# On oscillation of second-order noncanonical neutral differential equations 

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#### Abstract

In the present work, we study the second-order neutral differential equation and formulate new oscillation criteria for this equation. Our conditions differ from the earlier ones. Also, our results are expansions and generalizations of some previous results. Examples to illustrate the main results are included.


MSC: 34C10; 34K11
Keywords: Sufficient conditions; Second-order noncanonical differential equation; Riccati transformation

## 1 Introduction

In this work, we introduce new oscillatory criteria for the second-order differential equations of the form

$$
\begin{equation*}
\left(r(l)\left(x^{\alpha}(l)+q(l) x(\lambda(l))\right)^{\prime}\right)^{\prime}+\sum_{i=1}^{m} h_{i}(l) g\left(x\left(\sigma_{i}(l)\right)\right)=0, \tag{1.1}
\end{equation*}
$$

where $l \geq l_{0}$. Throughout this work, the next conditions are satisfied:
(M1) $\alpha$ is a ratio of odd natural numbers, $\alpha>1$ and $m$ is positive integer;
(M2) $r \in C^{1}\left(\left[l_{0}, \infty\right),(0, \infty)\right), r^{\prime} \geq 0, h_{i} \in C\left(\left[l_{0}, \infty\right),[0, \infty)\right), \quad q \in C\left(\left[l_{0}, \infty\right),(0,1)\right)$, $\inf _{l \geq l_{0}} q(l) \neq 0, q, h$ are not identically zero for large $l$;
(M3) $\lambda, \sigma_{i} \in C^{1}\left(\left[l_{0}, \infty\right),(0, \infty)\right), \quad \lambda(l) \leq l, \sigma_{i}(l) \geq l, \sigma_{i}^{\prime}(l)>0$ and $\lim _{l \rightarrow \infty} \lambda(l)=$ $\lim _{l \rightarrow \infty} \sigma_{i}(l)=\infty ;$
(M4) $g \in C(R, R)$ and there exists $k>0$ where $k$ is a constant, such that $g(x) \geq k x^{\alpha}$ for $x \neq 0$.
We will assume that (1.1) is in the so-called noncanonical form

$$
\begin{equation*}
\int_{l_{0}}^{\infty} \frac{1}{r(s)} \mathrm{d} s<\infty \tag{1.2}
\end{equation*}
$$

By a solution of (1.1), we mean a real-valued function $x \in C\left(\left[l_{x}, \infty\right), R\right) l_{x} \geq l_{0}$, which satisfies (1.1) on $\left[l_{x}, \infty\right)$. and has the property $x^{\alpha}(l)+q(l) x(\lambda(l))$ and $r(l)\left(x^{\alpha}(l)+q(l) x(\lambda(l))\right)$ are continuously differentiable for $l \in\left[l_{x}, \infty\right)$. We only consider those solutions $x(l)$ of (1.1)
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satisfying $\sup \left\{|x(l)|: l \geq l_{a}\right\}>0$ for all $l_{a} \geq l_{x}$, and we assume that (1.1) possesses such solutions.

A solution of (1.1) is called oscillatory if it has arbitrarily many zeros on $\left[l_{0}, \infty\right)$, and is called nonoscillatory otherwise. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

Recently, the oscillatory theory of functional differential equations has received great attention due to the existence of a number of applications in engineering and the natural sciences. There are some contributions in the field of oscillatory behavior of different classes of differential equations, we refer the reader to [1-12] and the references mentioned therein. The neutral delay differential equations have applications in electrical networks containing lossless transmission lines; these networks appear in high-speed computers. See [13].
Several scholars have studied the oscillatory behavior of second-order differential equations under various conditions. See [14-23].

