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Some remarks on the sparse dominations for commutators of multi(sub)linear operator

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Abstract

We establish pointwise sparse dominations for the iterated commutators of multi(sub)linear operators satisfying the W_q condition. As consequences, we present some quantitative weighted estimates for the commutators. In addition, we also obtain the Fefferman–Stein inequality, the Coifman–Fefferman inequality, and the local decay estimates regarding the iterated commutators.

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1 Introduction

Sparse domination is a relatively new tool to prove weighted estimates for singular integrals. The sparse method has been developed over the past five years by many researchers in harmonic analysis, with significant works by Lerner [1], Lacey [2], Conde-Alonso [3], and the references therein.

A collection $\mathcal{S}$ is an η-sparse family of cubes in $\mathbb{R}^d$ if for every $Q \in \mathcal{S}$ there exists $E_Q \subset Q$ such that $|E_Q| \geq \eta|Q|$ (here $0 < \eta < 1$), and $E_Q \cap E_{Q'} = \varnothing$ when $Q \neq Q'$. For an η-sparse collection of cubes $\mathcal{S}$, we use the notation

$$\langle f \rangle_{p,Q} := |Q|^{-\frac{1}{p}} \|f\chi_Q\|_{L^p}.$$

This averaging form is then easily controlled, facilitating the proof of weighted A_p-type estimates.

Lerner [1] proved a pointwise sparse domination for ω Calderón–Zygmund operators. The key role in his proof is played by the grand maximal operator

$$\mathcal{M}_T(f)(x) := \sup_{Q \ni x} \operatorname*{esssup}_{\xi \in Q} \left| T\big(f\mathbb{R}^d \backslash 3Q\big)(\xi)\right|.$$

Later, Li [4] established sparse domination theorem for multilinear singular integral operators with the kernel satisfying the L^r-Hörmander condition. Cao and Yabuta [5] developed sparse dominations for the multilinear Littlewood–Paley operators with the same kernel

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condition. Wen, Wu, and Xue [6] gave a sparse domination for the iterated commutators of multilinear pseudo-differential operators. Recently Lerner and Ombrosi [7] improved the results in [1] by weakening the assumption on T and by replacing $\mathcal{M}_T(f)(x)$ with a more flexible operator. Motivated by the above works, the purpose of this paper is to establish a sparse domination for the iterated commutators of multi(sub)linear operator with weaker hypotheses than [4, 5].

Motivated by Lerner and Ombrosi [7], we assume that T is an operator satisfying the following W_q property instead of assuming T is bounded from $L^q \times \cdots \times L^q \to L^{q/m,\infty}$: there is a nonincreasing function $\psi_{T,q}$ such that, for any $f_j \in L^q(Q)$ with $j = 1, \dots, m$ and any cube Q,

$$\left| \left\{ x \in Q : \left| T(\vec{f}\chi_Q)(x) \right| > \psi_{T,q}(\lambda) \prod_{j=1}^{m} \langle f_j \rangle_{q,Q} \right\} \right| \le \lambda |Q| \quad (0 < \lambda < 1). \tag{1.1}$$

It is easy to see that $L^q \times \cdots \times L^q \to L^{q/m,\infty}$ implies that T satisfies the W_q property with $\psi_{T,q} = \|T\|_{L^q \times \cdots \times L^q \to L^{q/m,\infty}} \lambda^{-m/q}$.

Let $\alpha > 0$, we define

$$\mathcal{M}_{T,\alpha}^{\sharp}(\vec{f})(x) := \sup_{Q \ni x} \operatorname*{esssup}_{\xi,\xi' \in Q} \left| T(\vec{f}\chi_{\mathbb{R}^d \setminus \alpha Q})(\xi) - T(\vec{f}\chi_{\mathbb{R}^d \setminus \alpha Q})(\xi') \right|.$$

Given an operator T, $\alpha > 0$, the iterated commutators of T are defined by

$$T_{\vec{b}}^{\ell}(\vec{f})(x) = \left[b_1, \left[b_2, \dots [b_\ell, T]_\ell \cdots \right]_2 \right]_1 (\vec{f})(x), \tag{1.2}$$

where $\vec{b} = \vec{b}^{(\ell)} = (b_1, \dots, b_\ell)$ $(1 \le \ell \le m)$. Throughout this paper, $\tau_m = \{1, \dots, m\}$. The symbol $|\tau|$ denotes the number of the elements in τ. $\tau' = \tau_m \setminus \tau$ is the complementary set.

Our main results of this paper are as follows.

Theorem 1.1 *Assume that the multi(sub)linear integral T satisfies the W_q condition and $\mathcal{M}_{T,\alpha}^{\sharp}$ is bounded from $L^r \times \cdots \times L^r \to L^{r/m,\infty}$ for some $\alpha \ge 3$. Let $1 \le q, r < \infty$, and $s = \max\{q, r\}$. Then, for any compactly supported functions $f_i \in L^s(\mathbb{R}^d)$, $i = 1, \dots, m$, there exist 3^d sparse families $\mathcal{S}_j$ such that, for a.e. $x \in \mathbb{R}^d$,*

$$\left| T_{\vec{b}}^{\ell}(\vec{f})(x) \right|$$
$$\le C \sum_{j=1}^{3^d} \sum_{\tau \subset \tau_\ell} \sum_{Q \in \mathcal{S}_j} \prod_{i \in \tau} \langle f_i \rangle_{s,Q} \left| b_i(x) - b_{i,Q} \right| \times \prod_{k \in \tau_\ell \setminus \tau} \left\langle (b_k - b_{k,Q}) f_k \right\rangle_{s,Q} \times \prod_{t \in \tau_m \setminus \tau_\ell} \langle f_t \rangle_{s,Q} \chi_Q,$$

where $C = c_{d,s,\alpha}(\psi_{T,q}(1/12 \cdot (2\alpha)^d) + \|\mathcal{M}_{T,\alpha}^{\sharp}\|_{L^r \times \cdots \times L^r \to L^{r/m,\infty}})$.

