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An improvement of convergence rate in the local limit theorem for integral-valued random variables

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Abstract

Let $X_1, X_2, \dots, X_n$ be independent integral-valued random variables, and let $S_n = \sum_{j=1}^{n} X_j$. One of the interesting probabilities is the probability at a particular point, i.e., the density of S_n. The theorem that gives the estimation of this probability is called the local limit theorem. This theorem can be useful in finance, biology, etc. Petrov (Sums of Independent Random Variables, 1975) gave the rate $O(\frac{1}{n})$ of the local limit theorem with finite third moment condition. Most of the bounds of convergence are usually defined with the symbol O. Giuliano Antonini and Weber (Bernoulli 23(4B):3268–3310, 2017) were the first who gave the explicit constant C of error bound $\frac{C}{\sqrt{n}}$. In this paper, we improve the convergence rate and constants of error bounds in local limit theorem for S_n. Our constants are less complicated than before, and thus easy to use.

MSC: 60F05

Keywords: Local limit theorem; Normal density function; Lattice random variable; Convergence rate

1 Introduction

Let $X_1, X_2, \dots, X_n$ be independent integral-valued random variables with means μ_j and variances σ_j^2 for $j = 1, 2, \dots, n$. Let $S_n = \sum_{j=1}^{n} X_j$, $\mu = \sum_{j=1}^{n} \mu_j$, and $\sigma^2 = \sum_{j=1}^{n} \sigma_j^2$. One of the interesting probabilities is the probability at a particular point, i.e., $P(S_n = k)$, where $k = 1, 2, \dots$. There are two density functions, i.e., discretized normal and normal, to approximate this probability. The discretized normal random variable ($\widetilde{Z}_{\mu,\sigma^2}$) has the probability mass function

$$P(\widetilde{Z}_{\mu,\sigma^2} = k) = P\left(\frac{k-\mu-\frac{1}{2}}{\sigma} < Z_{\mu,\sigma^2} \le \frac{k-\mu+\frac{1}{2}}{\sigma}\right)$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\frac{k-\mu-\frac{1}{2}}{\sigma}}^{\frac{k-\mu+\frac{1}{2}}{\sigma}} e^{-\frac{x^2}{2}}\, dx,$$

where Z_{μ,σ^2} is a normal distribution with mean μ and variance σ^2.

To approximate $P(S_n = k)$ by using the discretized normal density function, we can apply the Berry–Esseen theorem. Berry [3] and Esseen [4] were the first two mathematicians who gave the bound between $P(S_n \leq k)$ and the normal distribution. Here is their result.

If $E|X_j|^3 \leq \infty$ for $j = 1, 2, \dots, n$, then

$$\sup_{k\in\mathbb{R}} \left| P\left(\frac{S_n - \mu}{\sigma} \leq k\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{k} e^{-\frac{x^2}{2}}\, dx \right| \leq \frac{C_0}{\sigma^3} \sum_{j=1}^{n} E|X_j - \mu_j|^3, \tag{1}$$

where C_0 is an absolute constant.

We can apply (1) to show that

$$\left| P(S_n = k) - \frac{1}{\sqrt{2\pi}} \int_{\frac{k-\mu-\frac{1}{2}}{\sigma}}^{\frac{k-\mu+\frac{1}{2}}{\sigma}} e^{-\frac{x^2}{2}}\, dx \right| \leq \frac{2C_0}{\sigma^3} \sum_{j=1}^{n} E|X_j - \mu_j|^3. \tag{2}$$

The constant C_0 in (2) was found and improved by many mathematicians (see, [3–10] for examples). The best C_0 obtained by Shevtsova [8] in 2013 was 0.5583 for the case of non-identically and 0.469 for the case of identically.

The local limit theorem describes how the probability mass function of a sum of independent discrete random variables approaches the normal density.

Let

$$\epsilon_n(k) = \left| P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right|.$$

De Moivre and Laplace (see [11]) established the local limit theoremDe Moivre and Laplace (see [11]) established the local limit theorem for the binomial case in 1754. For sums of independent random variables, we can prove the local limit theorem by using the Berry–Esseen theorem and get the rate convergence $O(\frac{1}{\sqrt{n}})$ (see [2]).

In 1971, Ibragimov and Linnik improved the rate of convergence from $O(\frac{1}{\sqrt{n}})$ to $O(\frac{1}{n^{\frac{1}{2}+\alpha}})$, $0 < \alpha < \frac{1}{2}$, in the case of X_js being identical and square integrable random variables.

For the non-identical case, Petrov (1975, [1]) showed that if

1 $\sigma^2 \to \infty$ as $n \to \infty$,
2 $\sum_{j=1}^{n} E|X_j - \mu_j|^3 = O(\sigma^2)$,
3 $P(X_j = 0) \geq P(X_j = m)$ for all j and m and
4 $\gcd\{m : \frac{1}{\log n} \sum_{j=1}^{n} P(X_j = 0)P(X_j = m) \to \infty \text{ as } n \to \infty\} = 1$,

then

$$\epsilon_n(k) \leq \frac{C_1}{\sigma^2}.$$

Furthermore, Petrov ([1], see also [2]) improved the rate of convergence from $O(\frac{1}{\sigma^2})$ to $O(\frac{1}{n\sqrt{n}})$ in the case of a symmetric binomial.

