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A characterization of nonhomogeneous wavelet bi-frames for reducing subspaces of Sobolev spaces

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Abstract

For nonhomogeneous wavelet bi-frames in a pair of dual spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ with $s \neq 0$, smoothness and vanishing moment requirements are separated from each other, that is, one system is for smoothness and the other one for vanishing moments. This gives us more flexibility to construct nonhomogeneous wavelet bi-frames than in $L^2(\mathbb{R}^d)$. In this paper, we introduce the reducing subspaces of Sobolev spaces, and characterize the nonhomogeneous wavelet bi-frames under the setting of a general pair of dual reducing subspaces of Sobolev spaces.

MSC: 42C15; 42C40

Keywords: Nonhomogeneous wavelet system; Reducing subspace; Sobolev space; Nonhomogeneous wavelet bi-frames

1 Introduction

Most classical nonhomogeneous wavelet systems are derived from a refinable structure (see [2, 5, 7, 9, 21] and the references therein). To obtain the stability of frames or bi-frames, some technical restrictions are imposed on refinable masks in this literature. Observe that for wavelet systems derived from refinable structures, one of the most important features is their associated fast wavelet transform. Due to lack of a refinable function, the correspondence between the homogeneous systems and fast wavelet transforms is not exact, while the nonhomogeneous systems are different. Moreover, Han in [12] showed that nonhomogeneous wavelet systems are closely related to nonstationary wavelets (see [6, 14]). Based on these considerations, in this paper, we will dicuss the nonhomogeneous wavelet bi-frames under the setting of the reducing subspaces of Sobolev spaces.

The notion of frames was first introduced in [10], which dealt with nonharmonic Fourier series. Let $\mathcal{I}$ be a countable set, and $\mathcal{H}$ be a separable Hilbert space. The sequence $\{e_i\}_{i\in\mathcal{I}} \subset \mathcal{H}$ is called a Bessel sequence in $\mathcal{H}$ if there exists $C > 0$ such that

$$\sum_{i\in\mathcal{I}} \left|\langle f, e_i\rangle\right|^2 \le C\|f\|^2 \quad \text{for } f \in \mathcal{H};$$

this is called a frame for $\mathcal{H}$ if there exist $0 < C_1 \le C_2 < +\infty$ such that

$$C_1\|f\|^2 \le \sum_{i\in\mathcal{I}} \big|\langle f, e_i\rangle\big|^2 \le C_2\|f\|^2 \quad \text{for } f \in \mathcal{H}.$$

Given two frames $\{e_i\}_{i\in\mathcal{I}}$ and $\{\tilde{e}_i\}_{i\in\mathcal{I}}$ for $\mathcal{H}$, we call $\{\tilde{e}_i\}_{i\in\mathcal{I}}$ a dual of $\{e_i\}_{i\in\mathcal{I}}$ if

$$f = \sum_{i\in\mathcal{I}} \langle f, \tilde{e}_i\rangle e_i \quad \text{for } f \in \mathcal{H}. \tag{1.1}$$

It is easy to check that (1.1) is equivalent to

$$f = \sum_{i\in\mathcal{I}} \langle f, e_i\rangle \tilde{e}_i \quad \text{for } f \in \mathcal{H}.$$

So, in the case, we also say $(\{e_i\}_{i\in\mathcal{I}}, \{\tilde{e}_i\}_{i\in\mathcal{I}})$ is a pair of bi-frames. It is well known that $(\{e_i\}_{i\in\mathcal{I}}, \{\tilde{e}_i\}_{i\in\mathcal{I}})$ is a pair of bi-frames for $\mathcal{H}$ if $\{e_i\}_{i\in\mathcal{I}}$ and $\{\tilde{e}_i\}_{i\in\mathcal{I}}$ are Bessel sequences satisfying (1.1).

Let d be a positive integer. The Fourier transform of an integrable function $f \in L^1(\mathbb{R}^d)$ is defined by

$$\hat{f}(\cdot) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i\langle x,\cdot\rangle}\, dx$$

and is naturally extended to the tempered distribution spaces, where $\langle\cdot,\cdot\rangle$ means the Euclidean inner product in $\mathbb{R}^d$. Similarly, its inverse Fourier transform is defined as

$$\check{f} = \int_{\mathbb{R}^d} f(x) e^{2\pi i\langle x,\cdot\rangle}\, dx.$$

For functions f and g on $\mathbb{R}^d$, we define

$$[f,g]_t(\cdot) = \sum_{k\in\mathbb{Z}^d} f(\cdot + k)\overline{g(\cdot + k)}\big(1 + |\cdot + k|^2\big)^t, \quad t \in \mathbb{R},$$

if it is well-defined in some sense, where $|\cdot|$ denotes its Euclidean norm. We denote by χ_E the characteristic function of a Lebesgue measurable set E and by δ the Dirac sequence. The support of a distribution f on $\mathbb{R}^d$ is defined by

$$\operatorname{supp}(f) = \big\{x \in \mathbb{R}^d : f(x) \neq 0\big\}$$

which is well-defined up to a null set. Given $s \in \mathbb{R}$, let $H^s(\mathbb{R}^d)$ be the Sobolev space consisting of all tempered distributions f such that

$$\|f\|^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \big|\hat{f}(\xi)\big|^2\big(1 + |\xi|^2\big)^s\, d\xi < \infty.$$

It is easy to check that $H^s(\mathbb{R}^d)$ is a Hilbert space under the inner product:

$$\langle f, g\rangle_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}\big(1 + |\xi|^2\big)^s\, d\xi \quad \text{for } f, g \in H^s\big(\mathbb{R}^d\big).$$

In particular, $H^0(\mathbb{R}^d)$ is the usual Hilbert space $L^2(\mathbb{R}^d)$ by the Plancherel theorem. Moreover, for every $g \in H^{-s}(\mathbb{R}^d)$,

$$\langle f, g\rangle = \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi \quad \text{for } f \in H^s(\mathbb{R}^d)$$

defines a continuous linear functional on $H^s(\mathbb{R}^d)$. So the spaces $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$ constitute a pair of dual spaces.

We say that an integer matrix A is expansive if all its eigenvalues are greater than 1 in module. Throughout this paper, we always assume that A is isotropic, i.e. A is similar to a diagonal matrix $\operatorname{diag}(\lambda_1, \lambda_2, \cdot, \lambda_d)$ satisfying $|\lambda_1| = |\lambda_2| = \cdot = |\lambda_d| = |\det A|^{\frac{1}{d}}$. We always denote by A^* its conjugate transpose for a matrix A. Define a function $\kappa : \mathbb{Z}^d \to \mathbb{Z}$ by

$$\kappa(n) = \sup\{j \in \mathbb{Z}_+ : A^{*-j} n \in \mathbb{Z}^d\}, \tag{1.2}$$

where $\mathbb{Z}_+$ denotes the set of the natural integers. It is obvious that $\kappa(0) = +\infty$. Define the shift operator T_k with $k \in \mathbb{Z}^d$ and the dilation operator by

$$T_k f(\cdot) = f(\cdot - k) \quad \text{and} \quad Df(\cdot) = |\det A|^{\frac{1}{2}} f(A\cdot)$$

for a distribution f, respectively. For convenience, we write $m = |\det A|^{\frac{1}{d}}$, and write

$$f_{j,k} = D^j T_k f \quad \text{and} \quad f_{j,k}^s = m^{-js} f_{j,k}$$

for $s \in \mathbb{R}$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$. Given $L \in \mathbb{N}$. Let $\psi_0 \in H^s(\mathbb{R}^d)$ be a tempered distribution, and $\Psi = \{\psi_1, \dots, \psi_L\} \subset H^s(\mathbb{R}^d)$ a finite set of tempered distributions, we denote the homogeneous wavelet system $X^s(\Psi)$ and the nonhomogeneous wavelet system $X^s(\psi_0; \Psi)$ in $H^s(\mathbb{R}^d)$, respectively, by

$$X^s(\Psi) = \{\psi_{l,j,k}^s : j \in \mathbb{Z}, k \in \mathbb{Z}^d, l = 1, \dots, L\} \tag{1.3}$$

and

$$X^s(\psi_0; \Psi) = \{\psi_{0,0,k} : k \in \mathbb{Z}^d\} \cup \{\psi_{l,j,k}^s : j \in \mathbb{Z}_+, k \in \mathbb{Z}^d, l = 1, \dots, L\}. \tag{1.4}$$

In particular, we write

$$X^0(\Psi) = X(\Psi) \quad \text{and} \quad X^0(\psi_0; \Psi) = X(\psi_0; \Psi)$$

for simplicity.