Remark 1.1 Our Theorem 1.1 is the commutators result of [7]. Compared with the hypotheses in [4, 5], the $\mathcal{M}_{T,\alpha}^{\sharp}$ makes our proof concise and clear. Further, the W_q condition of T is weaker than the assumption $L^q \times \cdots \times L^q \to L^{q/m,\infty}$ of T in [4, 5].

Let $1 \le p_1,\ldots,p_m \le \infty$, $\vec{w} = (w_1,\ldots,w_m)$, and each w_i is a nonnegative function on $\mathbb{R}^d$. $\vec{w}$ is said to satisfy the following $A_{\vec{p}/r}$ condition if

$$[\vec{w}]_{A_{\vec{p}/r}} := \sup_Q \left(\frac{1}{|Q|} \int_Q v_{\vec{w}} \, dx \right) \prod_{k=1}^m \left(\frac{1}{|Q|} \int_Q w_k^{-\frac{r}{p_i-r}}(x) \, dx \right)^{\frac{p(p_i-r)}{p_i r}},$$

where $v_{\vec{w}} = \prod_{k=1}^m w_k^{p/p_k}$. When $r = 1$, $A_{\vec{p}/r}$ is the $A_{\vec{p}}$ weight class defined by Lerner et al. [8].

With the pointwise sparse domination result, we can obtain the following quantitative weighted estimates and endpoint estimates from Sect. 3 in [5] immediately.

Corollary 1.2 *Assume that the multi(sub)linear integral T satisfies the W_q condition and $\mathcal{M}_{T,\alpha}^{\sharp}$ is bounded from $L^r \times \cdots \times L^r \to L^{r/m,\infty}$ for some $\alpha \ge 3$. Let $1 \le q, r < \infty$, and $s = \max\{q,r\}$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 \le s < p_i < \infty$. Set $\sigma_i = w_i^{1-(p_i/s)'}$, $i = 1,\ldots,m$. Let $T_{\vec{b}}^{\ell}(\vec{f})$ be defined as* (1.2).

(1) *For $\vec{w} \in A_{\vec{p}/s}$, there exists a constant $C = C_{m,\vec{p},d,s}$ such that*

$$\left\| T_{\vec{b}}^{\ell}(\vec{f}) \right\|_{L^p(v_{\vec{w}})}$$
$$\le C[\vec{w}]_{A_{\vec{p}/s}}^{\max_{1\le i \le m}\{1,\frac{1}{p}(\frac{p_i}{s})'\}} \left(\sum_{\tau \subset \tau_\ell} [v_{\vec{w}}]_{A_\infty}^{|\tau|} \prod_{j \in \tau_\ell \setminus \tau} [\sigma_j]_{A_\infty} \right) \prod_{i=1}^{\ell} \|b_i\|_{BMO} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}.$$

(2) *If $\vec{w} \in A_{\vec{1}}$, then for any $\lambda > 0$ and $\Phi_{s,\ell} = t^s(1 + \log^+ t)^{s\ell}$ it holds that*

$$v_{\vec{w}}\left(\{x : \left| T_{\vec{b}}^{\ell}(\vec{f}) \right| > \lambda^m \} \right) \le C_{\vec{b},s,\ell} \prod_{i=1}^m \left(\int_{\mathbb{R}^d} \Phi_{s,\ell}\left(\frac{|f_i(x)|}{\lambda} \right) w_i \, dx \right)^{1/m}.$$

We also obtain the local decay estimate, the Coifman–Fefferman inequality with $w \in A_\infty$ weight, and the Fefferman–Stein inequality with arbitrary weights regarding the iterated commutators. To the best knowledge of the author, these results are new for the iterated commutators of multi(sub)linear operator.

Theorem 1.3 *Assume that the multi(sub)linear integral T satisfies the W_q condition and $\mathcal{M}_{T,\alpha}^{\sharp}$ is bounded from $L^r \times \cdots \times L^r \to L^{r/m,\infty}$ for some $\alpha \ge 3$. Let $1 \le q, r < \infty$, $s = \max\{q,r\}$, and $a > s$, then for any functions $\operatorname{supp} f_i \subset Q$, $i = 1,\ldots,m$, there exist constants $\alpha_{d,m}$ and $c_{d,m}$ such that*

$$\left| \left\{ x \in Q : \frac{|T_{\vec{b}}^{\ell}(\vec{f})|}{M_a(\vec{f})} > \lambda \right\} \right| \le \alpha_{d,m} e^{-2c_{d,m} \frac{\lambda}{\prod_{i \in \tau_\ell} \|b_i\|_{BMO^{c_T}}}} |Q|, \quad \lambda > 0.$$