Some new oscillation criteria for the neutral nonlinear differential equation

$$
\left(r(l)(x(l)+q(l) x(\lambda(l)))^{\prime}\right)^{\prime}+\sum_{i=1}^{m} h_{i}(l) g\left(x\left(\sigma_{i}(l)\right)\right)=0
$$

are established by Xu et al. [21], where $\alpha=1, \sigma_{i}(l) \leq l$ and

$$
\begin{equation*}
\int_{l_{0}}^{\infty} \frac{1}{r(s)} \mathrm{d} s=\infty \tag{1.3}
\end{equation*}
$$

Agarwal et al. [22] investigated the second-order differential equations with a sublinear neutral term

$$
\left(r(l)\left(x(l)+q(l) x^{\alpha}(\lambda(l))\right)^{\prime}\right)^{\prime}+h(l) x(\sigma(l))=0,
$$

where $0<\alpha \leq 1, \lambda(l) \leq l$ and $\sigma(l) \leq l$. They established some oscillation criteria under the condition (1.2) and (1.3).
Dzurina [23] established a new comparison theorem for deducing oscillation of the nonlinear differential equation

$$
\left(r(l)\left(x^{\prime}(l)\right)^{\alpha}\right)^{\prime}+h(l) x^{\alpha}(\sigma(l))=0
$$

where $\alpha$ is a quotient of odd positive integers and $\sigma(l) \leq l$.
The objective of this paper is to study the oscillatory properties of the second-order neutral differential equations in noncanonical form. By using Riccati transformations, we present a new conditions for oscillation of the studied equation. The results obtained here extend and complement to some known results in the literature. See for example [21-23]. Some examples are provided to illustrate the relevance of new theorems.

## 2 Main results

In the rest of the work, we will adopt the following notation:

$$
v(l):=x^{\alpha}(l)+q(l) x(\lambda(l)),
$$

$$
\pi(l)=\int_{l}^{\infty} \frac{1}{r(s)} \mathrm{d} s
$$

and

$$
\sigma(l)=\min \left\{\sigma_{i}(l), i=1,2, \ldots, m\right\} .
$$

In order to prove our results, we will present the following lemma.

Lemma 2.1 Assume that $x(l)$ is a positive solution of $(1.1)$ on $\left[l_{1}, \infty\right)$, where $l_{1} \geq l_{0}$. Moreover, assume that (1.2) holds and

$$
\begin{equation*}
\int_{l_{0}}^{\infty} \sum_{i=1}^{m} h_{i}(s) \mathrm{d} s=\infty \tag{2.1}
\end{equation*}
$$

Then

$$
\begin{equation*}
v(l)>0, \quad v^{\prime}(l)<0, \quad\left(r(l) v^{\prime}(l)\right)^{\prime} \leq 0, \tag{2.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\frac{v(l)}{\pi(l)}\right)^{\prime} \geq 0 \tag{2.3}
\end{equation*}
$$

for $l \geq l_{1}$.

Proof Let $x(l)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(l)>0, x(\lambda(l))>0$ and $x(\sigma(l))>0$ for $l \geq l_{1} \geq l_{0}$. From (1.1), we have

$$
\begin{align*}
\left(r(l) v^{\prime}(l)\right)^{\prime} & =-\sum_{i=1}^{m} h_{i}(l) g\left(x\left(\sigma_{i}(l)\right)\right) \\
& \leq-k \sum_{i=1}^{m} h_{i}(l) x^{\alpha}\left(\sigma_{i}(l)\right) \leq 0 . \tag{2.4}
\end{align*}
$$

Hence, the function $r(l) v^{\prime}(l)$ is decreasing and therefore we shall consider the following two cases, either $v^{\prime}(l)<0$ or $v^{\prime}(l)>0$. Assume that there exists $l_{2} \geq l_{1}$ such that $v^{\prime}(l)>0$ on $\left[l_{2}, \infty\right)$. Then

$$
x^{\alpha}(l)=v(l)-q(l) x(\lambda(l)) \geq v(l)-q(l) v(\lambda(l)) \geq v(l)(1-q(l)), \quad \text { for } l \geq l_{2},
$$

and so

$$
\begin{equation*}
x^{\alpha}\left(\sigma_{i}(l)\right) \geq v\left(\sigma_{i}(l)\right)\left(1-q\left(\sigma_{i}(l)\right)\right), \tag{2.5}
\end{equation*}
$$

which together with (2.4) implies that

$$
\left(r(l) v^{\prime}(l)\right)^{\prime} \leq-k \sum_{i=1}^{m} h_{i}(l)\left(1-q\left(\sigma_{i}(l)\right)\right) v\left(\sigma_{i}(l)\right)
$$

$$
\begin{equation*}
\leq-k v(\sigma(l)) \sum_{i=1}^{m} h_{i}(l)\left(1-q\left(\sigma_{i}(l)\right)\right) . \tag{2.6}
\end{equation*}
$$