Han in [15] studied nonhomogeneous wavelet frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. In particular, when $s \neq 0$ and A is the dyadic matrix $2I_d$, [15, Theorem 1.1], not only established the mixed extension principle for nonhomogeneous wavelet bi-frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$, but also characterized the functions in $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$ using such bi-frames. The characterization is different from the one in [3, 4], using homogeneous wavelet bi-frames in $L^2(\mathbb{R}^d)$. The homogeneous wavelet bi-frames used in [3, 4] are required to have vanishing moments and positive regularity simultaneously, however, this pair of competing requirements can be completely separated for two wavelet systems in nonhomogeneous wavelet bi-frames used in [15, Theorem 1.1]. Without loss of generality, assuming that $s > 0$, then one can demand the synthesis system to adapt the desired order of regularity, while requiring the analysis system to have the desired order of vanishing moments to achieve the sparsity. This gives great flexibility in constructing bi-frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. Two variations of [15, Theorem 1.1] are obtained in [11, Theorem 2.1] and [20, Theorem 4.1]. Li and Zhang in [18] obtained the following characterization for a nonhomogeneous wavelet bi-frames of $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$.

Proposition 1.1 *Let $X^s(\psi_0;\Psi)$ and $X^{-s}(\tilde{\psi}_0;\tilde{\Psi})$ be Bessel sequences in $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$, respectively. Then $(X^s(\psi_0;\Psi), X^{-s}(\tilde{\psi}_0;\tilde{\Psi}))$ is a nonhomogeneous wavelet bi-frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ if and only if, for every $k \in \mathbb{Z}^d$,*

$$\hat{\psi}_0(\cdot)\overline{\hat{\tilde{\psi}}_0(\cdot+k)} + \sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)} \hat{\psi}_l\big((A^*)^{-j}\cdot\big)\overline{\hat{\tilde{\psi}}_l\big((A^*)^{-j}(\cdot+k)\big)} = \delta_{0,k} \quad \text{a.e. on } \mathbb{R}^d. \tag{1.5}$$

In particular, when $s \neq 0$, Proposition 1.1 reduces to the one in [19, Lemma 2.5], with taking $\Omega = \mathbb{R}^d$. [19, Lemma 2.5], is a variation of [13, Theorems 9, 11] and [1, Proposition 2.3]. [13, Theorems 9, 11] are for frequency-based nonhomogeneous wavelet bi-frames in space of distribution. And [1, Proposition 2.3], is for wavelet bi-frames in $L^2(\mathbb{R}^d)$. Observe that all the above work concerns the whole space $H^s(\mathbb{R}^d)$ or $L^2(\mathbb{R}^d)$. This paper addresses nonhomogeneous wavelet bi-frames under the setting of reducing subspaces of $H^s(\mathbb{R}^d)$ which is more general than $H^s(\mathbb{R}^d)$. Now, we introduce the definition of reducing subspaces of $H^s(\mathbb{R}^d)$.

Definition 1.1 Given $s \in \mathbb{R}$ and a $d \times d$ expansive matrix A, a nonzero closed linear subspace X of $H^s(\mathbb{R}^d)$ is called a *reducing subspace* if $DX = X$ and $T_kX = X$ for every $k \in \mathbb{Z}^d$, and

$$\big(1 + \big|(A^*)^{-1}\cdot\big|^2\big)^{\frac{s}{2}}\widehat{X} = \big(1 + |\cdot|^2\big)^{\frac{s}{2}}\widehat{X}, \tag{1.6}$$

where $\widehat{X} = \{\hat{f} : f \in X\}$.

Observe that (1.6) is trivial if $s = 0$. Definition 1.1 is a generalization of the notion of reducing subspaces of $L^2(\mathbb{R}^d)$. The following proposition gives a Fourier-domain characterization for reducing subspaces of $L^2(\mathbb{R}^d)$.

Proposition 1.2 ([8, Theorem 1]) *For a $d \times d$ expansive matrix A, X is a reducing subspace of $L^2(\mathbb{R}^d)$ if and only if $X = FL^2(\Omega)$ for some $\Omega \subset \mathbb{R}^d$ with nonzero measure satisfying $\Omega = A^*\Omega$, where*

$$FL^2(\Omega) := \big\{f \in L^2\big(\mathbb{R}^d\big) : \operatorname{supp}(\hat{f}) \subset \Omega\big\}.$$

Before proceeding, let us introduce some notations and notions. For $\Omega \subset \mathbb{R}^d$ with nonzero measure, we write

$$FH^s(\Omega) = \big\{f \in H^s\big(\mathbb{R}^d\big) : \operatorname{supp}(\hat{f}) \subset \Omega\big\},$$

and $FH^0(\Omega) = FL^2(\Omega)$. Then $FH^s(\mathbb{R}^d) = H^s(\mathbb{R}^d)$. Obviously, for each $g \in FH^{-s}(\Omega)$,

$$\langle f, g\rangle = \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi \quad \text{for } f \in FH^s(\Omega)$$

defines a continuous linear functional on $FH^s(\Omega)$. Then $(FH^s(\Omega), FH^{-s}(\Omega))$ constitutes a pair of dual spaces, as we discussed above as regards $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$.

Theorem 2.1 in Sect. 2 claims that X is a reducing subspace of $H^s(\mathbb{R}^d)$ if and only if $X = FH^s(\Omega)$ for some nonzero measure set $\Omega \subset \mathbb{R}^d$ satisfying $\Omega = A^*\Omega$. So, specifically, we denote as $FH^s(\Omega)$ a reducing subspace of $H^s(\mathbb{R}^d)$ in place of X.

Next, we introduce the definition of a nonhomogeneous wavelet bi-frames in reducing subspaces of Sobolev spaces. Let $FH^s(\Omega)$ and $FH^{-s}(\Omega)$ be reducing subspaces of $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$, respectively. $\psi_0 \in FH^s(\Omega)$, $\tilde{\psi}_0 \in FH^{-s}(\Omega)$, Ψ and $\tilde{\Psi}$ be finite subsets of $FH^s(\Omega)$ and $FH^{-s}(\Omega)$, respectively. We say that $X^s(\psi_0;\Psi)$ is a nonhomogeneous wavelet frame (Bessel sequence) in $FH^s(\Omega)$ if it is a frame (Bessel sequence) in $FH^s(\Omega)$, and that $(X^s(\psi_0;\Psi), X^{-s}(\tilde{\psi}_0;\tilde{\Psi}))$ is a nonhomogeneous wavelet bi-frames (*NWBF*s) in $(FH^s(\Omega), FH^{-s}(\Omega))$ if

(1) $X^s(\psi_0;\Psi)$ is a frame for $FH^s(\Omega)$ and $X^{-s}(\tilde{\psi}_0;\tilde{\Psi})$ is a frame in $FH^{-s}(\Omega)$;

(2) the identity

$$\langle f, g\rangle = \sum_{k\in\mathbb{Z}^d} \langle f, \tilde{\psi}_{0,0,k}\rangle\langle \psi_{0,0,k}, g\rangle + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \langle f, \tilde{\psi}^{-s}_{l,j,k}\rangle\langle \psi^{s}_{l,j,k}, g\rangle \tag{1.7}$$

holds for all $f \in FH^s(\Omega)$ and $g \in FH^{-s}(\Omega)$.