Theorem 1.4 *Assume that the multi(sub)linear integral T satisfies the W_q condition and $\mathcal{M}_{T,\alpha}^{\sharp}$ is bounded from $L^r \times \cdots \times L^r \to L^{r/m,\infty}$ for some $\alpha \ge 3$. Let $1 \le q, r < \infty$, $s = \max\{q,r\}$, and $a > s$, then for any $1 \le p < \infty$ and any weight $w \in A_\infty$,*

$$\left\| T_{\vec{b}}^{\ell}(\vec{f}) \right\|_{L^p(w)} \le c_{d,m} \sum_{\tau \subset \tau_\ell} [w]_{A_\infty}^{|\tau|} \prod_{i \in \tau_\ell} \|b_i\|_{BMO} \left\| M_a(\vec{f}) \right\|_{L^p(w)}.$$

Theorem 1.5 *Assume that the multi(sub)linear integral T satisfies the W_q condition and $\mathcal{M}^{\sharp}_{T,\alpha}$ is bounded from $L^r \times \cdots \times L^r \to L^{r/m,\infty}$ for some $\alpha \geq 3$. Let $1 \leq q, r < \infty$, the exponents $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $p > s$ and $1 < p_1, \ldots, p_m < \infty$, then for all weights $\vec{w} = (w_1, \ldots, w_m)$, $\nu_{\vec{w}} = \prod_{i=1}^{m} w_i^{p/p_i}$,*

$$\big\| T_b^{\ell}(\vec{f}) \big\|_{L^p(\nu_{\vec{w}})} \lesssim \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(Mw_i)}. \tag{1.3}$$

The article is organized as follows. Section 2 contains some definitions and main lemmas. The proof of theorems is given in Sect. 3. We present some variations of Theorem 1.1 in Sect. 4.

2 Definitions and main lemmas

We begin by introducing some definitions and notations.

Lemma 2.1 (Generalized Hölder inequality, [9]) *Let $\Phi_0, \Phi_1, , \ldots, \Phi_k$ be Young functions. If*

$$\Phi_1^{-1}(t)\Phi_2^{-1}(t) \cdots \Phi_k^{-1}(t) \leq \kappa \Phi_0^{-1}(t),$$

then the following inequality holds:

$$\|f_1 f_2 \cdots f_k\|_{\Phi_0,Q} \leq \kappa \|f_1\|_{\Phi_1,Q} \|f_2\|_{\Phi_2,Q} \cdots \|f_k\|_{\Phi_k,Q}$$

for all functions $f_1, \ldots, f_m$ and all cubes Q.

In particular, if $\sum_{i=1}^{k} \frac{1}{s_i} = \frac{1}{s}$ with each $s_i \geq 1$, then it holds that

$$\frac{1}{Q} \int_Q |f_1 f_2 \cdots f_k g| \leq C_s \|f_1\|_{\exp L^{s_1},Q} \|f_2\|_{\exp L^{s_2},Q} \cdots \|f_k\|_{\exp L^{s_k},Q} \|g\|_{L(\log L)^{\frac{1}{s}},Q}.$$

We introduce the weighted maximal operator and the multi(sub)linear maximal operator which will be used in the proof of our theorem. Let w be a weight and $a \geq 1$,

$$M_w f(x) = \sup_{Q \ni x} \frac{1}{w(Q)} \int_Q |f| w \, dy, \qquad M_a(\vec{f}) = \sup_{Q \ni x} \prod_{i=1}^{m} \left(\frac{1}{|Q|} \int_Q |f_i|^a dy_i \right)^{\frac{1}{a}}.$$

Lemma 2.2 ([5]) *Let $s > 1$, $t > 0$, and $w \in A_\infty$, then there holds that*

$$\|fw\|_{L(\log L)^t,Q} \lesssim [w]_{A_\infty}^t \inf_{x \in Q} M_w(|f|^s)^{1/s} \langle w \rangle_Q.$$

Definition 2.1 The sidelength of Q is denoted by $\ell(Q)$. Given a cube $Q_0 \subset \mathbb{R}^d$, let $\mathcal{D}(Q_0)$ denote the generation of Q_0, that is, the cubes obtained by repeated subdivision of Q_0. Given dyadic grids $\mathcal{D}$, for any $j \in \mathbb{Z}$, the set $\mathcal{D}_j = \{Q \in \mathcal{D}, \ell(Q) = 2^j\}$ forms a partition of $\mathbb{R}^d$.

3 Proof of theorems

Now we devote to proving Theorem 1.1 with the case $\ell = m$. The other cases are similar so we omit their proof here. The basic idea of our proof is borrowed from Sect. 3 in [5], but the definition of W_q and $\mathcal{M}_{T,\alpha}^{\sharp}$ makes our proof more convenient than [5].

Proof of Theorem 1.1 Fix a cube $Q_0 \in \mathbb{R}^d$ and let $Q_0^* = \alpha Q_0$. Set

$$\widetilde{M}_T(\vec{f}) = \max\big(\big|T(\vec{f})\big|, \mathcal{M}_{T,\alpha}^{\sharp}(\vec{f})\big).$$

We define the set E as follows:

$$E = \left\{ x \in Q_0 : \max\left(\frac{M_s(\vec{f}\chi_{Q_0^*})(x)}{c}, \frac{\widetilde{M}_T(\vec{f})}{A} \right) > \prod_{j=1}^{m} \langle f_j \rangle_{s, Q_0^*} \right\}.$$