Define the function $\omega(l)$ by the Riccati substitution

$$
\begin{equation*}
\omega(l)=\frac{r(l) \nu^{\prime}(l)}{v(\sigma(l))} . \tag{2.7}
\end{equation*}
$$

Then $\omega(l)>0$. Differentiating (2.7), using (1.1) and (2.5) we see that

$$
\begin{align*}
\omega^{\prime}(l) & =\frac{\left(r(l) v^{\prime}(l)\right)^{\prime}}{v(\sigma(l))}-\frac{r(l) v^{\prime}(l) v^{\prime}(\sigma(l)) \sigma^{\prime}(l)}{v^{2}(\sigma(l))} \\
& \leq-k \sum_{i=1}^{m} h_{i}(l)\left(1-q\left(\sigma_{i}(l)\right)\right)-\frac{v^{\prime}(\sigma(l)) \sigma^{\prime}(l)}{v(\sigma(l))} \omega(l) \\
& \leq-k \sum_{i=1}^{m} h_{i}(l)\left(1-q\left(\sigma_{i}(l)\right)\right) . \tag{2.8}
\end{align*}
$$

Integrating (2.8) from $l_{2}$ to $l$, we obtain

$$
\begin{align*}
\omega(l) & \leq \omega\left(l_{2}\right)-k \int_{l_{2}}^{l} \sum_{i=1}^{m} h_{i}(s)\left(1-q\left(\sigma_{i}(s)\right)\right) \mathrm{d} s \\
& \leq \omega\left(l_{2}\right)-k \min _{1 \leq i \leq m} \inf _{l \geq l_{2}}\left(1-q\left(\sigma_{i}(l)\right)\right) \int_{l_{2}}^{l} \sum_{i=1}^{m} h_{i}(s) \mathrm{d} s . \tag{2.9}
\end{align*}
$$

The above inequality, taking assumption (2.1) into account, implies that $\omega(l) \rightarrow-\infty$ as $l \rightarrow \infty$, which is a contradiction. Hence, the case $v^{\prime}(l)>0$ is impossible. Thus, $v(l)$ satisfies (2.2) for $l \geq l_{1}$. On the other hand, it follows from the monotonicity of $r(l) v^{\prime}(l)$ that

$$
\begin{equation*}
v(l) \geq-\int_{l}^{\infty} \frac{r(s) v^{\prime}(s)}{r(s)} \mathrm{d} s \geq-r(l) v^{\prime}(l) \int_{l}^{\infty} \frac{1}{r(s)} \mathrm{d} s \geq-r(l) \nu^{\prime}(l) \pi(l), \tag{2.10}
\end{equation*}
$$

that is,

$$
\begin{equation*}
v(l)+r(l) v^{\prime}(l) \pi(l) \geq 0 . \tag{2.11}
\end{equation*}
$$

Now

$$
\begin{equation*}
\left(\frac{v(l)}{\pi(l)}\right)^{\prime}=\frac{\pi(l) v^{\prime}(l)-v(l) \pi^{\prime}(l)}{\pi^{2}(l)} . \tag{2.12}
\end{equation*}
$$

From (2.11) and (2.12), we conclude that

$$
\left(\frac{v(l)}{\pi(l)}\right)^{\prime}=\frac{r(l) \pi(l) v^{\prime}(l)+v(l)}{r(l) \pi^{2}(l)} \geq 0 .
$$

The proof of the lemma is complete.