The notion of *NWBF*s herein is a direct generalization of the one in [16], which deals with $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. Observe that $X^s(\psi_0;\Psi)$ and $X^{-s}(\tilde{\psi}_0;\tilde{\Psi})$ cannot be replaced by $X(\psi_0,\Psi)$ and $X(\tilde{\psi}_0,\tilde{\Psi})$ in the above definitions when $s \neq 0$. An argument for this can be found in [17].

Denote

$$\mathcal{D} = \{f : \hat{f} \in L^\infty(\mathbb{R}^d), \operatorname{supp}(\hat{f}) \text{ is bounded}\}.$$

It is well known that $\mathcal{D}$ is dense in $H^s(\mathbb{R}^d)$ and $\mathcal{D} \cap FH^s(\Omega)$ is dense in $FH^s(\Omega)$ for every $s \in \mathbb{R}$, respectively.

The paper is organized as follows. In Sect. 2, we characterize the reducing subspaces of $H^s(\mathbb{R}^d)$ and give some auxiliary lemmas used later. In Sect. 3, we establish a characterization of a *NWBF*s in $(FH^s(\Omega), FH^{-s}(\Omega))$ via a pair of equations.

2 Reducing subspaces of $H^s(\mathbb{R}^d)$ and some auxiliary lemma

In this section, we characterize the reducing subspaces of Sobolev spaces, and give some auxiliary lemmas used later.

By a careful computation, we get the following lemma.

Lemma 2.1 *Let $s \in \mathbb{R}$, and let Ω be a measurable set in $\mathbb{R}^d$ with nonzero measure. Define λ by*

$$\widehat{\lambda f}(\cdot) = \left(1 + |\cdot|^2\right)^{\frac{s}{2}}\hat{f}(\cdot)$$

for $f \in H^s(\mathbb{R}^d)$. *Then*

(i) λ *and* λ^2 *are unitary operators from* $H^s(\mathbb{R}^d)$ *onto* $L^2(\mathbb{R}^d)$ *and onto* $H^{-s}(\mathbb{R}^d)$, *respectively*;

(ii) $\lambda(FH^s(\Omega)) = FL^2(\Omega)$, *and* $\lambda^2(FH^s(\Omega)) = FH^{-s}(\Omega)$;

(iii)

$$\begin{aligned}(\lambda f_{j,k})\widehat{\ }(\cdot) &= |\det A|^{-\frac{j}{2}}\left(1+|\cdot|^2\right)^{\frac{s}{2}} e^{-2\pi i\langle k,(A^*)^{-j}\cdot\rangle}\hat{f}\left((A^*)^{-j}\cdot\right)\\ &= \left(\frac{1+|\cdot|^2}{1+|(A^*)^{-j}\cdot|^2}\right)^{\frac{s}{2}}\left[(\lambda f)_{j,k}\right]\widehat{\ }(\cdot)\end{aligned}$$

for $f \in H^s(\mathbb{R}^d)$.

Theorem 2.1 *Given* $s \in \mathbb{R}$ *and a* $d \times d$ *expansive matrix* A, X *is a reducing subspace of* $H^s(\mathbb{R}^d)$ *if and only if* $X = FH^s(\Omega)$ *for some* $\Omega \subset \mathbb{R}^d$ *with nonzero measure satisfying* $\Omega = A^*\Omega$.

Proof Necessity. Suppose X is a reducing subspace of $H^s(\mathbb{R}^d)$. Defined λ as in Lemma 2.1, and denote $X_1 = \lambda X$., Then we only need to prove that

$$X_1 = FL^2(\Omega)$$

for some measurable set Ω in $\mathbb{R}^d$ with $\Omega = A^*\Omega$ by Lemma 2.1. By the unitarity of λ, X_1 is a linear closed subspace of $L^2(\mathbb{R}^d)$. So it is sufficient to prove that

$$DX_1 = X_1 \quad \text{and} \quad T_kX_1 = X_1 \quad \text{for } k \in \mathbb{Z}^d \tag{2.1}$$

by Proposition 1.2. A simple computation shows that

$$\begin{aligned}&\widehat{DX_1} = \left(1+\left|(A^*)^{-1}\cdot\right|^2\right)^{\frac{s}{2}}\widehat{DX} = \widehat{DX_1} = \left(1+\left|(A^*)^{-1}\cdot\right|^2\right)^{\frac{s}{2}}\widehat{X} = \left(1+|\cdot|^2\right)^{\frac{s}{2}}\widehat{X} = \widehat{X_1},\\ &\widehat{T_kX_1} = \left(1+|\cdot|^2\right)^{\frac{s}{2}}\widehat{T_kX} = \widehat{T_kX_1} = \left(1+|\cdot|^2\right)^{\frac{s}{2}}\widehat{X} = \widehat{X_1} \quad \text{for } k \in \mathbb{Z}^d,\end{aligned}$$

according to Definition 1.1 and the fact that X is a reducing subspace of $H^s(\mathbb{R}^d)$. Hence (2.1) holds.

Sufficiency. Assume that $X = FH^s(\Omega)$, $A^*\Omega = \Omega$. Obviously, (1.6) holds. By Lemma 2.1, λ is a unitary operator, $FH^s(\Omega) = \lambda^{-1}(FL^2(\Omega))$. Furthermore, $FL^2(\Omega)$ is a linear closed subspace of $L^2(\mathbb{R}^d)$. So X is a linear closed subspace of $H^s(\mathbb{R}^d)$. For $k \in \mathbb{Z}^d$, we have

$$\widehat{T_kX} = e^{-2\pi i\langle k,\cdot\rangle}\widehat{X} = \widehat{X} \quad \text{and} \quad \widehat{DX} = D^{-1}\widehat{X} = \widehat{X}$$

due to the fact that $A^*\Omega = \Omega$. It follows that

$$DX = X \quad \text{and} \quad T_kX = X \quad \text{for } k \in \mathbb{Z}^d.$$

Therefore, X is a reducing subspace of $H^s(\mathbb{R}^d)$. The lemma is proved. □

The following three lemmas are borrowed from [18, Lemmas 3.6, 3.9 and 3.11].

Lemma 2.2 *Let $s \in \mathbb{R}$ and $\phi \in H^s(\mathbb{R}^d)$. Then*

(i) $\{T_k\phi : k \in \mathbb{Z}^d\}$ is a Bessel sequence in $H^s(\mathbb{R}^d)$ if and only if

$$[\hat{\phi}, \hat{\phi}]_s \in L^\infty(\mathbb{T}^d).$$

In this case, $\|[\hat{\phi}, \hat{\phi}]_s\|_{L^\infty(\mathbb{T}^d)}$ is a Bessel bound.

(ii) If $\{T_k\phi : k \in \mathbb{Z}^d\}$ is a Bessel sequence in $H^s(\mathbb{R}^d)$, then $\{\phi^s_{j,k} : k \in \mathbb{Z}^d\}$ is a Bessel sequence in $H^s(\mathbb{R}^d)$.

Lemma 2.3 *Let $s \in \mathbb{R}$ and $j \in \mathbb{Z}$. Given $\phi \in H^s(\mathbb{R}^d)$ and $f \in H^{-s}(\mathbb{R}^d)$. Then the kth Fourier coefficient of $[|\det A|^{\frac{j}{2}}\hat{f}((A^*)^j\cdot), \hat{\phi}(\cdot)]_0(\xi)$ is $\langle f, \phi_{j,k}\rangle$ for $k \in \mathbb{Z}^d$. In particular,*

$$\bigl[|\det A|^{\frac{j}{2}}\hat{f}\bigl(\bigl(A^*\bigr)^j\cdot\bigr), \hat{\phi}(\cdot)\bigr]_0(\xi) = \sum_{k\in\mathbb{Z}^d}\langle f, \phi_{j,k}\rangle e^{-2\pi i\langle k,\xi\rangle} \tag{2.2}$$

if $\{T_k\phi : k \in \mathbb{Z}^d\}$ is a Bessel sequence in $H^s(\mathbb{R}^d)$.