By the weak endpoint estimates of $M_s(\vec{f})$ and (1.1), we can choose $c = c_{d,s,\alpha}$ and $A = 2\psi_{T,q}(1/12 \cdot (2\alpha)^d) + \|\mathcal{M}_{T,\alpha}^{\sharp}\|_{L^r \to L^{r,\infty}}$ such that $|E| \leq \frac{1}{2^{d+2}}|Q_0|$. Then, applying the local Calderón–Zygmund decomposition to χ_E on Q_0 at $\lambda = \frac{1}{2^{d+1}}$, we can get a family of pairwisely disjoint cubes $\{P_l\} \subset \mathcal{D}(Q_0)$ such that

$$\frac{1}{2^{d+1}|P_l|} < |E \cap P_l| \leq \frac{1}{2}|P_l|, \qquad \left| E \setminus \bigcup_l P_l \right| = 0.$$

It is easy to have that $\sum_l |P_l| \leq \frac{1}{2}|Q_0|$ and $P_l \bigcap E^c \neq \emptyset$.

$$
\begin{aligned}
&\big|T_{\vec{b}}(\vec{f}\chi_{Q_0^*})(x)\big|\chi_{Q_0}(x) \\
&\quad \leq \big|T_{\vec{b}}(\vec{f}\chi_{Q_0^*})(x)\big|\chi_{Q_0 \setminus \cup_l P_l} + \sum_l \big|T_{\vec{b}}(\vec{f}\chi_{Q_0^* \setminus P_l^*})(x)\big|\chi_{P_l} + \sum_l \big|T_{\vec{b}}(\vec{f}\chi_{P_l^*})(x)\big|\chi_{P_l} \\
&\quad := I + II + III.
\end{aligned}
\tag{3.1}
$$

From [10, Remark 5.1], there exist 3^d dyadic lattices $\mathcal{D}_j$ such that for every cube $Q \subset \mathbb{R}^d$ we can find a cube $R_Q \in \mathcal{D}_j$ satisfying $3Q \subset R_Q$ and $|R_Q| \leq 9^n|Q|$. Note that $\prod_{i=1}^{m}(b_i(x) - b_i(y_i)) = \sum_{\tau \subset \tau_m} \prod_{i \in \tau}(b_i(x) - b_{i,R_{Q_0}}) \prod_{j \in \tau'}(b_{j,R_{Q_0}} - b_j(y_j))$. Then we can write

$$T_{\vec{b}}(\vec{f})(x) = \sum_{\tau \subset \tau_m} \prod_{i \in \tau}\big(b_i(x) - b_{i,R_{Q_0}}\big) T\left(\prod_{i \in \tau} f_i(y_i) \prod_{j \in \tau'}(b_{j,R_{Q_0}} - b_j(y_j))f_j(y_j) \right)(x).$$

Since $|E \setminus \bigcup_l P_l| = 0$, it follows that

$$I \lesssim A \sum_{\tau \subset \tau_m} \prod_{i \in \tau} \langle f_i \rangle_{s, Q_0^*} \big|b_i(x) - b_{i,R_{Q_0}}\big| \prod_{j \in \tau'} \langle (b_j - b_{j,R_{Q_0}})f_j \rangle_{s, Q_0^*} \chi_{Q_0}.
\tag{3.2}$$

Now we calculate II in (3.1). For $x \in P_l$ and $x' \in P_l \setminus E$,

$$
\begin{aligned}
&\sum_l \big|T_{\vec{b}}(\vec{f}\chi_{Q_0^* \setminus P_l^*})(x)\big|\chi_{P_l} \\
&\quad \leq \sum_l \sum_{\tau \subset \tau_m} \prod_{i \in \tau}\big|b_i(x) - b_{i,R_{Q_0}}\big| \left| T\left(\prod_{i \in \tau} f_i \chi_{Q_0^* \setminus P_l^*} \prod_{j \in \tau'}(b_{j,R_{Q_0}} - b_j)f_j \chi_{Q_0^* \setminus P_l^*} \right)(x) \right|.
\end{aligned}
$$

Denoting $\vec{g}\chi_{Q_0^*\setminus P_l^*}(x) = \prod_{i\in\tau} f_i \chi_{Q_0^*\setminus P_l^*} \prod_{j\in\tau'}(b_{j,R_{Q_0}} - b_j)f_j \chi_{Q_0^*\setminus P_l^*}(x)$, we can write

$$\left| T\vec{g}\chi_{Q_0^*\setminus P_l^*}(x) \right| \leq \inf_{P_l} \mathcal{M}_{T,\alpha}^\sharp (\vec{g}\chi_{Q_0^*\setminus P_l^*})(x) + \left| T(\vec{g}\chi_{Q_0^*\setminus P_l^*})(x') \right|$$

$$\lesssim A \prod_{i\in\tau} \langle f_i \rangle_{s,Q_0^*} \prod_{j\in\tau'} \langle (b_j - b_{j,R_{Q_0}})f_j \rangle_{s,Q_0^*} + \left| T(\vec{g}\chi_{Q_0^*})(x') \right| + \left| T(\vec{g}\chi_{P_l^*})(x') \right|$$

$$\lesssim 2A \prod_{i\in\tau} \langle f_i \rangle_{s,Q_0^*} \prod_{j\in\tau'} \langle (b_j - b_{j,R_{Q_0}})f_j \rangle_{s,Q_0^*} + \left| T(\vec{g}\chi_{P_l^*})(x') \right|. \tag{3.3}$$