Theorem 2.1 Let condition (1.2) be satisfied. If

$$
\begin{equation*}
0<1-q(l) \frac{\pi(\lambda(l))}{\pi(l)}<1, \quad \inf _{l \geq l_{1}}\left(1-q(l) \frac{\pi(\lambda(l))}{\pi(l)}\right)>0 \tag{2.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{l_{0}}^{\infty} \frac{1}{r(u)} \int_{l_{0}}^{u} \sum_{i=1}^{m} h_{i}(s) \pi\left(\sigma_{i}(s)\right) \mathrm{d} s \mathrm{~d} u=\infty \tag{2.14}
\end{equation*}
$$

then every solution $x(l)$ of $(1.1)$ is oscillatory.

Proof Let $x(l)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(l)>0, x(\lambda(l))>0$ and $x(\sigma(l))>0$ for $l \geq l_{1} \geq l_{0}$. It is well known that (2.1) is necessary to verify (2.14). Since the function

$$
\int_{l_{0}}^{\infty} \sum_{i=1}^{m} h_{i}(s) \pi\left(\sigma_{i}(s)\right) \mathrm{d} s
$$

is unbounded due to (1.2) and $\pi^{\prime}(l)<0$, (2.1) must hold. Using Lemma 2.1, $v(l)$ satisfies (2.2) for $l \geq l_{1}$. It follows from (2.3) that there is $c>0$ such that

$$
\begin{equation*}
\frac{v(l)}{\pi(l)} \geq c \tag{2.15}
\end{equation*}
$$

and

$$
\begin{align*}
x^{\alpha}(l) & =v(l)-q(l) x(\lambda(l)) \geq v(l)-q(l) v(\lambda(l)) \\
& \geq v(l)-q(l) \frac{\pi(\lambda(l)) v(l)}{\pi(l)}=v(l)\left(1-q(l) \frac{\pi(\lambda(l))}{\pi(l)}\right) . \tag{2.16}
\end{align*}
$$

Using (2.15) and (2.16) in (1.1), we obtain

$$
\begin{align*}
\left(r(l) v^{\prime}(l)\right)^{\prime} & \leq-k \sum_{i=1}^{m} h_{i}(l) v\left(\sigma_{i}(l)\right)\left(1-q\left(\sigma_{i}(l)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(l)\right)\right)}{\pi\left(\sigma_{i}(l)\right)}\right) \\
& \leq-k \sum_{i=1}^{m} h_{i}(l)\left(1-q\left(\sigma_{i}(l)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(l)\right)\right)}{\pi\left(\sigma_{i}(l)\right)}\right) c \pi\left(\sigma_{i}(l)\right) . \tag{2.17}
\end{align*}
$$

Integrating (2.17) from $l_{1}$ to $l$, we obtain

$$
r(l) \nu^{\prime}(l)-r\left(l_{1}\right) v^{\prime}\left(l_{1}\right) \leq-k c \int_{l_{1}}^{l} \sum_{i=1}^{m} h_{i}(s)\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \pi\left(\sigma_{i}(s)\right) \mathrm{d} s,
$$

that is,

$$
\begin{equation*}
v^{\prime}(l) \leq-\frac{k c}{r(l)} \int_{l_{1}}^{l} \sum_{i=1}^{m} h_{i}(s)\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \pi\left(\sigma_{i}(s)\right) \mathrm{d} s . \tag{2.18}
\end{equation*}
$$

Integrating (2.18) again from $l_{1}$ to $l$ and taking into account (2.13) and (2.14), we have

$$
\begin{aligned}
v(l) & \leq v\left(l_{1}\right)-\int_{l_{1}}^{l} \frac{k c}{r(u)} \int_{l_{1}}^{u} \sum_{i=1}^{m} h_{i}(s)\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \pi\left(\sigma_{i}(s)\right) \mathrm{d} s \mathrm{~d} u \\
& \leq v\left(l_{1}\right)-k c \min _{1 \leq i \leq m} \inf _{l \geq l_{1}}\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \int_{l_{1}}^{l} \frac{1}{r(u)} \int_{l_{1}}^{u} \sum_{i=1}^{m} h_{i}(s) \pi\left(\sigma_{i}(s)\right) \mathrm{d} s \mathrm{~d} u .
\end{aligned}
$$

The above inequality, taking assumptions (2.13) and (2.14) into account, implies that $v(l) \rightarrow-\infty$ as $l \rightarrow \infty$, which is a contradiction. The proof of the theorem is complete.