Lemma 2.4 *Given $s \in \mathbb{R}$, let $X^s(\psi_0; \Psi)$ be a Bessel sequence in $H^s(\mathbb{R}^d)$. Then*

$$\bigl|\hat{\psi}_0(\cdot)\bigr|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty} m^{-2js}\bigl|\hat{\psi}_l\bigl(\bigl(A^*\bigr)^{-j}\cdot\bigr)\bigr|^2 \le B\bigl(1 + |\cdot|^2\bigr)^{-s} \tag{2.3}$$

holds a.e. on $\mathbb{R}^d$.

Lemma 2.5 *Given $s \in \mathbb{R}$. If $g \in \mathcal{D}$, then $[\hat{g}, \hat{g}]_s(\cdot) \le C$.*

Proof By $g \in \mathcal{D}$, we have $\hat{g} \in L^\infty(\mathbb{R}^d)$ and $\operatorname{supp}(\hat{g}) \subset K$ for some bounded set K. Observe that

$$[\hat{g}, \hat{g}]_s(\xi) = \sum_{k\in\mathbb{Z}^d}\bigl|\hat{g}(\xi + k)\bigr|^2\bigl(1 + |\xi + k|^2\bigr)^s$$

is $\mathbb{Z}^d$-periodic. So we only need to prove that $[\hat{g}, \hat{g}]_s(\cdot) \le C$ on $\mathbb{T}^d$. Combining this with the boundedness of $\operatorname{supp}(\hat{g})$, we can deduce that there are only finitely many nonzero terms $|\hat{g}(\xi + k)|^2(1 + |\xi + k|^2)^s$ in $\sum_{k\in\mathbb{Z}^d}|\hat{g}(\xi + k)|^2(1 + |\xi + k|^2)^s$ for $\xi \in \mathbb{T}^d$ and thus $[\hat{g}, \hat{g}]_s(\cdot) \le C$. The lemma is completed. □

Lemma 2.6 *Let $K \subset \mathbb{R}^d$ be a bounded set. Then there exist finite sets $F_1 \subset \mathbb{Z}_+$ and $F_2 \subset \mathbb{Z}^d \setminus \{0\}$ such that*

$$K \cap \bigl(K + A^*k\bigr) = \emptyset \tag{2.4}$$

for $(j, k) \notin F_1 \times F_2$ with $k \neq 0$.

Proof Since A is expansive, we have

$$\lim_{j\to\infty}\bigl\|\bigl(A^*\bigr)^{-j}\bigr\|^{\frac{1}{j}} < r \quad \text{for some } 0 < r < 1.$$

Take $\delta >$ diamter(K). It follows that there exists $J_0 > -\log_r \delta$ such that

$$\|(A^*)^{-j}\| < r^j \quad \text{for } r > J_0.$$

So $1 \le |k| < r^j|(A^*)^j k|$ and it leads to

$$|(A^*)^j k| > r^{-j} > \delta > \text{diameter}(K) \tag{2.5}$$

for $j > J_0$ and $0 \ne k \in \mathbb{Z}^d$. Below we consider the case $0 \le j \le J_0$. By the definition of the operator norm, we get

$$|k| \le \Big(\max_{0\le j\le J_0} \|(A^*)^{-j}\|\Big)|(A^*)^j k|$$

for $k \in \mathbb{Z}^d$. Again using (2.5), we have

$$|(A^*)^j k| \ge \frac{|k|}{\max_{0\le j\le J_0} \|(A^*)^{-j}\|} > \text{diameter}(K) \tag{2.6}$$

if $|k| > (\max_{0\le j\le J_0} \|(A^*)^{-j}\|)\delta$. Take

$$F_1 = \{j \in \mathbb{Z} : 0 \le j \le J_0\}$$

and

$$F_2 = \Big\{k \in \mathbb{Z}^d \setminus \{0\} : |k| \le \Big(\max_{0\le j\le J_0} \|(A^*)^{-j}\|\Big)\delta\Big\}.$$

Then (2.4) holds by (2.5) and (2.6). The lemma is proved. □

3 The characterization of *NWBF*s

In this section, we focus on characterizing a *NWBF*s in $(FH^s(\Omega), FH^{-s}(\Omega))$. For this purpose, we first give two lemmas.

Lemma 3.1 *Given $s \in \mathbb{R}$, let $\{T_k\psi_0 : k \in \mathbb{Z}^d\} \cup \{T_k\psi_l : k \in \mathbb{Z}^d, 1 \le l \le L\}$ be a Bessel sequence in $H^s(\mathbb{R}^d)$. Then*

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d} |\langle g, \psi_{0,0,k}\rangle|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} |\langle g, \psi^s_{l,j,k}\rangle|^2 \\
&\quad = \int_{\mathbb{R}^d} |\hat{g}(\xi)|^2 \Big(|\hat{\psi}_0(\xi)|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty} m^{-2js}|\hat{\psi}_l((A^*)^{-j}\xi)|^2\Big)\,d\xi \\
&\qquad + \int_{\mathbb{R}^d} \overline{\hat{g}(\xi)} \sum_{0\ne k\in\mathbb{Z}^d} \hat{g}(\xi+k) \\
&\qquad \times \Big(\hat{\psi}_0(\xi)\overline{\hat{\psi}_0(\xi+k)} + \sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)} m^{-2js}\hat{\psi}_l((A^*)^{-j}\xi)\overline{\hat{\psi}_l((A^*)^{-j}(\xi+k))}\Big)\,d\xi
\end{aligned} \tag{3.1}$$

for $g \in \mathcal{D}$.

Proof Applying Lemma 2.3, we obtain

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^d}\bigl|\langle g,\psi_{0,0,k}\rangle\bigr|^2+\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\bigl|\langle g,\psi^s_{l,j,k}\rangle\bigr|^2\\
&=\int_{\mathbb{T}^d}\Bigl|\sum_{k\in\mathbb{Z}^d}\hat{g}(\xi+k)\overline{\hat{\psi}_0(\xi+k)}\Bigr|^2\,d\xi\\
&\quad+\sum_{l=1}^{L}\sum_{j=0}^{\infty}m^{j(d-2s)}\int_{\mathbb{T}^d}\Bigl|\sum_{k\in\mathbb{Z}^d}\hat{g}\bigl((A^*)^j(\xi+k)\bigr)\overline{\hat{\psi}_l(\xi+k)}\Bigr|^2\,d\xi\\
&=\int_{\mathbb{T}^d}\Bigl(\sum_{k\in\mathbb{Z}^d}\hat{\psi}_0(\xi+k)\overline{\hat{g}(\xi+k)}\Bigr)\Bigl(\sum_{k\in\mathbb{Z}^d}\hat{g}(\xi+k)\overline{\hat{\psi}_0(\xi+k)}\Bigr)\,d\xi\\
&\quad+\sum_{l=1}^{L}\sum_{j=0}^{\infty}m^{j(d-2s)}\int_{\mathbb{T}^d}\Bigl(\sum_{k\in\mathbb{Z}^d}\hat{\psi}_l(\xi+k)\overline{\hat{g}\bigl((A^*)^j(\xi+k)\bigr)}\Bigr)\\
&\quad\times\Bigl(\sum_{k\in\mathbb{Z}^d}\hat{g}\bigl((A^*)^j(\xi+k)\bigr)\overline{\hat{\psi}_l(\xi+k)}\Bigr)\,d\xi\\
&=\int_{\mathbb{T}^d}\Bigl(\sum_{k\in\mathbb{Z}^d}\hat{\psi}_0(\xi+k)\overline{\hat{g}(\xi+k)}\Bigr)E_0(\xi)\,d\xi\\
&\quad+\sum_{l=1}^{L}\sum_{j=0}^{\infty}m^{j(d-2s)}\int_{\mathbb{T}^d}\Bigl(\sum_{k\in\mathbb{Z}^d}\hat{\psi}_l(\xi+k)\overline{\hat{g}\bigl((A^*)^j(\xi+k)\bigr)}\Bigr)E_{l,j}(\xi)\,d\xi\\
&=:I_1+I_2,
\end{aligned}
\tag{3.2}
$$