In the previous studies, no one gave the explicit constants of error bounds. Most of the theorems were usually presented in the form of O. Therefore, finding the constants has been interesting. In 2018, Zolotukhin, Nagaev, and Chebotarev [12] gave the convergence with a constant of error bound in the case that S_n is a binomial. They showed that

$$\epsilon_n(k) \leq \min\left\{ \frac{1}{\sigma\sqrt{2e}}, \frac{0.516}{\sigma^2} \right\}. \tag{3}$$

After that Siripraparat and Neammanee [13] relaxed the identically condition and obtained the convergence in the case of Poisson binomial in 2020. Their result is

$$\epsilon_n(k) \le \frac{0.1194}{\sigma^2(1-\frac{3}{4\sigma})^3} + \frac{0.0749}{\sigma^3} + \frac{0.2107}{\sigma^3(1-\frac{3}{4\sigma})^6} + \left(\frac{0.4579}{\sqrt{\sigma}} + \frac{0.4725}{\sigma\sqrt{\sigma}}\right)e^{-\frac{3\sigma}{2}}. \tag{4}$$

Furthermore, in the case of $S_n = \mathrm{Bi}(\frac{1}{2})$ being a symmetric binomial, i.e., $P(X_j = 1) = \frac{1}{2} = 1 - P(X_j = 0)$, they showed that

$$\epsilon_n(k) \le \frac{0.5992}{n\sqrt{n}} + \frac{3.3984}{n^2(1-\frac{3}{2\sqrt{n}})^4} + \frac{337.8048}{n^3\sqrt{n}(1-\frac{3}{2\sqrt{n}})^8} + \left(\frac{0.6476}{n^{\frac{1}{4}}} + \frac{1.3365}{n^{\frac{3}{4}}}\right)e^{-\frac{3\sqrt{n}}{4}}. \tag{5}$$

In 2017, Giuliano Antonini and Weber [2] gave the rate of convergence $O(\frac{1}{\sigma})$ with a constant of error bound in the case of sums of independent lattice random variables. X is a lattice random variable when the value of X is in $L(a,b) = \{v_k\}$, where $v_k = a + bk$, $k \in \mathbb{Z}$, a and $b > 0$ are real numbers. They gave the following theorem.

Theorem 1.1 (See [2]) *Let $X_1, X_2, \ldots, X_n$ be independent square integrable random variables taking values in a lattice $L(a,b)$ and $S_n = \sum_{j=1}^n X_j$. Let $\alpha_X = \sum_{k\in\mathbb{Z}} \min\{P(X = v_k), P(X = v_{k+1})\}$ and V_js, L_js, ϵ_js be such that*

$$V_j + \epsilon_j b L_j \stackrel{\mathcal{D}}{=} X_j \quad \text{for all } j = 1, 2, \ldots, n,$$

where $P(L_j = 0) = P(L_j = 1) = \frac{1}{2}$, $P(\epsilon_j = 1) = 1 - P(\epsilon_j = 0) = q_j$, where $0 < q_j \le \alpha_{X_j}$ for all $j = 1, 2, \ldots, n$, and (V_j, ϵ_j) and L_j are independent for each $j = 1, 2, \ldots, n$.

Assume that

1. $\frac{\log\lambda_n}{\lambda_n} \le \frac{1}{14}$, *where* $\lambda_n = \sum_{j=1}^n q_j$
2. $\frac{(k-ES_n)^2}{\mathrm{Var}(S_n)} \le (\frac{\lambda_n}{14\log\lambda_n})^{\frac{1}{2}}$ *for all* $k \in L(na, b)$.

Then

$$\left|P(S_n = k) - \frac{b}{\sqrt{2\pi\,\mathrm{Var}(S_n)}}e^{-\frac{(k-ES_n)^2}{2\,\mathrm{Var}\,S_n}}\right| \le C_2\left[b\left(\frac{\log\lambda_n}{\mathrm{Var}(S_n)\lambda_n}\right)^{\frac{1}{2}} + \frac{\delta_n + \lambda_n^{-1}}{\sqrt{\lambda_n}}\right],$$

where

$$C_2 = 2^{\frac{7}{2}}\max\left\{\frac{8}{\sqrt{2\pi}}, C_3\right\},$$

$$C_3 \text{ is the constant such that } \sup_z\left|P\left(\mathrm{Bi}\left(\frac{1}{2}\right) = z\right) - \sqrt{\frac{2}{\pi n}}e^{-\frac{(2z-n)^2}{2n}}\right| \le \frac{C_3}{n\sqrt{n}},$$

$$\delta_n = \sup_{x\in\mathbb{R}}\left|P\left(\frac{S_n' - ES_n'}{\sqrt{\mathrm{Var}(S_n')}} < x\right) - P(Z_{0,1} < x)\right|, \quad \text{and}$$

$$S_n' = W_n + \frac{b}{2}B_n,\ W_n = \sum_{j=1}^n V_j \quad \text{and} \quad B_n = \sum_{j=1}^n \epsilon_j.$$

Note that if we choose the constant of error bound C_3 in (5), then C_2 is 36.1082 and the rate of Theorem 1.1 is $O(\frac{1}{\sigma})$. We can see that the bound of [2] depends on C_3 and is still complicated. In this work, we improve the rate of convergence of [2] to be $O(\frac{1}{\sigma^2})$ and also give the constant of error bound. Our constants are not complicated and can be applied easily. The results are shown in the following.

Theorem 1.2 *Let $X_1, X_2, \dots, X_n$ be independent integral-valued random variables and $\alpha_j = 2\sum_{l=-\infty}^{\infty} p_{jl}p_{j(l+1)}$, where $p_{jl} = P(X_j = l)$. If $\alpha_j > 0$ for all $j = 1, 2, \dots, n$, then*

$$\epsilon_n(k) \leq \frac{2.2075e^{-\frac{\tau^2\alpha}{\pi^2}}}{\tau\alpha} + \frac{1.7898}{\sigma^4}\sum_{j=1}^{n} E|X_j|^3,$$

where $\tau = \frac{1}{10\sqrt[3]{\sum_{j=1}^{n} E|X_j|^3}}$ and $\alpha = \sum_{j=1}^{n} \alpha_j$.

b is said to be maximal when there are no other numbers a' and $b' > b$ for which $P(X \in L(a', b')) = 1$.

Theorem 1.3 *Let $X_1, X_2, \dots, X_n$ be independent random variables in a maximal lattice $L(a, b)$ and*

$$\delta_n(k) = \left| P(S_n = na + kb) - \frac{b}{\sigma\sqrt{2\pi}} e^{-\frac{(b(na+kb)-(\mu-na))^2}{2\sigma^2}} \right|.$$

Then

$$\delta_n(k) \leq \frac{2.2075e^{-\frac{\tau^2\alpha}{\pi^2}}}{\tau\alpha} + \frac{1.7898b^4}{\sigma^4}\sum_{j=1}^{n} E|X_j|^3,$$

where $\alpha_j = 2\sum_{l=-\infty}^{\infty} p_{jl}p_{j(l+1)}$, $p_{jl} = P(X_j = a + bl)$, and $\alpha = \sum_{j=1}^{n} \alpha_j$.