Theorem 2.2 Suppose that (1.2) and (2.1) hold. If

$$
\begin{equation*}
0<1-q(l) \frac{\pi(\lambda(l))}{\pi(l)}<1 \tag{2.19}
\end{equation*}
$$

and

$$
\begin{equation*}
W^{\prime}(l) \geq\left(k \sum_{i=1}^{m} h_{i}(l) M\left(\sigma_{i}(l)\right)\left(1-q\left(\sigma_{i}(l)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(l)\right)\right)}{\pi\left(\sigma_{i}(l)\right)}\right)\right) W(\sigma(l)) \tag{2.20}
\end{equation*}
$$

is oscillatory, where

$$
M(l)=k \int_{l}^{\infty} \pi(s) \sum_{i=1}^{m} h_{i}(s) \pi\left(\sigma_{i}(s)\right)\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \mathrm{d} s
$$

then every solution $x(l)$ of $(1.1)$ is oscillatory.

Proof Let $x(l)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(l)>0, x(\lambda(l))>0$ and $x(\sigma(l))>0$ for $l \geq l_{1} \geq l_{0}$. Because of (2.1), from Lemma 2.1, we can conclude that $v(l)$ satisfies (2.2). Now, since

$$
\begin{equation*}
\left(v(l)+r(l) v^{\prime}(l) \pi(l)\right)^{\prime}=v^{\prime}(l)+\left(r(l) v^{\prime}(l)\right)^{\prime} \pi(l)+r(l) v^{\prime}(l) \pi^{\prime}(l)=\left(r(l) v^{\prime}(l)\right)^{\prime} \pi(l), \tag{2.21}
\end{equation*}
$$

using (1.1) and (2.16), (2.21) becomes

$$
\begin{align*}
\left(v(l)+r(l) v^{\prime}(l) \pi(l)\right)^{\prime} & \leq-k \pi(l) \sum_{i=1}^{m} h_{i}(l) x^{\alpha}\left(\sigma_{i}(l)\right) \\
& \leq-k \pi(l) \sum_{i=1}^{m} h_{i}(l) v\left(\sigma_{i}(l)\right)\left(1-q\left(\sigma_{i}(l)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(l)\right)\right)}{\pi\left(\sigma_{i}(l)\right)}\right) \leq 0 \tag{2.22}
\end{align*}
$$

Hence, we observe that $\Theta(l)=v(l)+r(l) v^{\prime}(l) \pi(l) \geq 0$ is nonincreasing. By integrating (2.22) from $l$ to $\infty$ and using (2.11), we obtain

$$
\begin{aligned}
\Theta(l) & \geq \int_{l}^{\infty} k \pi(s) \sum_{i=1}^{m} h_{i}(s) v\left(\sigma_{i}(s)\right)\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \mathrm{d} s \\
& \geq k \int_{l}^{\infty} \pi(s) \sum_{i=1}^{m} h_{i}(s)\left(-r\left(\sigma_{i}(s)\right) v^{\prime}\left(\sigma_{i}(s)\right)\right) \pi\left(\sigma_{i}(s)\right)\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \mathrm{d} s
\end{aligned}
$$

$$
\begin{aligned}
& \geq k \int_{l}^{\infty} \pi(s) \sum_{i=1}^{m} h_{i}(s)\left(-r(s) v^{\prime}(s)\right) \pi\left(\sigma_{i}(s)\right)\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \mathrm{d} s \\
& \geq\left(-r(l) \nu^{\prime}(l)\right) k \int_{l}^{\infty} \pi(s) \sum_{i=1}^{m} h_{i}(s) \pi\left(\sigma_{i}(s)\right)\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \mathrm{d} s,
\end{aligned}
$$

since $r(l) v^{\prime}(l) \pi(l)<0$, we get

$$
\begin{align*}
v(l) & \geq\left(-r(l) v^{\prime}(l)\right) k \int_{l}^{\infty} \pi(s) \sum_{i=1}^{m} h_{i}(s) \pi\left(\sigma_{i}(s)\right)\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \mathrm{d} s \\
& \geq M(l)\left(-r(l) v^{\prime}(l)\right) . \tag{2.23}
\end{align*}
$$