where $E_0(\cdot)=\sum_{k\in\mathbb{Z}^d}\hat{g}(\xi+k)\overline{\hat{\psi}_0(\xi+k)}$ and $E_{l,j}(\cdot)=\sum_{k\in\mathbb{Z}^d}\hat{g}((A^*)^j(\xi+k))\overline{\hat{\psi}_l(\xi+k)}$. Note that $\{T_k\psi_0:k\in\mathbb{Z}^d\}$ is a Bessel sequence in $H^s(\mathbb{R}^d)$ and $g\in\mathcal{D}$, then we have $|E_0(\cdot)|\le[\hat{g},\hat{g}]^{\frac12}_{-s}(\cdot)[\hat{\psi}_0,\hat{\psi}_0]^{\frac12}_{s}(\cdot)<\infty$ by Lemma 2.2 (i) and Lemma 2.5. It follows that

$$
\int_{\mathbb{T}^d}\sum_{k\in\mathbb{Z}^d}\bigl|\overline{\hat{g}(\xi+k)}\hat{\psi}_0(\xi+k)E_0(\xi)\bigr|\,d\xi\le\|E_0\|_{L^\infty(\mathbb{T}^d)}\int_{\mathbb{T}^d}[\hat{g},\hat{g}]^{\frac12}_{-s}(\xi)[\hat{\psi}_0,\hat{\psi}_0]^{\frac12}_{s}(\xi)\,d\xi<\infty,
$$

and thus

$$
\begin{aligned}
&\int_{\mathbb{T}^d}\Bigl(\sum_{k\in\mathbb{Z}^d}\hat{\psi}_0(\xi+k)\overline{\hat{g}(\xi+k)}\Bigr)\Bigl(\sum_{k\in\mathbb{Z}^d}\hat{g}(\xi+k)\overline{\hat{\psi}_0(\xi+k)}\Bigr)\,d\xi\\
&=\int_{\mathbb{R}^d}\hat{\psi}_0(\xi)\overline{g(\xi)}\sum_{k\in\mathbb{Z}^d}\hat{g}(\xi+k)\overline{\hat{\psi}_0(\xi+k)}\,d\xi
\end{aligned}
$$

by the Fubini–Tonelli theorem. Furthermore, we have

$$
\int_{\mathbb{R}^d}\bigl|\hat{\psi}_0(\xi)\overline{g(\xi)}\bigr|\sum_{k\in\mathbb{Z}^d}\bigl|\hat{g}(\xi+k)\overline{\hat{\psi}_0(\xi+k)}\,d\xi\bigr|\le\int_{\operatorname{supp}(\hat{g})}\Bigl(\sum_{k\in\mathbb{Z}^d}\bigl|\hat{g}(\xi+k)\overline{\hat{\psi}_0(\xi+k)}\,d\xi\bigr|\Bigr)^2\,d\xi
$$

$$\begin{aligned} &\leq \int_{\operatorname{supp}(\hat{g})} [\hat{g},\hat{g}]_{-s}(\xi)[\hat{\psi}_0,\hat{\psi}_0]_s(\xi)\,d\xi \\ &< \infty \end{aligned}$$

since $[\hat{g},\hat{g}]_{-s}(\cdot)[\hat{\psi}_0,\hat{\psi}_0]_s(\cdot)$ is essentially bounded by Lemma 2.2. It follows that

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^d} \hat{\psi}_0(\xi)\overline{g(\xi)} \sum_{k\in\mathbb{Z}^d} \hat{g}(\xi+k)\overline{\hat{\psi}_0(\xi+k)}\,d\xi \\ &= \int_{\mathbb{R}^d} \left|\hat{\psi}_0(\xi)\right|^2 \left|\hat{g}(\xi)\right|^2 d\xi + \int_{\mathbb{R}^d} \hat{\psi}_0(\xi)\overline{\hat{g}(\xi)} \sum_{0\neq k\in\mathbb{Z}^d} \hat{g}(\xi+k)\overline{\hat{\psi}_0(\xi+k)}\,d\xi. \end{aligned} \tag{3.3}$$

Below we calculate I_2 to finish the proof. Define $\tilde{g}$ by $\hat{\tilde{g}}(\cdot) = \hat{g}((A^*)^j\cdot)$, then we deduce that

$$\left[\hat{g}\left((A^*)^j\cdot\right),\hat{g}\left((A^*)^j\cdot\right)\right]_{-s}(\cdot) \leq C$$

by $g \in \mathcal{D}$ and Lemma 2.5. So $|E_{l,j}(\cdot)| \leq [\hat{g}((A^*)^j\cdot),\hat{g}((A^*)^j\cdot)]_{-s}^{\frac{1}{2}}(\cdot)[\hat{\psi}_l,\hat{\psi}_l]_s^{\frac{1}{2}}(\cdot) < \infty$ and thus

$$I_2 = \sum_{l=1}^{L}\sum_{j=0}^{\infty} m^{j(d-2s)} \int_{\mathbb{R}^d} \hat{\psi}_l(\xi)\overline{\hat{g}\left((A^*)^j\xi\right)} \sum_{k\in\mathbb{Z}^d} \hat{g}\left((A^*)^j(\xi+k)\right)\overline{\hat{\psi}_l(\xi+k)}\,d\xi.$$

Take K as a bounded set in $\mathbb{R}^d$ such that $\operatorname{supp}(\hat{g}) \subset K$. By Lemma 2.6, we have

$$K \cap \left(K + (A^*)^j k\right) = \emptyset \quad \text{for } (j,k) \notin F_1 \times F_2 \text{ with } k \neq 0,$$

where $F_1 \subset \mathbb{Z}_+$ and $F_2 \subset \mathbb{Z}^d \setminus \{0\}$ are two finite sets. It follows that

$$I_2 = \sum_{l=1}^{L}\sum_{j\in F_1} m^{j(d-2s)} \int_{\mathbb{R}^d} \hat{\psi}_l(\xi)\overline{\hat{g}\left((A^*)^j\xi\right)} \sum_{k\in F_2} \hat{g}\left((A^*)^j(\xi+k)\right)\overline{\hat{\psi}_l(\xi+k)}\,d\xi.$$

Denote $S = \bigcup_{k\in F_2\cup\{0\}}(\bigcup_{j\in F_1}(A^*)^{-j}K + k)$. Then we deduce that

$$\begin{aligned} &\int_{\mathbb{R}^d} \left|\hat{\psi}_l(\xi)\overline{\hat{g}\left((A^*)^j\xi\right)}\hat{g}\left((A^*)^j\xi + (A^*)^j k\right)\overline{\hat{\psi}_l(\xi+k)}\right| d\xi \\ &\quad\leq \|\hat{g}\|_{L^\infty(\mathbb{R}^d)}^2 \int_{(A^*)^{-j}K} \left|\hat{\psi}_l(\xi)\hat{\psi}_l(\xi+k)\right| d\xi \\ &\quad\leq \|\hat{g}\|_{L^\infty(\mathbb{R}^d)}^2 \left(\int_{(A^*)^{-j}K} \left|\hat{\psi}_l(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{(A^*)^{-j}K} \left|\hat{\psi}_l(\xi+k)\right|^2 d\xi\right)^{\frac{1}{2}} \\ &\quad\leq \|\hat{g}\|_{L^\infty(\mathbb{R}^d)}^2 \int_S \left|\hat{\psi}_l(\xi)\right|^2 d\xi \end{aligned}$$

for each $(j,k) \in F_1 \times F_2$. Also observe the fact $1 \leq (\max_{\xi\in S}(1+|\xi|^2)^{-s})(1+|\xi|^2)^s$ for $\xi \in S$. It follows that

$$\int_{\mathbb{R}^d} \left|\hat{\psi}_l(\xi)\overline{\hat{g}\left((A^*)^j\xi\right)}\hat{g}\left((A^*)^j\xi + (A^*)^j k\right)\overline{\hat{\psi}_l(\xi+k)}\right| d\xi$$

$$\begin{aligned}
&\leq \left(\max_{\xi\in S}\left(1+|\xi|^2\right)^{-s}\right)\|\hat{g}\|^2_{L^\infty(\mathbb{R}^d)}\int_S\left|\hat{\psi}_l(\xi)\right|^2\left(1+|\xi|^2\right)^s d\xi \\
&\leq \left(\max_{\xi\in S}\left(1+|\xi|^2\right)^{-s}\right)\|\hat{g}\|^2_{L^\infty(\mathbb{R}^d)}\|\psi_l\|^2_{H^s(\mathbb{R}^d)} \\
&< \infty.
\end{aligned}$$