Theorem 1.4 *If $X_1, X_2, \dots, X_n$ in Theorem 1.3 are independent identically distributed (i.i.d.), then*

$$\delta_n(k) \leq \frac{2.2075e^{-\frac{\tau^2\alpha}{\pi^2}}}{\tau\alpha} + \frac{1.7898b^4}{n\sigma_1^4} E|X_1|^3,$$

where $\tau = \frac{1}{10\sqrt[3]{nE|X_1|^3}}$ and $\alpha = 2n\sum_{l=-\infty}^{\infty} p_l p_{l+1}$, $p_l = P(X_1 = a + bl)$.

Observe that the constant in Theorems 1.2–1.4 is easier than the constant in Theorem 1.1.

We organize this paper as follows. In Sect. 2, we give the exponential bounds of a characteristic function which will be used to prove the main theorems in Sect. 3. After that we give some examples in Sect. 4.

2 Exponential bounds of a characteristic function

In this section, we let X be an integral-valued random variable with characteristic function ψ and $\theta(t)$ = argument of $\psi(t)$. Then

$$\psi(t) = \sum_{j=-\infty}^{\infty} p_j e^{ijt}, \quad \text{where } p_j = P(X = j) \text{ for } t \in \mathbb{R}$$

and

$$\theta(t) = \arctan\left(\frac{\sum_{j=-\infty}^{\infty} p_j \sin(jt)}{\sum_{j=-\infty}^{\infty} p_j \cos(jt)}\right). \tag{6}$$

Characteristic functions are important in probability theory and statistics, especially in local limit theorems, stability problems, etc. In the study of local limit theorems, it is required to estimate the bounds for modulus $|\psi(t)|$ of a characteristic function ψ. The various bounds for $|\psi(t)|$ play a key role in the investigation of the rate of convergence in the local limit theorems. Previous studies have shown the bounds for $|\psi(t)|$ in the case of continuous and bounded random variable in a variety of versions (see [14–18] for example). In addition, the bounds for $|\psi(t)|$ of a lattice random variable have been shown in a number of research works (see [18–21] for example). Furthermore, there is the exponential bound for $|\psi(t)|$ of a Poisson binomial distribution as shown in Neammanee [22]. In this section, we use the idea of Neammanee [22] to obtain the exponential bound for $|\psi(t)|$ of an integral-valued random variable. The following lemmas are our results.

Lemma 2.1 *Let $t \in [0,\pi)$ and $\alpha = 2\sum_{j=-\infty}^{\infty} p_j p_{j+1}$. Then $|\psi(t)| \le e^{-\frac{1}{\pi^2}\alpha t^2}$.*

Proof Let $t \in [0,\pi)$. If $|\psi(t)| = 0$, then Lemma 2.1 holds. Assume that $|\psi(t)| > 0$.

Note that

$$\begin{aligned}
\left|\psi(t)\right|^2 &= \psi(t)\overline{\psi(t)} \\
&= \sum_{j=-\infty}^{\infty} p_j e^{ijt} \sum_{l=-\infty}^{\infty} p_l e^{-ilt} \\
&= \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} e^{it(j-l)} p_j p_l \\
&= \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \cos\bigl((j-l)t\bigr) p_j p_l + i\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \sin\bigl((j-l)t\bigr) p_j p_l.
\end{aligned}$$

Since $|\psi(t)|^2$ is real, we get

$$\begin{aligned}
\left|\psi(t)\right|^2 &= \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \cos\bigl((j-l)t\bigr) p_j p_l \\
&= \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \left(1 - 2\sin^2\left((j-l)\frac{t}{2}\right)\right) p_j p_l \\
&= \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} p_j p_l - 2\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \sin^2\left((j-l)\frac{t}{2}\right) p_j p_l \\
&= 1 - 2\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \sin^2\left((j-l)\frac{t}{2}\right) p_j p_l.
\end{aligned} \tag{7}$$

From this fact and the fact that $|\psi(t)| > 0$, we have

$$0 \leq 2\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2\left((j-l)\frac{t}{2}\right)p_jp_l < 1. \tag{8}$$

By (7) and (8), we get

$$\begin{aligned}
\ln\left|\psi(t)\right| &= \frac{1}{2}\ln\left(1-2\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2\left((j-l)\frac{t}{2}\right)p_jp_l\right) \\
&= -\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\left[2\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2\left((j-l)\frac{t}{2}\right)p_jp_l\right]^k \qquad (9) \\
&\leq -\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2\left((j-l)\frac{t}{2}\right)p_jp_l \\
&= -\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2\left(|j-l|\frac{t}{2}\right)p_jp_l \\
&\leq -\sum_{j=-\infty}^{\infty}\sum_{\substack{l=-\infty \\ |j-l|\leq 1}}^{\infty}\frac{(j-l)^2t^2}{\pi^2}p_jp_l \\
&= -\frac{1}{\pi^2}\alpha t^2,
\end{aligned}$$

where we use the fact that $\sin(\frac{t}{2}) \geq \frac{t}{\pi}$ on $[0,\pi)$ in the last inequality.

Hence, $|\psi(t)| \leq e^{-\frac{1}{\pi^2}\alpha t^2}$. □

Lemma 2.2 *For* $t \in [0,\pi]$,

$$\left|\psi(t)\right| \leq e^{-\frac{1}{2}\sigma^2(X)t^2+\frac{2}{3}E|X|^3t^3}.$$

Proof The lemma holds if $|\psi(t)| = 0$. Assume that $|\psi(t)| > 0$.