From (2.17), we have

$$
\begin{equation*}
\left(r(l) v^{\prime}(l)\right)^{\prime} \leq-k \sum_{i=1}^{m} h_{i}(l) v\left(\sigma_{i}(l)\right)\left(1-q\left(\sigma_{i}(l)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(l)\right)\right)}{\pi\left(\sigma_{i}(l)\right)}\right) . \tag{2.24}
\end{equation*}
$$

Using (2.23) and (2.24), we see that $W(l)=-r(l) v^{\prime}(l)$ is a positive solution of the differential inequality

$$
W^{\prime}(l) \geq k \sum_{i=1}^{m} h_{i}(l) M\left(\sigma_{i}(l)\right)\left(1-q\left(\sigma_{i}(l)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(l)\right)\right)}{\pi\left(\sigma_{i}(l)\right)}\right) W\left(\sigma_{i}(l)\right) .
$$

From the increasing property of $W(l)$, we get

$$
W^{\prime}(l) \geq k \sum_{i=1}^{m} h_{i}(l) M\left(\sigma_{i}(l)\right)\left(1-q\left(\sigma_{i}(l)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(l)\right)\right)}{\pi\left(\sigma_{i}(l)\right)}\right) W(\sigma(l))
$$

that is,

$$
W^{\prime}(l) \geq\left(k \sum_{i=1}^{m} h_{i}(l) M\left(\sigma_{i}(l)\right)\left(1-q\left(\sigma_{i}(l)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(l)\right)\right)}{\pi\left(\sigma_{i}(l)\right)}\right)\right) W(\sigma(l)),
$$

which is a contradiction. The proof of the theorem is complete.

Theorem 2.3 Suppose that (1.2), (2.1) and (2.19)hold. If

$$
\begin{equation*}
\int_{l_{0}}^{\infty}\left(k \pi(s) \sum_{i=1}^{m} h_{i}(s)\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \frac{\pi\left(\sigma_{i}(s)\right)}{\pi(s)}-\frac{1}{4 r(s) \pi(s)}\right) \mathrm{d} s=\infty, \tag{2.25}
\end{equation*}
$$

then every solution $x(l)$ of $(1.1)$ is oscillatory.

Proof Let $x(l)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(l)>0, x(\lambda(l))>0$ and $x(\sigma(l))>0$ for $l \geq l_{1} \geq l_{0}$. Because of (2.1), from Lemma 2.1, we can conclude that $v(l)$ satisfies (2.2). We now define the following function:

$$
\begin{equation*}
\phi(l)=\frac{r(l) v^{\prime}(l)}{v(l)} \tag{2.26}
\end{equation*}
$$

for $l \geq l_{1}$. Differentiating (2.26), we have

$$
\begin{equation*}
\phi^{\prime}(l)=\frac{\left(r(l) v^{\prime}(l)\right)^{\prime}}{v(l)}-\frac{r(l)\left(v^{\prime}(l)\right)^{2}}{v^{2}(l)}, \tag{2.27}
\end{equation*}
$$

from (2.24) and (2.27), we have

$$
\phi^{\prime}(l) \leq \frac{-k \sum_{i=1}^{m} h_{i}(l) v\left(\sigma_{i}(l)\right)\left(1-q\left(\sigma_{i}(l)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(l)\right)\right)}{\pi\left(\sigma_{i}(l)\right)}\right)}{v(l)}-\frac{r(l)\left(v^{\prime}(l)\right)^{2}}{v^{2}(l)} .
$$

Because of (2.3) and (2.26), we conclude

$$
\phi^{\prime}(l) \leq-k \sum_{i=1}^{m} h_{i}(l)\left(1-q\left(\sigma_{i}(l)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(l)\right)\right)}{\pi\left(\sigma_{i}(l)\right)}\right) \frac{\pi\left(\sigma_{i}(l)\right)}{\pi(l)}-\frac{\phi^{2}(l)}{r(l)} .
$$