Combining the above formula, then we have

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^d}\sum_{l=1}^{L}\sum_{j=0}^{\infty} m^{j(d-2s)}\left|\hat{\psi}_l(\xi)\right|^2\left|\hat{g}\left((A^*)^j\xi\right)\right|^2 d\xi \\
&\quad + \int_{\mathbb{R}^d}\sum_{l=1}^{L}\sum_{j=0}^{\infty} m^{j(d-2s)}\overline{\hat{g}\left((A^*)^j\xi\right)}\hat{\psi}_l(\xi)\sum_{0\neq k\in\mathbb{Z}^d}\hat{g}\left((A^*)^j\xi+(A^*)^jk\right)\overline{\hat{\tilde{\psi}}_l(\xi+k)}\,d\xi \\
&= \int_{\mathbb{R}^d}\sum_{l=1}^{L}\sum_{j=0}^{\infty} m^{-2js}\left|\hat{\psi}_l\left((A^*)^{-j}\xi\right)\right|^2\left|\hat{g}(\xi)\right|^2 d\xi \\
&\quad + \int_{\mathbb{R}^d}\sum_{l=1}^{L}\sum_{j=0}^{\infty} m^{-2js}\overline{\hat{g}(\xi)}\hat{\psi}_l\left((A^*)^{-j}\xi\right)\sum_{0\neq k\in\mathbb{Z}^d}\hat{g}\left(\xi+(A^*)^jk\right)\overline{\hat{\tilde{\psi}}_l\left((A^*)^{-j}\xi+k\right)}\,d\xi \\
&= \int_{\mathbb{R}^d}\sum_{l=1}^{L}\sum_{j=0}^{\infty} m^{-2js}\left|\hat{\psi}_l\left((A^*)^{-j}\xi\right)\right|^2\left|\hat{g}(\xi)\right|^2 d\xi \\
&\quad + \int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\sum_{0\neq k\in\mathbb{Z}^d}\hat{g}(\xi+k)\sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)} m^{-2js}\hat{\psi}_l\left((A^*)^{-j}\xi\right)\overline{\hat{\tilde{\psi}}_l\left((A^*)^{-j}\xi+k\right)}\,d\xi
\end{aligned} \tag{3.4}$$

by the definition of $\kappa(k)$. It leads to (3.1) by (3.2), (3.3) and (3.4). The lemma is proved. □

Lemma 3.2 *Given $s\in\mathbb{R}$, let $X^s(\psi_0;\Psi)$ and $X^{-s}(\tilde{\psi}_0;\tilde{\Psi})$ be Bessel sequences in $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$, respectively. Then*

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\psi}_{0,0,k}\rangle\langle\psi_{0,0,k},g\rangle+\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\psi}^{-s}_{l,j,k}\rangle\langle\psi^s_{l,j,k},g\rangle \\
&\quad = \int_{\mathbb{R}^d}\hat{f}(\xi)\overline{\hat{g}(\xi)}\left(\hat{\psi}_0(\xi)\overline{\hat{\tilde{\psi}}_0(\xi)}+\sum_{l=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}_l\left((A^*)^{-j}\xi\right)\overline{\hat{\tilde{\psi}}_l\left((A^*)^{-j}\xi\right)}\right)d\xi \\
&\qquad + \int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\sum_{0\neq k\in\mathbb{Z}^d}\hat{f}(\xi+k) \\
&\qquad \times\left(\hat{\psi}_0(\xi)\overline{\hat{\tilde{\psi}}_0(\xi+k)}+\sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)}\hat{\psi}_l\left((A^*)^{-j}\xi\right)\overline{\hat{\tilde{\psi}}_l\left((A^*)^{-j}(\xi+k)\right)}\right)d\xi
\end{aligned} \tag{3.5}$$

for $f,g\in\mathcal{D}$.

Proof Since $X^s(\psi_0;\Psi)$ and $X^{-s}(\tilde{\psi}_0;\tilde{\Psi})$ are Bessel sequences in $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$, respectively, the expression of (3.5) is meaningful. According to an argument similar to

Lemma 3.1, we can deduce that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\psi}_{0,0,k}\rangle\langle\psi_{0,0,k},g\rangle+\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\psi}_{l,j,k}^{-s}\rangle\langle\psi_{l,j,k}^{s},g\rangle\\
&=\int_{\mathbb{R}^d}\hat{\psi}_0(\xi)\overline{\hat{g}(\xi)}\sum_{k\in\mathbb{Z}^d}\hat{f}(\xi+k)\overline{\hat{\tilde{\psi}}_0(\xi+k)}\,d\xi\\
&\quad+\sum_{l=1}^{L}\sum_{j=0}^{\infty}|\det A|^j\int_{\mathbb{R}^d}\hat{\psi}_l(\xi)\overline{\hat{g}\big((A^*)^j\xi\big)}\sum_{k\in\mathbb{Z}^d}\hat{f}\big((A^*)^j(\xi+k)\big)\overline{\hat{\tilde{\psi}}_l(\xi+k)}\,d\xi\\
&=:J_1+J_2.
\end{aligned}\tag{3.6}$$

Observe that

$$\big|\hat{\psi}_0(\cdot)\overline{\hat{g}(\cdot)}\big|\sum_{k\in\mathbb{Z}^d}\big|\hat{f}(\cdot+k)\overline{\hat{\tilde{\psi}}_0(\cdot+k)}\big|\le[\hat{f},\hat{f}]_s^{\frac{1}{2}}(\cdot)[\hat{\tilde{\psi}}_0,\hat{\tilde{\psi}}_0]_{-s}^{\frac{1}{2}}(\cdot)[\hat{g},\hat{g}]_{-s}^{\frac{1}{2}}(\cdot)[\hat{\psi}_0,\hat{\psi}_0]_s^{\frac{1}{2}}(\cdot),$$

which is bounded by Lemma 2.2 (i). Then we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\big|\hat{\psi}_0(\xi)\overline{\hat{g}(\xi)}\big|\sum_{k\in\mathbb{Z}^d}\big|\hat{f}(\xi+k)\overline{\hat{\tilde{\psi}}_0(\xi+k)}\big|\,d\xi\\
&\le\int_{\operatorname{supp}(\hat{g})}\big|\hat{\psi}_0(\xi)\overline{\hat{g}(\xi)}\big|\sum_{k\in\mathbb{Z}^d}\big|\hat{f}(\xi+k)\overline{\hat{\tilde{\psi}}_0(\xi+k)}\big|\,d\xi<\infty,
\end{aligned}$$

and thus

$$J_1=\int_{\mathbb{R}^d}\hat{f}(\xi)\overline{\hat{g}(\xi)}\hat{\psi}_0(\xi)\overline{\hat{\tilde{\psi}}_0(\xi)}\,d\xi+\int_{\mathbb{R}^d}\hat{\psi}_0(\xi)\overline{\hat{g}(\xi)}\sum_{0\ne k\in\mathbb{Z}^d}\hat{f}(\xi+k)\overline{\hat{\tilde{\psi}}_0(\xi+k)}\,d\xi.\tag{3.7}$$