By $|P_l \setminus E| \geq \frac{|P_l|}{2}$ and $|\{x \in P_l : |T(\vec{f}\chi_{P_l^*})(x')| > A \prod_{j=1}^m \langle f_j \rangle_{s,P_l^*}\}| \leq \frac{|P_l|}{2^{d+2}}$, we can get that

$$\inf_{P_l\setminus E} \left| T(\vec{g}\chi_{P_l^*}) \right| \leq A \prod_{j=1}^m \langle g_j \rangle_{s,P_l^*} \leq cA \prod_{j=1}^m \langle g_j \rangle_{s,Q_0^*}.$$

This allows us to continue (3.3) with

$$\left| T\vec{g}\chi_{Q_0^*\setminus P_l^*}(x) \right| \lesssim (2+c)A \prod_{i\in\tau} \langle f_i \rangle_{s,Q_0^*} \prod_{j\in\tau'} \langle (b_j - b_{j,R_{Q_0}})f_j \rangle_{s,Q_0^*}.$$

Therefore

$$II \lesssim (2+c)A \sum_{\tau\subset\tau_m} \prod_{i\in\tau} \langle f_i \rangle_{s,Q_0^*} \left| b_i(x) - b_{i,R_{Q_0}} \right| \prod_{j\in\tau'} \langle (b_j - b_{j,R_{Q_0}})f_j \rangle_{s,Q_0^*} \chi_{Q_0}. \tag{3.4}$$

III in (3.1) is the term we need. Combining (3.2) with (3.4), it follows that

$$\left| T_{\vec{b}}(\vec{f}\chi_{Q_0^*}) \right| \chi_{Q_0}(x) \lesssim (3+c)A \sum_{\tau\subset\tau_m} \prod_{i\in\tau} \langle f_i \rangle_{s,Q_0^*} \left| b_i(x) - b_{i,R_{Q_0}} \right| \prod_{j\in\tau'} \langle (b_j - b_{j,R_{Q_0}})f_j \rangle_{s,Q_0^*} \chi_{Q_0}$$

$$+ \sum_l \left| T_{\vec{b}}(\vec{f}\chi_{P_l^*})(x) \right| \chi_{P_l}.$$

Integrating the above estimates, we can get a $\frac{1}{2}$-sparse family $\mathcal{F} \subset \mathcal{D}(Q_0)$ such that for a.e. $x \in Q_0$

$$\left| T_{\vec{b}}(\vec{f}\chi_{Q_0^*}) \right| \lesssim (3+c)A \sum_{Q\in\mathcal{F}} \sum_{\tau\subset\tau_m} \prod_{i\in\tau} \langle f_i \rangle_{s,Q^*} \left| b_i(x) - b_{i,R_Q} \right| \prod_{j\in\tau'} \langle (b_j - b_{j,R_Q})f_j \rangle_{s,Q^*} \chi_Q. \tag{3.5}$$

The remaining procedure which transfers local setting Q_0 to global setting $\mathbb{R}^d$ can be referred to Sect. 4.2 in [5]. We omit the details to avoid redundancy. The proof of Theorem 1.1 is finished now. $\qquad\square$

Proof of Theorem 1.3 Let $\ell = m$. The proof of other cases is similar. Assume supp$f_i \subset Q_0$, $i = 1,\ldots,m$, and denote

$$\mathcal{A}_{\mathcal{F},\tau}(\vec{b},\vec{f}) := \sum_{Q\in\mathcal{F}} \prod_{i\in\tau} \langle f_i \rangle_{s,Q^*} \left| b_i(x) - b_{i,R_Q} \right| \prod_{j\in\tau'} \langle (b_j - b_{j,R_Q})f_j \rangle_{s,Q^*} \chi_Q.$$

It is straightforward to see that we can replace b_{R_Q} with b_{Q^*} in $\mathcal{A}_{\mathcal{F},\tau}(\vec{b},\vec{f})$. Then, by (3.5), we can get a $\frac{1}{2}$-sparse family $\mathcal{F} \subset \mathcal{D}(Q_0)$ such that, for almost every $x \in Q_0$,

$$\left| T_{\vec{b}}(\vec{f}) \right| = \left| T_{\vec{b}}(\vec{f}\chi_{Q_0^*}) \right| \lesssim \sum_{\tau \subset \tau_m} \mathcal{A}_{\mathcal{F},\tau}(\vec{b},\vec{f}).$$

First we observe that $|b_i(x) - b_{i,Q^*}| \lesssim \|b_i\|_{BMO} + |b_i(x) - b_{i,Q}|$. From [11, Lemma 5.1], we can construct a sparse family $\tilde{\mathcal{F}} \subset \mathcal{D}(Q_0)$ such that, for every $Q \in \mathcal{F} \subset \tilde{\mathcal{F}}$,

$$\left| b_i(x) - b_{i,Q} \right| \lesssim c_d \sum_{P \in \tilde{\mathcal{F}}, P \subseteq Q} \left(\frac{1}{|P|} \int_P |b_i - b_{i,P}| \, dx \right) \chi_P(x) \lesssim c_d \|b_i\|_{BMO} \sum_{P \in \tilde{\mathcal{F}}, P \subseteq Q_0} \chi_P(x).$$

For any $a > s$, Lemma 2.1 gives that

$$\left\langle (b_j - b_{j,Q^*}) f_j \right\rangle_{s,Q^*} \lesssim \|b_j\|_{BMO} \langle f_j \rangle_{a,Q^*}.$$