Note that

$$\begin{aligned}
\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(j-l)^2p_jp_l &= \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sum_{m=0}^{2}\binom{2}{m}j^m(-l)^{2-m}p_jp_l \\
&= \sum_{m=0}^{2}(-1)^{2-m}\binom{2}{m}\sum_{j=-\infty}^{\infty}j^mp_j\sum_{l=-\infty}^{\infty}l^{2-m}p_l \\
&= \sum_{m=0}^{2}(-1)^{2-m}\binom{2}{m}EX^mEX^{2-m} \\
&= EX^2-2(EX)^2+EX^2 \\
&= 2\sigma^2(X) \qquad (10)
\end{aligned}$$

and

$$\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\big(|j|+|l|\big)^3 p_j p_l \le 4\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\big(|j|^3+|l|^3\big)p_j p_l = 8E|X|^3, \tag{11}$$

where we use the fact that $(a+b)^k \le 2^{k-1}(a^k+b^k)$, $a,b\ge 0$, and $k\in\mathbb{N}$ in the first inequality.

From the fact that

$$\cos(at) = 1-\frac{1}{2}a^2t^2+\frac{1}{6}a^3t^3\sin(t_1) \quad \text{for some } t_1$$

and (9), (10), (11), we get

$$\begin{aligned}
&\ln\big|\psi(t)\big| \\
&\le -\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2\left((j-l)\frac{t}{2}\right)p_j p_l \\
&= -\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left[\frac{1}{2}-\frac{1}{2}\cos\big((j-l)t\big)\right]p_j p_l \\
&= -\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left[\frac{1}{2}-\frac{1}{2}\left(1-\frac{1}{2}(j-l)^2t^2+\frac{1}{6}(j-l)^3t^3\sin\big((j-l)t_1\big)\right)\right]p_j p_l \\
&= -\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left[\frac{1}{4}(j-l)^2t^2-\frac{1}{12}(j-l)^3t^3\sin\big((j-l)t_1\big)\right]p_j p_l \\
&\le -\frac{t^2}{4}\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(j-l)^2p_j p_l+\frac{t^3}{12}\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}|j-l|^3p_j p_l \\
&\le -\frac{1}{2}\sigma^2(X)t^2+\frac{t^3}{12}\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\big(|j|+|l|\big)^3p_j p_l \\
&\le -\frac{1}{2}\sigma^2(X)t^2+\frac{2}{3}E|X|^3t^3.
\end{aligned}$$

Hence, $|\psi(t)| \le e^{-\frac{1}{2}\sigma^2(X)t^2+\frac{2}{3}E|X|^3t^3}$. □

Lemma 2.3 *Let* $\tau_1 = \frac{1}{10\sqrt[3]{E|X|^3}}$. *Then*

$$\big|\psi(t)\big| \ge e^{-\frac{1}{2}\sigma^2(X)t^2-\frac{2}{3}E|X|^3t^3} \quad \textit{for } t\in[0,\tau_1].$$

Proof Since $|\sin(\theta)|\le|\theta|$ and (10),

$$2\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2\left((j-l)\frac{t}{2}\right)p_j p_l \le \frac{t^2}{2}\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(j-l)^2p_j p_l = t^2\sigma^2(X). \tag{12}$$

Note that

$$\sigma^2(X) \le E\big(X^2\big) \le \big(E|X|^3\big)^{\frac{2}{3}}. \tag{13}$$

From (12), (13), and the fact that $t^2 \le \frac{1}{100(E|X|^3)^{\frac{2}{3}}}$,

$$0 \le 2\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2\left((j-l)\frac{t}{2}\right)p_jp_l \le \frac{1}{100}.$$

Therefore,

$$1-2\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2\left((j-l)\frac{t}{2}\right)p_jp_l \ge \frac{99}{100} \tag{14}$$

and

$$-\frac{1}{1-2\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2((j-l)\frac{t}{2})p_jp_l} \ge -\frac{100}{99}. \tag{15}$$

By (9), (12), (13), and (15), we get

$$\begin{aligned}
\ln\left|\psi(t)\right| &= -\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2\left((j-l)\frac{t}{2}\right)p_jp_l \\
&\quad -\frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{k}\left[2\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2\left((j-l)\frac{t}{2}\right)p_jp_l\right]^k \\
&\ge -\frac{1}{2}\sigma^2(X)t^2 - \frac{1}{4}\frac{[2\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2((j-l)\frac{t}{2})p_jp_l]^2}{1-2\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2((j-l)\frac{t}{2})p_jp_l} \\
&\ge -\frac{1}{2}\sigma^2(X)t^2 - \frac{1}{4}\left(\frac{100}{99}\right)\left[2\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2\left((j-l)\frac{t}{2}\right)p_jp_l\right]^2 \\
&\ge -\frac{1}{2}\sigma^2(X)t^2 - \frac{25}{99}\sigma^4(X)t^4 \\
&\ge -\frac{1}{2}\sigma^2(X)t^2 - \frac{25}{99}\left(E|X|^3\right)^{\frac{4}{3}}t^4 \\
&\ge -\frac{1}{2}\sigma^2(X)t^2 - \frac{25}{99}\left(E|X|^3\right)^{\frac{4}{3}}\frac{1}{10\sqrt[3]{E|X|^3}}t^3 \\
&\ge -\frac{1}{2}\sigma^2(X)t^2 - \frac{2}{3}E|X|^3t^3.
\end{aligned}$$

Hence, $|\psi(t)| \ge e^{-\frac{1}{2}\sigma^2(X)t^2-\frac{2}{3}E|X|^3t^3}$ for $t \in [0,\tau_1]$. □

Lemma 2.4

1 $\theta^{(1)}(0) = EX$.

2 $\theta^{(2)}(0) = 0$.

3 $|\theta^{(3)}(t)| \le 4.2874E|X|^3$ *for* $t \in [0,\tau_1]$.

Proof 1. By (6), we get

$$\theta^{(1)}(0) = \frac{\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}jp_jp_l}{\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}p_jp_l} = EX.$$

2. Let $A(t) = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} j\cos((j-l)t)p_j p_l$ and $B(t) = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \cos((j-l)t)p_j p_l$. Observe that

$$\theta^{(1)}(t) = \frac{A(t)}{B(t)} \quad \text{and} \quad \theta^{(2)}(t) = \frac{B(t)A'(t) - A(t)B'(t)}{(B(t))^2}, \tag{16}$$

where

$$A'(t) = -\sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} j(j-l)\sin\bigl((j-l)t\bigr)p_j p_l \quad \text{and}$$

$$B'(t) = -\sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (j-l)\sin\bigl((j-l)t\bigr)p_j p_l.$$

Since $A'(0) = 0$ and $B'(0) = 0$, $\theta^{(2)}(0) = 0$.