Next we discuss J_2 into two parts: the $k=0$ term and $k\ne0$ term. By Lemma 2.4 and the Cauchy–Schwartz inequality, we have

$$\begin{aligned}
&\sum_{l=1}^{L}\sum_{j=0}^{\infty}\big|\hat{\psi}_l\big((A^*)^{-j}\xi\big)\overline{\hat{\tilde{\psi}}_l\big((A^*)^{-j}\xi\big)}\big|\\
&\le\left(\sum_{l=1}^{L}\sum_{j=0}^{\infty}m^{-2js}\big|\hat{\psi}_l\big((A^*)^{-j}\xi\big)\big|^2\right)^{\frac{1}{2}}\left(\sum_{l=1}^{L}\sum_{j=0}^{\infty}m^{2js}\big|\hat{\tilde{\psi}}_l\big((A^*)^{-j}\xi\big)\big|^2\right)^{\frac{1}{2}}\\
&\le B_1B_2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\big|\hat{f}(\xi)\overline{\hat{g}(\xi)}\big|\sum_{l=1}^{L}\sum_{j=0}^{\infty}\big|\hat{\psi}_l\big((A^*)^{-j}\xi\big)\overline{\hat{\tilde{\psi}}_l\big((A^*)^{-j}\xi\big)}\big|\,d\xi\\
&\le B_1B_2\big|\operatorname{supp}(\hat{f})\cap\operatorname{supp}(\hat{g})\big|\|\hat{f}\|_{L^\infty(\mathbb{R}^d)}\|\hat{g}\|_{L^\infty(\mathbb{R}^d)}\\
&<\infty.
\end{aligned}\tag{3.8}$$

Take a compact set $K \in \mathbb{R}^d$ such that $\operatorname{supp}(\hat{f}) \cap \operatorname{supp}(\hat{g}) \subset K$. Applying Lemma 2.6, we have

$$K \cap \big(K + (A^*)^j k\big) = \emptyset \quad \text{for } (j,k) \notin F_1 \times F_2 \text{ with } k \neq 0, \tag{3.9}$$

where $F_1 \subset \mathbb{Z}_+$ and $F_2 \subset \mathbb{Z}^d \setminus \{0\}$ are two finite sets. Using an argument similar to I_2 in Lemma 3.1, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big|\overline{\hat{g}\big((A^*)^j\xi\big)}\hat{f}\big((A^*)^j(\xi+k)\big)\hat{\psi}_l(\xi)\overline{\hat{\tilde{\psi}}_l(\xi+k)}\big|\,d\xi \\
&\quad\le \|\hat{g}\|_{L^\infty(\mathbb{R}^d)}\|\hat{f}\|_{L^\infty(\mathbb{R}^d)}\bigg(\int_{(A^*)^{-j}K}\big|\hat{\psi}_l(\xi)\big|^2\,d\xi\bigg)^{\frac{1}{2}}\bigg(\int_{(A^*)^{-j}K}\big|\hat{\tilde{\psi}}_l(\xi+k)\big|^2\,d\xi\bigg)^{\frac{1}{2}} \\
&\quad\le \|\hat{g}\|_{L^\infty(\mathbb{R}^d)}\|\hat{f}\|_{L^\infty(\mathbb{R}^d)}\bigg(\int_S\big|\hat{\psi}_l(\xi)\big|^2\,d\xi\bigg)^{\frac{1}{2}}\bigg(\int_S\big|\hat{\tilde{\psi}}_l(\xi)\big|^2\,d\xi\bigg)^{\frac{1}{2}} \\
&\quad\le \|\hat{g}\|_{L^\infty(\mathbb{R}^d)}\|\hat{f}\|_{L^\infty(\mathbb{R}^d)}\Big(\max_{\xi\in S}\big(1+|\xi|^2\big)^{-\frac{s}{2}}\Big)\Big(\max_{\xi\in S}\big(1+|\xi|^2\big)^{\frac{s}{2}}\Big)\|\psi_l\|_{H^s(\mathbb{R}^d)}\|\tilde{\psi}_l\|_{H^{-s}(\mathbb{R}^d)} \\
&\quad< \infty
\end{aligned} \tag{3.10}$$

for $(j,k) \in F_1 \times F_2$, where $S = \bigcup_{k\in F_2\cup\{0\}}(\bigcup_{j\in F_1}(A^*)^{-j}K + k)$. According to (3.8) and (3.10), we have

$$\begin{aligned}
J_2 &= \sum_{l=1}^{L}\sum_{j=0}^{\infty}\int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\hat{\psi}_l\big((A^*)^{-j}\xi\big)\sum_{0\neq k\in\mathbb{Z}^d}\hat{f}\big(\xi+(A^*)^jk\big)\overline{\hat{\tilde{\psi}}_l\big((A^*)^{-j}\xi+k\big)}\,d\xi \\
&= \int_{\mathbb{R}^d}\sum_{l=1}^{L}\sum_{j=0}^{\infty}\overline{\hat{g}(\xi)}\hat{\psi}_l\big((A^*)^{-j}\xi\big)\sum_{0\neq k\in\mathbb{Z}^d}\hat{f}\big(\xi+(A^*)^jk\big)\overline{\hat{\tilde{\psi}}_l\big((A^*)^{-j}\xi+k\big)}\,d\xi \\
&= \int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\sum_{0\neq k\in\mathbb{Z}^d}\hat{f}(\xi+k)\sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)}\hat{\psi}_l\big((A^*)^{-j}\xi\big)\overline{\hat{\tilde{\psi}}_l\big((A^*)^{-j}(\xi+k)\big)}\,d\xi,
\end{aligned} \tag{3.11}$$

where we replace $(A^*)^jk$ by k' in the last step. Collecting (3.6), (3.7) and (3.11), we obtain (3.5). The lemma is completed. □

The next theorem presents a characterization of *NWBF*s in $(FH^s(\Omega), FH^{-s}(\Omega))$ via a pair of equations.

Theorem 3.1 *Given $s \in \mathbb{R}$, let $FH^s(\Omega)$ and $FH^{-s}(\Omega)$ be reducing subspaces of $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$, respectively, $\psi_0 \in FH^s(\Omega)$, $\tilde{\psi}_0 \in FH^{-s}(\Omega)$, and $\Psi \subset FH^s(\Omega)$, $\tilde{\Psi} \subset FH^{-s}(\Omega)$. Suppose that $X^s(\psi_0;\Psi)$ and $X^{-s}(\tilde{\psi}_0;\tilde{\Psi})$ are Bessel sequences in $FH^s(\Omega)$ and $FH^{-s}(\Omega)$, respectively. Then $(X^s(\psi_0;\Psi), X^{-s}(\tilde{\psi}_0;\tilde{\Psi}))$ is an* NWBFs *in $(FH^s(\Omega), FH^{-s}(\Omega))$ if and only if*

$$\hat{\psi}_0(\cdot)\overline{\hat{\tilde{\psi}}_0(\cdot+k)} + \sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)}\hat{\psi}_l\big((A^*)^{-j}\cdot\big)\overline{\hat{\tilde{\psi}}_l\big((A^*)^{-j}(\cdot+k)\big)} = \delta_{0,k} \quad \textit{a.e. on } \Omega. \tag{3.12}$$

Proof Since $\mathcal{D} \cap FH^s(\Omega)$ is dense in $FH^s(\Omega)$,

$$\big(X^s(\psi_0;\Psi), X^{-s}(\tilde{\psi}_0;\tilde{\Psi})\big)$$

is an $NWBF$s in $(FH^s(\Omega), FH^{-s}(\Omega))$ if and only if

$$\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\psi}_{0,0,k}\rangle\langle\psi_{0,0,k},g\rangle+\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\psi}_{l,j,k}^{-s}\rangle\langle\psi_{l,j,k}^{s},g\rangle=\langle f,g\rangle$$

for $f\in\mathcal{D}\cap FH^s(\Omega)$ and $g\in\mathcal{D}\cap FH^{-s}(\Omega)$, equivalently,

