text{loc}}} \leq 2\|M'\|_{\mathcal{B}(L^{p(\cdot)}(w))}$.

Fix $f \in L^{p(\cdot)}(w)$. Then, $fw^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}$. Thus, by duality there exists a nonnegative function $h \in L^{p(\cdot)}$ with $\|h\|_{L^{p(\cdot)}} = 1$ such that

$$\begin{aligned} \|f\|_{L^{p(\cdot)}(w)} &\lesssim \int_{\mathbb{R}^n} f(x)h(x)w(x)^{\frac{1}{p(x)}} dx \\ &\leq \int_{\mathbb{R}^n} f(x)\mathcal{R}h_1(x)^{-\frac{1}{p_0}} \mathcal{R}h_1(x)^{\frac{1}{p_0}} \mathcal{R}'[hw^{-\frac{1}{p(\cdot)}}](x)^{\frac{1}{p_0}} \mathcal{R}'[hw^{-\frac{1}{p(\cdot)}}](x)^{\frac{1}{p_0}} w(x) dx \\ &\leq \left(\int_{\mathbb{R}^n} f(x)^{p_0} \mathcal{R}h_1(x)^{1-p_0} \mathcal{R}'[hw^{-\frac{1}{p(\cdot)}}](x)w(x) dx \right)^{\frac{1}{p_0}} \\ &\quad \times \left(\int_{\mathbb{R}^n} \mathcal{R}h_1(x)\mathcal{R}'[hw^{-\frac{1}{p(\cdot)}}](x)w(x) dx \right)^{\frac{1}{p_0}} \\ &\equiv I_1 \times I_2. \end{aligned}$$

We estimate I_1 . Since $\mathcal{R}h_1, \mathcal{R}'[hw^{-\frac{1}{p(\cdot)}}]w \in A_1^{\text{loc}}$, according to Lemma 5.1, we have

$$(\mathcal{R}h_1)^{1-p_0} (\mathcal{R}'[hw^{-\frac{1}{p(\cdot)}}]w) \in A_{p_0}.$$

Using our assumption (1.3) and Hölder's inequality, we have

$$\begin{aligned} I_1^{p_0} &\leq \int_{\mathbb{R}^n} g(x)^{p_0} \mathcal{R}h_1(x)^{1-p_0} \mathcal{R}'[hw^{-\frac{1}{p(\cdot)}}](x)w(x) dx \\ &\leq \int_{\mathbb{R}^n} g(x)^{p_0} \left(\frac{g(x)}{\|g\|_{L^{p(\cdot)}(w)}} \right)^{1-p_0} \mathcal{R}'[hw^{-\frac{1}{p(\cdot)}}](x)w(x) dx \end{aligned}$$

$$\begin{aligned}
 &= \|g\|_{L^{p(\cdot)}(w)}^{p_0-1} \int_{\mathbb{R}^n} g(x) \mathcal{R}'[hw^{-\frac{1}{p'(\cdot)}}](x) w(x) \, dx \\
 &= \|g\|_{L^{p(\cdot)}(w)}^{p_0-1} \|g\|_{L^{p(\cdot)}(w)} \|\mathcal{R}'[hw^{-\frac{1}{p'(\cdot)}}]\|_{L^{p'(\cdot)}(w)} \\
 &\lesssim \|g\|_{L^{p(\cdot)}(w)}^{p_0} \|hw^{-\frac{1}{p'(\cdot)}}\|_{L^{p'(\cdot)}(w)} \\
 &= \|g\|_{L^{p(\cdot)}(w)}^{p_0}.
 \end{aligned}$$

Next, we estimate I_2 . Using Hölder’s inequality, we have

$$\begin{aligned}
 I_2^{p'_0} &\lesssim \|(\mathcal{R}h_1)w^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}} \|\mathcal{R}'[hw^{-\frac{1}{p'(\cdot)}}] \cdot w^{\frac{1}{p'(\cdot)}}\|_{L^{p'(\cdot)}} \\
 &= \|\mathcal{R}h_1\|_{L^{p(\cdot)}(w)} \|\mathcal{R}'[hw^{-\frac{1}{p'(\cdot)}}]\|_{L^{p'(\cdot)}(w)} \\
 &\lesssim \|h_1\|_{L^{p(\cdot)}(w)} \|hw^{-\frac{1}{p'(\cdot)}}\|_{L^{p'(\cdot)}(w)} \\
 &= \|h_1\|_{L^{p(\cdot)}(w)} \|h\|_{L^{p'(\cdot)}} \sim 1.
 \end{aligned}$$

Combining these two estimates gives the desired result.

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Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

TN suggested an application (Sect. 5) and was a major contributor in writing the manuscript. YS has drafted the work and investigated some properties of the dyadic cubes. All authors read and approved the final manuscript.

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