Multiplying this inequality by $\pi(l)$ and integrating the resulting inequality from $l_{1}$ to $l$ we find

$$
\begin{align*}
\pi(l) \phi(l)-\pi\left(l_{1}\right) \phi\left(l_{1}\right) \leq & -k \int_{l_{1}}^{l} \pi(s) \sum_{i=1}^{m} h_{i}(s)\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \frac{\pi\left(\sigma_{i}(s)\right)}{\pi(s)} \mathrm{d} s \\
& -\int_{l_{1}}^{l} \pi(s) \frac{\phi^{2}(s)}{r(s)} \mathrm{d} s-\int_{l_{1}}^{l} \frac{\phi(s)}{r(s)} \mathrm{d} s, \tag{2.28}
\end{align*}
$$

using the inequality

$$
\begin{equation*}
-B \Omega+A \Omega^{(\alpha+1) / \alpha} \geq \frac{-\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}}, \quad A, B>0 \tag{2.29}
\end{equation*}
$$

where

$$
A=\pi(s) / r(s), \quad B=1 / r(s) \quad \text { and } \quad \Omega=-\phi(l),
$$

thus (2.28) becomes

$$
\begin{align*}
\pi(l) \phi(l)-\pi\left(l_{1}\right) \phi\left(l_{1}\right) \leq & -k \int_{l_{1}}^{l} \pi(s) \sum_{i=1}^{m} h_{i}(s)\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \frac{\pi\left(\sigma_{i}(s)\right)}{\pi(s)} \mathrm{d} s \\
& +\frac{1}{4} \int_{l_{1}}^{l} \frac{1}{r(s) \pi(s)} \mathrm{d} s . \tag{2.30}
\end{align*}
$$

From (2.10) and (2.26), we have

$$
\begin{equation*}
1 \geq-\frac{r(l) v^{\prime}(l) \pi(l)}{v(l)}=-\phi(l) \pi(l) \tag{2.31}
\end{equation*}
$$

In view of (2.30) and (2.31), we obtain

$$
1+\pi\left(l_{1}\right) \phi\left(l_{1}\right) \geq \int_{l_{1}}^{l}\left(k \pi(s) \sum_{i=1}^{m} h_{i}(s)\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \frac{\pi\left(\sigma_{i}(s)\right)}{\pi(s)}-\frac{1}{4 r(s) \pi(s)}\right) \mathrm{d} s,
$$

which is a contradiction. The proof of the theorem is complete.

Now, we will present examples to illustrate our main results.

Example 2.1 Consider the second-order neutral differential equation

$$
\begin{equation*}
\left(l^{2}\left(x^{3}(l)+\frac{1}{3} x\left(\frac{l}{2}\right)\right)^{\prime}\right)^{\prime}+81 l^{2} x^{3}(2 l)+l^{4} x^{3}(3 l)=0 \tag{2.32}
\end{equation*}
$$

where $k=1$ and $m=2$. Now, we note that $\alpha=3>1, r(l)=l^{2}, q(l)=1 / 3, \lambda(l)=l / 2, h_{1}(l)=$ $81 l^{2}, h_{2}(l)=l^{4}, \sigma_{1}(l)=2 l$ and $\sigma_{2}(l)=3 l$. Then it is easy to see that

$$
\begin{aligned}
& \pi(l)=\int_{l}^{\infty} \frac{1}{r(s)} \mathrm{d} s=\int_{l}^{\infty} s^{-2} \mathrm{~d} s=\frac{1}{l}, \\
& 0<1-q(l) \frac{\pi(\lambda(l))}{\pi(l)}=1-\left(\frac{1}{3}\right) \frac{1 /(l / 2)}{1 / l}=\frac{1}{3}<1,
\end{aligned}
$$

and

$$
\int_{l_{0}}^{\infty} \frac{1}{r(u)} \int_{l_{0}}^{u} \sum_{i=1}^{m} h_{i}(s) \pi\left(\sigma_{i}(s)\right) \mathrm{d} s \mathrm{~d} u=\int_{l_{0}}^{\infty} \frac{1}{u^{2}} \int_{l_{0}}^{u}\left(81 s^{2} \frac{1}{2 s}+s^{4} \frac{1}{3 s}\right) \mathrm{d} s \mathrm{~d} u=\infty .
$$