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^d}\langle(\hat{f}\chi_\Omega)^\vee,\tilde{\psi}_{0,0,k}\rangle\langle\psi_{0,0,k},(\hat{g}\chi_\Omega)^\vee\rangle+\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\langle(\hat{f}\chi_\Omega)^\vee,\tilde{\psi}_{l,j,k}^{-s}\rangle\langle\psi_{l,j,k}^{s},(\hat{f}\chi_\Omega)^\vee\rangle\\&\quad=\langle(\hat{f}\chi_\Omega)^\vee,(\hat{g}\chi_\Omega)^\vee\rangle\end{aligned}\tag{3.13}$$

for $f,g\in\mathcal{D}$ due to the fact $\mathcal{D}\cap FH^s(\Omega)=\{(\hat{h}\chi_\Omega)^\vee : h\in\mathcal{D}\}$. In view of $X^s(\psi_0;\Psi)$ and $X^{-s}(\tilde{\psi}_0;\tilde{\Psi})$ being Bessel sequences in $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$, respectively, we know that the expression of (3.13) is well-defined. By Lemma 3.2, (3.13) can be rewritten as

$$\begin{aligned}&\int_{\mathbb{R}^d}\hat{f}(\xi)\overline{\hat{g}(\xi)}\chi_\Omega(\xi)\left(\hat{\psi}_0(\xi)\overline{\hat{\tilde{\psi}}_0(\xi)}+\sum_{l=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}_l\big((A^*)^{-j}\xi\big)\overline{\hat{\tilde{\psi}}_l\big((A^*)^{-j}\xi\big)}\right)d\xi\\&\quad+\int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\chi_\Omega(\xi)\sum_{0\neq k\in\mathbb{Z}^d}(\hat{f}\chi_\Omega)(\xi+k)\\&\quad\times\left(\hat{\psi}_0(\xi)\overline{\hat{\tilde{\psi}}_0(\xi+k)}+\sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)}\hat{\psi}_l\big((A^*)^{-j}\xi\big)\overline{\hat{\tilde{\psi}}_l\big((A^*)^{-j}(\xi+k)\big)}\right)d\xi\\&=\int_{\mathbb{R}^d}\hat{f}(\xi)\overline{\hat{g}(\xi)}\chi_\Omega(\xi)\,d\xi\end{aligned}\tag{3.14}$$

for $f,g\in\mathcal{D}$. Obviously, (3.12) leads to (3.14). Now, to finish the proof, we prove the converse statement. Assume that (3.14) holds. Applying the Cauchy–Schwartz inequality, we get

$$\begin{aligned}&\big|\hat{\psi}_0(\cdot)\overline{\hat{\tilde{\psi}}_0(\cdot+k)}\big|+\sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)}\big|\hat{\psi}_l\big((A^*)^{-j}\cdot\big)\overline{\hat{\tilde{\psi}}_l\big((A^*)^{-j}(\cdot+k)\big)}\big|\\&\leq\left(\big|\hat{\psi}_0(\cdot)\big|^2+\sum_{l=1}^{L}\sum_{j=0}^{\infty}m^{-2js}\big|\hat{\psi}_l\big((A^*)^{-j}\cdot\big)\big|^2\right)^{\frac{1}{2}}\\&\quad\times\left(\big|\hat{\tilde{\psi}}_0(\cdot+k)\big|^2+\sum_{l=1}^{L}\sum_{j=0}^{\infty}m^{2js}\big|\hat{\tilde{\psi}}_l\big((A^*)^{-j}(\xi+k)\big)\big|^2\right)^{\frac{1}{2}}\\&\leq B_1B_2\big(1+|\cdot|^2\big)^{-s}\big(1+|\cdot+k|^2\big)^{s}\\&=C_k<\infty\end{aligned}$$

for each $k\in\mathbb{Z}^d$ by Lemma 2.4. Thus the series $\hat{\psi}_0(\cdot)\overline{\hat{\tilde{\psi}}_0(\cdot+k)}+\sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)}\hat{\psi}_l((A^*)^{-j}\cdot)\times\overline{\hat{\tilde{\psi}}_l((A^*)^{-j}(\cdot+k))}$ converges absolutely a.e. on $\mathbb{R}^d$ and belongs to $L^\infty(\mathbb{R}^d)$ for every $k\in\mathbb{Z}^d$. So almost all points in $\mathbb{R}^d$ are its Lebesgue points. Next, we deal with it for two cases.

When $k = 0$. Let $\xi_0 \neq 0$ be a Lebesgue point of $\hat{\psi}_0(\cdot)\overline{\hat{\tilde{\psi}}_0(\cdot)} + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}_l((A^*)^{-j}\cdot)\overline{\hat{\tilde{\psi}}_l((A^*)^{-j}\cdot)}$ and $\chi_\Omega(\cdot)$. For $0 < \epsilon < \frac{1}{2}$, take f and g such that

$$\hat{f}(\cdot) = \hat{g}(\cdot) = \frac{\chi_{B(\xi_0,\epsilon)}(\cdot)}{\sqrt{|B(\xi_0,\epsilon)|}}$$

in (3.14), where $B(\xi_0,\epsilon) = \{\xi \in \mathbb{R}^d : |\xi - \xi_0| < \epsilon\}$. Then

$$\begin{aligned}&\frac{1}{|B(\xi_0,\epsilon)|}\int_{B(\xi_0,\epsilon)}\chi_\Omega(\xi)\left(\hat{\psi}_0(\xi)\overline{\hat{\tilde{\psi}}_0(\xi)} + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}_l\big((A^*)^{-j}\xi\big)\overline{\hat{\tilde{\psi}}_l\big((A^*)^{-j}\xi\big)}\right)d\xi\\ &\quad= \frac{1}{|B(\xi_0,\epsilon)|}\int_{B(\xi_0,\epsilon)}\chi_\Omega(\xi)\,d\xi,\end{aligned}$$

and letting $\epsilon \to 0$ we obtain

$$\hat{\psi}_0(\xi_0)\overline{\hat{\tilde{\psi}}_0(\xi_0)} + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}_l\big((A^*)^{-j}\xi_0\big)\overline{\hat{\tilde{\psi}}_l\big((A^*)^{-j}\xi_0\big)} = 1.$$

The case of $k \neq 0$: we fix $0 \neq k_0 \in \mathbb{Z}^d$, take f and g such that

$$\hat{f}(\cdot + k_0) = \hat{g}(\cdot) = \frac{\chi_{B(\xi_0,\epsilon)}(\cdot)}{\sqrt{|B(\xi_0,\epsilon)|}}$$

in (3.14), where $0 < \epsilon < \frac{1}{2}$. Then

$$\begin{aligned}&\frac{1}{|B(\xi_0,\epsilon)|}\int_{B(\xi_0,\epsilon)}\chi_\Omega(\xi)\\ &\quad\times\left(\hat{\psi}_0(\xi)\overline{\hat{\tilde{\psi}}_0(\xi + k_0)} + \sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)}\hat{\psi}_l\big((A^*)^{-j}\xi\big)\overline{\hat{\tilde{\psi}}_l\big((A^*)^{-j}(\xi + k_0)\big)}\right)d\xi = 0,\end{aligned}$$

letting $\epsilon \to 0$ and applying the Lebesgue differentiation theorem, we obtain

$$\hat{\psi}_0(\xi_0)\overline{\hat{\tilde{\psi}}_0(\xi_0 + k_0)} + \sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)}\hat{\psi}_l\big((A^*)^{-j}\xi_0\big)\overline{\hat{\tilde{\psi}}_l\big((A^*)^{-j}(\xi_0 + k_0)\big)} = 0.$$

Due to the arbitrariness of ξ_0 and k_0, we obtain (3.12). The theorem is proved. □

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