Now we can write

$$\mathcal{A}_{\mathcal{F},\tau}(\vec{b},\vec{f})$$

$$\lesssim \sum_{Q \in \mathcal{F}} \prod_{i=1}^{m} \langle f_i \rangle_{a,Q^*} \|b_i\|_{BMO} \chi_Q + c_{d,m} \sum_{Q \in \mathcal{F}} \prod_{i=1}^{m} \langle f_i \rangle_{a,Q^*} \|b_i\|_{BMO} \sum_{P \in \tilde{\mathcal{F}}, P \subseteq Q_0} \chi_P(x) \chi_Q.$$

Hence, it is straightforward to have that

$$\frac{\mathcal{A}_{\mathcal{F},\tau}(\vec{b},\vec{f})(x)}{M_a\vec{f}(x)} \le c_{d,m} \sum_{Q \in \mathcal{F}} \prod_{i=1}^{m} \|b_i\|_{BMO} \chi_Q + c_{d,m} \sum_{Q \in \mathcal{F}} \prod_{i=1}^{m} \|b_i\|_{BMO} \sum_{P \in \tilde{\mathcal{F}}, P \subseteq Q_0} \chi_P(x) \chi_Q$$

$$\le c_{d,m} \prod_{i=1}^{m} \|b_i\|_{BMO} \sum_{P \in \tilde{\mathcal{F}}, P \subseteq Q_0} \chi_P(x).$$

[12, Lemma 2.1] gives that

$$\left| \left\{ x \in Q_0 : \frac{\mathcal{A}_{\mathcal{F},\tau}(\vec{b},\vec{f})}{M_a(\vec{f})} > \lambda \right\} \right| \le \left| \left\{ x \in Q_0 : \frac{\sum_{P \in \tilde{\mathcal{F}}, P \subseteq Q_0} \chi_P(x)}{M_a\vec{f}} > \frac{\lambda}{c_{d,m} \prod_{i=1}^{m} \|b_i\|_{BMO} c_T} \right\} \right|$$

$$\le c e^{-\alpha_{d,m} \frac{\lambda}{\prod_{i=1}^{m} \|b_i\|_{BMO} c_T}} |Q_0|.$$

This finishes the proof of Theorem 1.3. □

Proof of Theorem 1.4 Let $\ell = m$. From Theorem 1.1, it only needs to control $\mathcal{A}_{\mathcal{S},\tau}(\vec{b},\vec{f})$. By duality, there exists a nonnegative function $g \in L^{p'}(w)$ satisfying $\|g\|_{L^{p'}(w)} = 1$. Then we can write

$$\int_{\mathbb{R}^n} \mathcal{A}_{\mathcal{S},\tau}(\vec{b},\vec{f}) g w \, dx$$

$$= \sum_{Q \in \mathcal{S}} \prod_{i \in \tau} \langle f_i \rangle_{s,Q} \int_Q \prod_{i \in \tau} |b_i(x) - b_{i,Q}| g(x) w(x) \, dx \prod_{j \in \tau'} \langle (b_j - b_{j,Q}) f_j \rangle_{s,Q}$$

$$\lesssim \sum_{Q\in\mathcal{S}} \prod_{i\in\tau} \langle f_i\rangle_{s,Q} |Q| \left\|(b_i - b_{i,Q})\right\|_{\exp L,Q} \|gw\|_{L(\log L)^{|\tau|},Q} \prod_{j\in\tau'} \|b_j\|_{BMO} \langle f_j\rangle_{a,Q}$$

$$\lesssim \prod_{i=1}^{m} \|b_i\|_{BMO} \langle f_i\rangle_{a,Q} \sum_{Q\in\mathcal{S}} [w]_{A_\infty}^{|\tau|} \inf_{x\in Q} M_w\big(|g|^q\big)(x)^{1/q} w(Q),$$

where $s < a$. The second inequality follows from Lemma 2.1, and we have used Lemma 2.2 in the last inequality. By [13], we know $M_w(|g|^q)(x)^{1/q}$ is $L^{p'}(w_j)$ bounded when $1 < q < p'$. We can continue writing above display as

$$\int_{\mathbb{R}^n} \mathcal{A}_{\mathcal{S},\tau}(\vec{b},\vec{f}) gw \, dx$$

$$\lesssim [w]_{A_\infty}^{|\tau|} \prod_{i=1}^{m} \|b_i\|_{BMO} \sum_{Q\in\mathcal{S}} \int_Q M_a(\vec{f})(x) M_w\big(|g|^q\big)(x)^{1/q} w \, dx$$

$$\lesssim [w]_{A_\infty}^{|\tau|} \prod_{i=1}^{m} \|b_i\|_{BMO} \int_{\mathbb{R}^d} M_a(\vec{f})(x) M_w\big(|g|^q\big)(x)^{1/q} w \, dx$$

$$\lesssim [w]_{A_\infty}^{|\tau|} \prod_{i=1}^{m} \|b_i\|_{BMO} \left\|M_a(\vec{f})(x)\right\|_{L^p(w)} \left\|M_w\big(|g|^q\big)(x)^{1/q}\right\|_{L^{p'}(w)}$$

$$\lesssim [w]_{A_\infty}^{|\tau|} \prod_{i=1}^{m} \|b_i\|_{BMO} \left\|M_a(\vec{f})(x)\right\|_{L^p(w)} \|g\|_{L^{p'}(w)},$$

where $1 < q < p'$. This finishes the proof of Theorem 1.4. □

Proof of Theorem 1.5. The basic idea of our proof is borrowed from Sect. 4.3 in [14] or [15]. From Theorem 1.1, it is enough to control $\mathcal{A}_{\mathcal{S},\tau}(\vec{b},\vec{f})$. Let $\ell = m$.