3. Note that

$$\bigl|A(t)\bigr| = \left|\sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} j\cos\bigl((j-l)t\bigr)p_j p_l\right| \le E|X|, \tag{17}$$

similarly to (10), we get

$$\sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} j(j-l)^2 p_j p_l = EX^3 - EX^2 EX.$$

Therefore,

$$\left|\sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} j(j-l)^2 p_j p_l\right| \le 2E|X|^3.$$

Hence,

$$\begin{aligned}
\bigl|A'(t)\bigr| &= \left|-\sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} j(j-l)\sin\bigl((j-l)t\bigr)p_j p_l\right| \\
&\le \left|\sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} j(j-l)^2 t p_j p_l\right| \\
&\le \tau_1 \left|\sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} j(j-l)^2 p_j p_l\right| \\
&\le 2\tau_1 E|X|^3 \\
&\le \frac{2}{10\sqrt[3]{E|X|^3}} E|X|^3 \\
&= \frac{1}{5}\bigl(E|X|^3\bigr)^{\frac{2}{3}}
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
\left|A''(t)\right| &= \left|-\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} j(j-l)^2\cos\bigl((j-l)t\bigr)p_jp_l\right| \\
&\le \left|\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} j(j-l)^2 p_jp_l\right| \\
&\le 2E|X|^3. \qquad (19)
\end{aligned}$$

By (14), we get

$$\begin{aligned}
B(t) &= \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\cos\bigl((j-l)t\bigr)p_jp_l \\
&= \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left(1-2\sin^2\left(\frac{(j-l)t}{2}\right)\right)p_jp_l \\
&= \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} p_jp_l - 2\sum_i\sum_j \sin^2\left(\frac{(j-l)t}{2}\right)p_jp_l \\
&= 1-2\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sin^2\left(\frac{(j-l)t}{2}\right)p_jp_l \\
&\ge \frac{99}{100}.
\end{aligned}$$

From this fact and $B(t)\le 1$,

$$\frac{99}{100}\le B(t)\le 1. \qquad (20)$$

By (10) and (13), we obtain

$$\begin{aligned}
\left|B'(t)\right| &= \left|-\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(j-l)\sin\bigl((j-l)t\bigr)p_jp_l\right| \\
&\le \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(j-l)^2 t p_jp_l \\
&= 2\tau_1\sigma^2(X) \\
&\le \frac{2}{10\sqrt[3]{E|X|^3}}\bigl(E|X|^3\bigr)^{\frac{2}{3}} \\
&= \frac{1}{5}\bigl(E|X|^3\bigr)^{\frac{1}{3}} \qquad (21)
\end{aligned}$$

and

$$\left|B''(t)\right| = \left|-\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(j-l)^2\cos\bigl((j-l)t\bigr)p_jp_l\right|$$

$$\leq \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(j-l)^2 p_j p_l$$
$$= 2\sigma^2(X)$$
$$\leq 2\big(E|X|^3\big)^{\frac{2}{3}}. \tag{22}$$

By (16), we obtain

$$\theta^{(3)}(t) = \frac{(B(t))^2 A''(t) - B(t)A(t)B''(t) - 2B'(t)B(t)A'(t) - 2A(t)(B'(t))^2}{(B(t))^3}.$$

From this fact and (17)–(22), we get $|\theta^{(3)}(t)| \leq 4.2874E|X|^3$. □

3 Proof of the main results

Let $X_1, X_2, \dots, X_n$ be independent integral-valued random variables. Let $S_n := \sum_{i=1}^n X_i$, $\mu := ES_n$ and $\sigma^2 := \operatorname{Var} S_n$. Let $\psi_1, \psi_2, \dots, \psi_n$ and ψ be the characteristic functions of $X_1, X_2, \dots, X_n$ and S_n, respectively. Then, for $j = 1, 2, \dots, n$,

$$\psi_j(t) = \sum_{l=-\infty}^{\infty} p_{jl} e^{ilt} = \sum_{l=-\infty}^{\infty} p_{jl}\cos(lt) + i\sum_{l=-\infty}^{\infty} p_{jl}\sin(lt)$$

and

$$\psi(t) = \prod_{j=1}^{n} \psi_j(t).$$

Note that $\psi_j(t) = |\psi_j(t)|e^{i\theta_j(t)}$,

where $\theta_j(t) :=$ argument of $\psi_j(t) = \arctan\big(\frac{\sum_{l=-\infty}^{\infty} p_{jl}\sin(lt)}{\sum_{l=-\infty}^{\infty} p_{jl}\cos(lt)}\big)$.

Hence, $\psi(t) = \rho(t)e^{i\theta(t)}$, where $\theta(t) = \sum_{j=1}^n \theta_j(t) (\operatorname{mod} 2\pi)$ and $\rho(t) = \prod_{j=1}^n |\psi_j(t)|$.

From Siripraparat and Neammanee [13], we know that

$$P(S_n = k) = \frac{1}{\pi}\int_0^{\pi} \rho(t)\cos\big((k-\mu)t - \alpha(t)\big)\,dt, \tag{23}$$

where $\alpha(t) = \theta(t) - \mu t$.

To prove our main theorems, we give the bound of $\rho(t)$ and $\cos((k-\mu)t - \alpha(t))$ in Lemma 3.1 and Lemma 3.2, respectively.