By using Theorem (2.1), we see that (2.32) is oscillatory.
Example 2.2 Consider the second-order neutral differential equation

$$
\begin{equation*}
\left(l^{3}\left(x^{5}(l)+q_{0} x\left(k_{1} l\right)\right)^{\prime}\right)^{\prime}+h_{0} l x^{5}\left(k_{2} l\right)+h_{1} l x^{5}\left(k_{3} l\right)=0 \tag{2.33}
\end{equation*}
$$

where $k=1, m=2, k_{1} \in(0,1], k_{2} \geq 1$ and $k_{3} \geq 1$. Now, we note that $q(l)=q_{0}, q_{0} \in\left(0, k_{1}^{2}\right)$, $\alpha=5>1, r(l)=l^{3}, \lambda(l)=k_{1} l, h_{1}(l)=h_{0} l, h_{0}>0, h_{2}(l)=h_{*} l, h_{*}>0, \sigma_{1}(l)=k_{2} l$ and $\sigma_{2}(l)=k_{3} l$. Then it is easy to see that

$$
\begin{aligned}
& \pi(l)=\int_{l}^{\infty} \frac{1}{r(s)} \mathrm{d} s=\int_{l}^{\infty} s^{-3} \mathrm{~d} s=\frac{1}{2 l^{2}}, \\
& \int_{l_{0}}^{\infty} \sum_{i=1}^{m} h_{i}(s) \mathrm{d} s=\int_{l_{0}}^{\infty}\left(h_{0} s+h_{*} s\right) \mathrm{d} s=\infty, \\
& 0<1-q(l) \frac{\pi(\lambda(l))}{\pi(l)}=1-q_{0} \frac{1}{k_{1}^{2}}<1,
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{l_{0}}^{\infty}\left(k \pi(s) \sum_{i=1}^{m} h_{i}(s)\left(1-q\left(\sigma_{i}(s)\right) \frac{\pi\left(\lambda\left(\sigma_{i}(s)\right)\right)}{\pi\left(\sigma_{i}(s)\right)}\right) \frac{\pi\left(\sigma_{i}(s)\right)}{\pi(s)}-\frac{1}{4 r(s) \pi(s)}\right) \mathrm{d} s \\
& \quad=\int_{l_{0}}^{\infty}\left(\frac{1}{2 s^{2}}\left(h_{0} s\left(1-q_{0} \frac{1}{k_{1}^{2}}\right) \frac{1}{k_{2}^{2}}+h_{*} s\left(1-q_{0} \frac{1}{k_{1}^{2}}\right) \frac{1}{k_{3}^{2}}\right)-\frac{1}{4 s^{3}\left(1 /\left(2 s^{2}\right)\right)}\right) \mathrm{d} s,
\end{aligned}
$$

therefore, we find that the condition (2.25) is satisfied if

$$
\begin{equation*}
h_{0} \frac{1}{k_{2}^{2}}+h_{*} \frac{1}{k_{3}^{2}}>\frac{1}{\left(1-q_{0} \frac{1}{k_{1}^{2}}\right)} . \tag{2.34}
\end{equation*}
$$

By using Theorem (2.3), we see that (2.33) is oscillatory if (2.34) holds.

## 3 Conclusions

In this work, we have obtained some new oscillation criteria for (1.1) in the case where $v(l):=x^{\alpha}(l)+q(l) x(\lambda(l))$. These results ensure that all solutions of the equation studied are oscillatory. The results obtained here extend and complement some known results in the literature. See for example [21-23]. It will be of interest to investigate the higher-order equations of the form

$$
\left(r(l)\left(x^{\alpha}(l)+q(l) x(\lambda(l))\right)^{(n-1)}\right)^{\prime}+\sum_{i=1}^{m} h_{i}(l) g\left(x\left(\sigma_{i}(l)\right)\right)=0
$$

where $g(x) \geq k x^{\alpha}$.

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