$$\left\|\mathcal{A}_{\mathcal{S},\tau}(\vec{b},\vec{f})\right\|_{L^p(v_{\vec{w}})}^p \le \sum_{Q\in\mathcal{S}} \prod_{i\in\tau} \langle f_i\rangle_{s,Q}^p \int_Q \prod_{i\in\tau} |b_i(x) - b_{i,Q}|^p v_w(x) \, dx \prod_{j\in\tau'} \langle (b_j - b_{j,Q}) f_j\rangle_{s,Q}^p.$$

We denote $v_i(x) := M w_i(x)$, then it is easy to see that $\langle w_i\rangle_Q \le v_i(x)$ for any cube Q containing x. We can choose constants a, b with $s < a < p < b$. [11, Lemma 5.1] gives us a sparse family $\tilde{\mathcal{S}}$ such that, for every $Q \in \mathcal{S} \subset \tilde{\mathcal{S}}$, it holds that

$$|b_i(x) - b_{i,Q}| \lesssim \|b_i\|_{BMO} \sum_{P\in\tilde{\mathcal{S}}, P\subseteq Q} \chi_P(x).$$

By Lemma 2.1, it holds that

$$\left\|\mathcal{A}_{\mathcal{S}}(\vec{b},\vec{f})\right\|_{L^p(v_{\vec{w}})}^p$$

$$\lesssim \sum_{Q\in\mathcal{S}} \prod_{i\in\tau} \langle f_i\rangle_{s,Q}^p \|b_i\|_{BMO}^p \sum_{P\in\tilde{\mathcal{S}}, P\subseteq Q} \chi_P(x) \prod_{j\in\tau'} \langle (b_j - b_{j,Q}) f_j\rangle_{s,Q}^p v_{\vec{w}}(P)$$

$$\lesssim \sum_{Q\in\mathcal{S}} \prod_{i\in\tau} \langle f_i\rangle_{s,Q}^p \|b_i\|_{BMO}^p \sum_{P\in\tilde{\mathcal{S}}, P\subseteq Q} \prod_{j\in\tau'} \|b_j\|_{BMO}^p \langle f_j\rangle_{a,Q}^p \langle v_{\vec{w}}\rangle_P |P|$$

$$\lesssim \sum_{Q\in\mathcal{S}}\prod_{i=1}^{m}\langle f_i v_i^{\frac{1}{p_i}}\rangle_{p,Q}^{p}\langle v_i^{-\frac{1}{p_i}}\rangle_{b,Q}^{p}\|b_i\|_{BMO}^{p}\sum_{P\in\tilde{\mathcal{S}},P\subseteq Q}\langle v_{\vec{w}}\rangle_P|P|$$

$$\lesssim \sum_{Q\in\mathcal{S}}\prod_{i=1}^{m}\langle f_i v_i^{\frac{1}{p_i}}\rangle_{p,Q}^{p}\|b_i\|_{BMO}^{p}\sum_{P\in\tilde{\mathcal{S}},P\subseteq Q}\langle w_i\rangle_P^{-\frac{p}{p_i}}\langle v_{\vec{w}}\rangle_P|P|.$$

The sparseness property of collection $\mathcal{S}$, $\tilde{\mathcal{S}}$ allows us to continue with

$$\|\mathcal{A}_{\mathcal{S}}(\vec{b},\vec{f})\|_{L^p(v_{\vec{w}})}^{p} \lesssim \sum_{Q\in\mathcal{S}}\prod_{i=1}^{m}\langle f_i v_i^{\frac{1}{p_i}}\rangle_{p,Q}^{p}\|b_i\|_{BMO}^{p}\sum_{P\in\tilde{\mathcal{S}},P\subseteq Q}|E_P|$$

$$\lesssim \sum_{Q\in\mathcal{S}}\prod_{i=1}^{m}\Big(\inf_Q M_p\big(f_i v_i^{\frac{1}{p_i}}\big)\Big)^{p}\|b_i\|_{BMO}^{p}|Q|$$

$$\lesssim \int_{\mathbb{R}^d}\prod_{i=1}^{m}\big(M_p\big(f_i v_i^{\frac{1}{p_i}}\big)\big)^{p}\,dx\prod_{i=1}^{m}\|b_i\|_{BMO}^{p}$$

$$\lesssim \prod_{i=1}^{m}\|b_i\|_{BMO}^{p}\|f_i\|_{L^{p_i}(Mw_i)}^{p}.$$

This finishes the proof of Theorem (1.3).   □

4 Variations

Let us introduce some notions first. T is a Calderón–Zygmund operator defined as

$$T\vec{f}(x) = \int_{(\mathbb{R}^d)^m} K(x,\vec{y})\prod_{j=1}^{m}f_j(y_j)\,d\vec{y}, \quad x\notin\bigcap_{j=1}^{m}\mathrm{supp}f_j. \tag{4.1}$$

Definition 4.1 ([4]) Let $1\le r<\infty$, $r'=\frac{r}{r-1}$. The kernel $K(x,\vec{y})$ is called to satisfy the multilinear L^r-Hörmander condition if

$$K_r := \sup_Q \sup_{x,z\in\frac{1}{2}Q}\sum_{k=1}^{\infty}|2^kQ|^{\frac{m}{r}}\bigg(\int_{(2^kQ)^m\setminus(2^{k-1}Q)^m}\big|K(x,\vec{y})-K(z,\vec{y})\big|^{r'}d\vec{y}\bigg)^{\frac{1}{r'}}<\infty.$$