Lemma 3.1 *Let* $\tau = \min\big(\frac{1}{10\sqrt[3]{\sum_{j=1}^n E|X_j|^3}}, \pi\big)$. *Then*

$$\big|\rho(t) - e^{-\frac{1}{2}\sigma^2 t^2}\big| \leq 0.6672\sum_{j=1}^{n} E|X_j|^3 t^3 e^{-\frac{1}{2}\sigma^2 t^2} \quad \textit{for } t \in [0, \tau).$$

Proof By Lemma 2.2 and Lemma 2.3, we get

$$e^{-\frac{1}{2}\sigma^2(X_j)t^2 - \frac{2}{3}E|X_j|^3 t^3} \leq \big|\psi_j(t)\big| \leq e^{-\frac{1}{2}\sigma^2(X_j)t^2 + \frac{2}{3}E|X_j|^3 t^3}. \tag{24}$$

By (24), we obtain

$$e^{-\frac{1}{2}\sigma^2 t^2-\frac{2}{3}\sum_{j=1}^n E|X_j|^3t^3} \le \rho(t) \le e^{-\frac{1}{2}\sigma^2 t^2+\frac{2}{3}\sum_{j=1}^n E|X_j|^3t^3}.$$

Thus,

$$\begin{aligned}\bigl(e^{-\frac{2}{3}\sum_{j=1}^n E|X_j|^3t^3}-1\bigr)e^{-\frac{1}{2}\sigma^2t^2} &\le \rho(t)-e^{-\frac{1}{2}\sigma^2t^2}\\ &\le \bigl(e^{\frac{2}{3}\sum_{j=1}^n E|X_j|^3t^3}-1\bigr)e^{-\frac{1}{2}\sigma^2t^2}.\end{aligned}$$

Hence,

$$\begin{aligned}-\frac{2}{3}\sum_{j=1}^n E|X_j|^3t^3e^{-\frac{1}{2}\sigma^2t^2} &\le \rho(t)-e^{-\frac{1}{2}\sigma^2t^2}\\ &\le \frac{2}{3}\sum_{j=1}^n E|X_j|^3t^3e^{\frac{2}{3}\sum_{j=1}^n E|X_j|^3t^3}e^{-\frac{1}{2}\sigma^2t^2},\end{aligned} \tag{25}$$

where we have used the fact

$$e^x-1\le xe^x \quad\text{and}\quad e^{-x}-1>-x \quad \text{for } x>0.$$

Since $t^3 \le \frac{1}{1000\sum_{j=1}^n E|X_j|^3}$ and (25), $|\rho(t)-e^{-\frac{1}{2}\sigma^2t^2}| \le 0.6672\sum_{j=1}^n E|X_j|^3t^3e^{-\frac{1}{2}\sigma^2t^2}$. □

Lemma 3.2 *For $t\in[0,\tau]$, we have* $\cos((k-\mu)t-\alpha(t)) = \cos((k-\mu)t)+\triangle$, *where* $|\triangle| \le 0.7152\sum_{j=1}^n E|X_j|^3t^3$.

Proof Using Taylor's expansion, we have

$$\cos\bigl(\alpha(t)\bigr) = 1-\frac{1}{2}\cos(t_2)\bigl(\alpha(t)\bigr)^2 \quad \text{for some } t_2, \tag{26}$$

$$\sin\bigl(\alpha(t)\bigr) = \alpha(t)-\frac{1}{2}\sin(t_3)\bigl(\alpha(t)\bigr)^2 \quad \text{for some } t_3, \quad\text{and} \tag{27}$$

$$\theta_j(t) = \theta_j^{(1)}(0)t+\frac{1}{2}\theta_j^{(2)}(0)t^2+\frac{1}{6}\theta_j^{(3)}(t_4)t^3 \quad \text{for some } t_4. \tag{28}$$

By Lemma 2.4, (28) and the fact that $\tau\le\tau_1$, we get

$$\begin{aligned}\bigl|\alpha(t)\bigr| &\le \frac{1}{6}\sum_{j=1}^n 2.1437\bigl(E|X_j|\sigma^2(X_j)+E|X_j|^3\bigr)t^3\\ &\le 0.7146\sum_{j=1}^n E|X_j|^3t^3.\end{aligned} \tag{29}$$

By (26) and (27), we obtain

$$\begin{aligned}&\cos\bigl((k-\mu)t-\alpha(t)\bigr)\\ &\quad = \cos\bigl((k-\mu)t\bigr)\cos\bigl(\alpha(t)\bigr)+\sin\bigl((k-\mu)t\bigr)\sin\bigl(\alpha(t)\bigr)\end{aligned}$$

$$= \cos\big((k-\mu)t\big)\left[1-\frac{1}{2}\cos(t_2)\alpha^2(t)\right] + \sin\big((k-\mu)t\big)\left[\alpha(t)-\frac{1}{2}\sin(t_3)\alpha^2(t)\right]$$
$$= \cos\big((k-\mu)t\big) + \triangle,$$

where

$$|\triangle| \le \big|\alpha(t)\big| + \alpha^2(t). \tag{30}$$

By (29) and $t^3 \le \frac{1}{1000\sum_{j=1}^{n} E|X_j|^3}$, we obtain $|\alpha(t)| \le \frac{0.7146}{1000}$.

From this fact, (29) and (30) imply that $|\triangle| \le 0.7152\sum_{j=1}^{n} E|X_j|^3 t^3$. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2 Note that

$$\frac{1}{\pi}\int_0^{\pi} \rho(t)\cos\big((k-\mu)t-\alpha(t)\big)\,dt = \frac{1}{\pi}\int_0^{\tau} \rho(t)\cos\big((k-\mu)t-\alpha(t)\big)\,dt + \triangle_1, \tag{31}$$

where $\triangle_1 = \frac{1}{\pi}\int_{\tau}^{\pi} \rho(t)\cos((k-\mu)t-\alpha(t))\,dt$.

By Lemma 2.1, $|\triangle_1| \le \frac{1}{\pi}\int_{\tau}^{\pi} \rho(t)\,dt \le \frac{1}{\pi}\int_{\tau}^{\infty} e^{-\frac{1}{\pi^2}\alpha t^2}\,dt \le \frac{\pi}{2\tau\alpha}e^{-\frac{\tau^2\alpha}{\pi^2}}$.