When $r=1$, it should be understood as

$$K_1 := \sup_Q \sup_{x,z\in\frac{1}{2}Q}\sum_{k=1}^{\infty}|2^kQ|^{\frac{m}{r}}\operatorname*{esssup}_{\vec{y}\in(2^kQ)^m\setminus(2^{k-1}Q)^m}\big|K(x,\vec{y})-K(z,\vec{y})\big|<\infty.$$

We first show that the multilinear L^r-Hörmander condition implies that $\mathcal{M}_{T,\alpha}^{\sharp}(\vec{f})$ is bounded from $L^r\times\cdots\times L^r\to L^{r/m,\infty}$ with $1\le r<\infty$.

Lemma 4.1 *Suppose* $1\le r<\infty$, $\alpha\ge 3$. *T is an operator whose kernel satisfies the bilinear L^r-Hörmander condition. Then $\mathcal{M}_{T,\alpha}^{\sharp}(\vec{f})$ is bounded from $L^r\times\cdots\times L^r\to L^{r/m,\infty}$. More specifically, for any $x\in\mathbb{R}^d$, it holds that*

$$\mathcal{M}_{T,\alpha}^{\sharp}(\vec{f})(x)\le K_r\mathcal{M}_r(\vec{f})(x), \tag{4.2}$$

where $\mathcal{M}_r(\vec{f})(x)=\mathrm{M}(|\vec{f}|^r)^{\frac{1}{r}}$ and M is the Hardy–Littlewood maximal function.

Proof Let $\xi, \xi' \in Q$.

$$\left| T(\vec{f}\chi_{\mathbb{R}^d \setminus \alpha Q})(\xi) - T(\vec{f}\chi_{\mathbb{R}^d \setminus \alpha Q})(\xi') \right|$$

$$\leq \left| \int_{(\mathbb{R}^d)^m \setminus (\alpha Q)^m} \left(K(\xi, \vec{y}) - K(\xi', \vec{y}) \right) \prod_{j=1}^m f_j(y_j) \, d\vec{y} \right|$$

$$\leq \sum_{k=1}^\infty \left| 2^k Q \right|^{\frac{m}{r}} \left(\int_{(2^k Q)^m \setminus (2^{k-1} Q)^m} \left| K(\xi, \vec{y}) - K(\xi', \vec{y}) \right|^{r'} d\vec{y} \right)^{\frac{1}{r'}} \left(\frac{1}{|2^k Q|^m} \int_{(2^k Q)^m} |\vec{f}|^r \, d\vec{y} \right)^{\frac{1}{r}}$$

$$\leq K_r \mathcal{M}_r(\vec{f})(x).$$

Since $\mathcal{M}_r(\vec{f})(x)$ is bounded from $L^r \times \cdots \times L^r \to L^{r/m,\infty}$, we can get the weak type estimates of $\mathcal{M}^\sharp_{T,\alpha}(\vec{f})$. □

Theorem 4.2 *Let the multi(sub)linear integral T satisfy the W_q condition and the kernel satisfy the multilinear L^r-Hörmander condition with $1 \leq q, r < \infty$, then all the estimates in Sect. 1 hold for this operator T.*

We give another variation of Theorem 1.1. In this case, we do not need to assume that T is a multi(sub)linear operator. Given an operator $T(\vec{f})$ and $\alpha > 0$, we define

$$\mathcal{M}^\sharp_{T,\alpha}(\vec{f})(x) := \sup_{Q \ni x} \operatorname*{esssup}_{\xi, \xi' \in Q} \left| \left(T\vec{f} - T(\vec{f}\chi_{\alpha Q}) \right)(\xi) - \left(T\vec{f} - T(\vec{f}\chi_{\alpha Q}) \right)(\xi') \right|.$$

Theorem 4.3 *Let $1 \leq q, r < \infty$, $\alpha \geq 3$. There are nonincreasing functions ψ and ϕ such that, for any cube Q,*

$$\left| \left\{ x \in Q : \left| T(\vec{f}\chi_Q)(x) \right| > \psi(\lambda) \prod_{j=1}^m \langle f_j \rangle_{q,Q} \right\} \right| \leq \lambda |Q| \quad (0 < \lambda < 1);$$

$$\left| \left\{ x \in Q : \left| \mathcal{M}^\sharp_{T,\alpha} \vec{f}\chi_Q(x) \right| > \phi(\lambda) \prod_{j=1}^m \langle f_j \rangle_{r,Q} \right\} \right| \leq \lambda |Q| \quad (0 < \lambda < 1).$$

Then all the estimates in Sect. 1 hold for this operator T.

Remark 4.2 The assumption of $\mathcal{M}^\sharp_{T,\alpha} \vec{f}$ satisfying the W_r condition allows us not to prove Lemma 4.1. In addition, we should use $T_{\vec{b}}(\vec{f}\chi_{Q_0})(x) - T_{\vec{b}}(\vec{f}\chi_{P_l^*})(x)$ instead of $T_{\vec{b}}(\vec{f}\chi_{Q_0^* \setminus P_l^*})(x)$ to avoid using the multi(sub)linear property.

Availability of data and materials
Not applicable.

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The author declares that they have no competing interests.

Authors' contributions

The authors worked jointly in drafting and approving the final manuscript. All authors read and approved the final manuscript.

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