From the fact that

$$\int_0^{\infty} t^3 e^{-\frac{1}{2}\sigma^2 t^2}\,dt = \frac{2}{\sigma^4} \tag{32}$$

and Lemma 3.1, we have

$$\frac{1}{\pi}\int_0^{\tau} \rho(t)\cos\big((k-\mu)t-\alpha(t)\big)\,dt = \frac{1}{\pi}\int_0^{\tau} e^{-\frac{1}{2}\sigma^2 t^2}\cos\big((k-\mu)t-\alpha(t)\big)\,dt + \triangle_2, \tag{33}$$

where

$$\begin{aligned}|\triangle_2| &\le 0.6672\sum_{j=1}^{n} E|X_j|^3\int_0^{\tau} t^3 e^{-\frac{1}{2}\sigma^2 t^2}\,dt \\ &\le 0.6672\sum_{j=1}^{n} E|X_j|^3\int_0^{\infty} t^3 e^{-\frac{1}{2}\sigma^2 t^2}\,dt \\ &= \frac{1.3344}{\sigma^4}\sum_{j=1}^{n} E|X_j|^3.\end{aligned} \tag{34}$$

From (32) and Lemma 3.2, we get

$$\frac{1}{\pi}\int_0^{\tau} e^{-\frac{1}{2}\sigma^2 t^2}\cos\big((k-\mu)t-\alpha(t)\big)\,dt = \frac{1}{\pi}\int_0^{\tau} e^{-\frac{1}{2}\sigma^2 t^2}\cos\big((k-\mu)t\big)\,dt + \triangle_3, \tag{35}$$

where

$$|\triangle_3| \le \frac{0.7152}{\pi}\sum_{j=1}^{n} E|X_j|^3\int_0^{\infty} t^3 e^{-\frac{1}{2}\sigma^2 t^2}\,dt = \frac{0.4554}{\sigma^4}\sum_{j=1}^{n} E|X_j|^3. \tag{36}$$

By(31) and (33)–(36), we obtain

$$\frac{1}{\pi}\int_0^{\pi}\rho(t)\cos\bigl((k-\mu)t-\alpha(t)\bigr)\,dt=\frac{1}{\pi}\int_0^{\tau}e^{-\frac{1}{2}\sigma^2t^2}\cos\bigl((k-\mu)t\bigr)\,dt+\triangle_4, \tag{37}$$

where

$$\begin{aligned}|\triangle_4|&\le|\triangle_1|+|\triangle_2|+|\triangle_3|\\&\le\frac{\pi}{2\tau\alpha}e^{-\frac{\tau^2\alpha}{\pi^2}}+\frac{1.3344}{\sigma^4}\sum_{j=1}^{n}E|X_j|^3+\frac{0.4554}{\sigma^4}\sum_{j=1}^{n}E|X_j|^3\\&=\frac{\pi}{2\tau\alpha}e^{-\frac{\tau^2\alpha}{\pi^2}}+\frac{1.7898}{\sigma^4}\sum_{j=1}^{n}E|X_j|^3.\end{aligned} \tag{38}$$

From (10), we can see that

$$\alpha=2\sum_{j=1}^{n}\sum_{l=-\infty}^{\infty}p_{jl}p_{j(l+1)}=\sum_{j=1}^{n}\left(\sum_{\substack{l=-\infty\\|l-m|\le1}}^{\infty}\sum_{m=-\infty}^{\infty}(l-m)^2p_{jl}p_{jm}\right)\le2\sigma^2,$$

which implies that $e^{-\frac{1}{2}\sigma^2t^2}\le e^{-\frac{1}{4}\alpha t^2}$. From this fact, we get

$$\begin{aligned}\frac{1}{\pi}\left|\int_{\tau}^{\infty}e^{-\frac{1}{2}\sigma^2t^2}\cos\bigl((k-\mu)t\bigr)\,dt\right|&\le\frac{1}{\pi}\int_{\tau}^{\infty}e^{-\frac{1}{2}\sigma^2t^2}\,dt\\&\le\frac{1}{\pi}\int_{\tau}^{\infty}e^{-\frac{1}{4}\alpha t^2}\,dt\\&\le\frac{1}{\pi\tau}\int_{\tau}^{\infty}te^{-\frac{1}{4}\alpha t^2}\,dt\\&=\frac{2}{\pi\tau\alpha}e^{-\frac{\tau^2\alpha}{4}}.\end{aligned}$$

From this fact and (37) and (38), we have

$$\frac{1}{\pi}\int_0^{\pi}\rho(t)\cos\bigl((k-\mu)t-\alpha(t)\bigr)\,dt=\frac{1}{\pi}\int_0^{\infty}e^{-\frac{1}{2}\sigma^2t^2}\cos\bigl((k-\mu)t\bigr)\,dt+\triangle_5, \tag{39}$$

where

$$\begin{aligned}|\triangle_5|&\le|\triangle_4|+\frac{1}{\pi}\left|\int_{\tau}^{\infty}e^{-\frac{1}{2}\sigma^2t^2}\cos\bigl((k-\mu)t\bigr)\,dt\right|\\&\le\frac{\pi}{2\tau\alpha}e^{-\frac{\tau^2\alpha}{\pi^2}}+\frac{1.7898}{\sigma^4}\sum_{j=1}^{n}E|X_j|^3+\frac{2}{\pi\tau\alpha}e^{-\frac{\tau^2\alpha}{4}}.\end{aligned} \tag{40}$$

Using the fact that

$$\int_0^{\infty}e^{-at^2}\cos(bt)\,dt=\frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}}\quad\text{for }a>0$$

(see [13], p. 7), we obtain

$$\frac{1}{\pi}\int_0^\infty e^{-\frac{1}{2}\sigma^2 t^2}\cos\bigl((k-\mu)t\bigr)\,dt = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(k-\mu)^2}{2\sigma^2}}. \tag{41}$$

By (23), (39), (40), and (41), we can conclude that

$$P(S_n = k) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(k-\mu)^2}{2\sigma^2}} + \triangle_6,$$

where $|\triangle_6| \le \frac{2.2075e^{-\frac{\tau^2\alpha}{\pi^2}}}{\tau\alpha} + \frac{1.7898}{\sigma^4}\sum_{j=1}^n E|X_j|^3$. □

Proof of Theorem 1.3 Let $Y_j = \frac{X_j}{b} - \frac{a}{b}$. Then

$$E\Biggl(\sum_{j=1}^n Y_j\Biggr) = \frac{\mu - na}{b}, \qquad \operatorname{Var}\Biggl(\sum_{j=1}^n Y_j\Biggr) = \frac{\sigma^2}{b^2},$$

$$P(S_n = na + kb) = P\Biggl(\sum_{j=1}^n Y_j = k\Biggr)$$

and

$$P(Y_j = k) = P\biggl(\frac{X_j}{b} - \frac{a}{b} = k\biggr) = P(X_j = a + bk).$$

Since b is maximal, we have $\alpha = \sum_{j=1}^n \alpha_j > 0$,
where $\alpha_j = 2\sum_{l=-\infty}^{\infty} p_{jl}p_{j(l+1)}$, $p_{jl} = P(X_j = a + bl)$.
From Theorem 1.2, we obtain Theorem 1.3. □

4 Examples of the main results

In this section, we give applications including Poisson binomial, binomial, and negative binomial that our main theorems can be applied as shown in Example 1–Example 3. In addition, the example that our main results can be applied to, unlike the result of Petrov [1], as shown in Example 4.

Example 1 If $X_1, X_2, \dots, X_n$ are independent Bernoulli random variables with $P(X_j = 1) = p_j$ and $P(X_j = 0) = q_j$, where $p_j + q_j = 1$ for $j = 1, 2, \dots, n$, S_n is a Poisson binomial random variable. Then

$$\biggl|P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(k-\mu)^2}{2\sigma^2}}\biggr| \le \frac{11.0375\sqrt[3]{\mu}e^{-\frac{\sigma^2}{50\pi^2(\sqrt[3]{\mu})^2}}}{\sigma^2} + \frac{1.7898\mu}{\sigma^4}, \tag{42}$$

where $\mu = \sum_{j=1}^n p_j$ and $\sigma^2 = \sum_{j=1}^n p_j q_j$.

Proof Note that $E|X_j|^3 = p_j$ and

$$\alpha = \sum_{j=1}^n \alpha_j = 2\sum_{j=1}^n\sum_{l=0}^{1} p_{j,l}p_{j,l+1}$$

$$= 2\sum_{j=1}^{n}\sum_{l=0}^{1} P(X_j = l)P(X_j = l+1)$$
$$= 2\sum_{j=1}^{n} P(X_j = 0)P(X_j = 1)$$
$$= 2\sum_{j=1}^{n} p_j q_j$$
$$= 2\sigma^2.$$

Hence, by Theorem 1.2, we see that (42) holds. □

Example 2 Let $S_n \sim \mathrm{Bi}(p)$. Then

$$\left|P(S_n = k) - \frac{1}{\sqrt{2\pi npq}} e^{-\frac{(k-np)^2}{2npq}}\right| \le \frac{11.0375\sqrt[3]{p} e^{-\frac{npq}{50\pi^2(\sqrt[3]{np})^2}}}{n^{\frac{2}{3}} pq} + \frac{1.7898}{npq^2}.$$

Proof We can apply Example 1 by letting $p_j = p$ and $q_j = q$. □

Observe that the results in Example 1 and Example 2 have the same order as (3) and (4) but the constants are bigger. However, (3) and (4) cannot be applied with the following example.

Example 3 If $X_1, X_2, \ldots, X_n$ are i.i.d. geometric random variables with parameter p. Then

$$\left|P(S_n = k) - \frac{p}{\sqrt{2\pi q}} e^{-\frac{(kp-1)^2}{2q}}\right|$$
$$\le \frac{11.0375(1+q)\sqrt[3]{p^2+6q} e^{-\frac{np^3 q}{50\pi^2(1+q)(\sqrt[3]{n(p^2+6q)})^2}}}{n^{\frac{2}{3}} p^2 q} + \frac{1.7898p(p^2+6q)}{nq^2}, \tag{43}$$

where $q = 1 - p$.

Proof Let ψ be the characteristic function of X_j. Then $\psi(t) = \frac{pe^{it}}{1-qe^{it}}$ and

$$\psi^{(3)}(t) = -\frac{ipe^{it}(q^2e^{2it} + 4qe^{it} + 1)}{(1-qe^{it})^4}.$$

Hence, $EX^3 = \frac{\psi^{(3)}(0)}{i^3} = \frac{p^2+6q}{p^3}$.

Note that

$$\alpha = 2n\sum_{l=1}^{\infty} p_{1,l}p_{1,l+1} = 2n\sum_{l=1}^{\infty} P(X_1 = l)P(X_1 = l+1)$$
$$= 2n\frac{p^2}{q}\sum_{l=1}^{\infty} q^{2l}$$
$$= \frac{2npq}{1+q}.$$

Hence, by Theorem 1.4, we get (43). □

Example 4 Let X_n be a sequence of independent random variables such that

$$P(X_j = 0) = \frac{1}{4}, P(X_j = 1) = \frac{3}{8} \quad \text{and} \quad P(X_j = 2) = \frac{3}{8}$$

for all $j = 1, 2, \dots, n$. Then

$$\left| P(S_n = k) - \frac{0.5111}{\sqrt{n}} e^{-\frac{(8k-9n)^2}{72n}} \right| \le \frac{0.069}{n^{\frac{2}{3}}} e^{-0.00022 n^{\frac{1}{3}}} + \frac{16.2671}{n}. \tag{44}$$

Proof Note that $E|X_j|^3 = \frac{27}{8}$, $ES_n = \frac{9n}{8}$, $\operatorname{Var} S_n = \frac{39n}{64}$, and

$$\begin{aligned}
\alpha &= \sum_{j=1}^{n} \alpha_j = 2 \sum_{j=1}^{n} \sum_{l=0}^{2} p_{j,l} p_{j,l+1} \\
&= 2 \sum_{j=1}^{n} \sum_{l=0}^{2} P(X_j = l) P(X_j = l+1) \\
&= 2 \sum_{j=1}^{n} \bigl(P(X_j = 0) P(X_j = 1) + P(X_j = 1) P(X_j = 2) \bigr) \\
&= 2 \sum_{j=1}^{n} \left(\frac{1}{4} \times \frac{3}{8} + \frac{3}{8} \times \frac{3}{8} \right) \\
&= \frac{15n}{32}.
\end{aligned}$$

Hence, by Theorem 1.2, we see that (44) holds. □

One can see that Theorem 1.2 can be applied to Example 4 and get the rate of convergence $O(\frac{1}{n})$, but Petrov's theorem [1] cannot be applied because this example does not satisfy its assumption 3.

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