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On a reverse Mulholland-type inequality in the whole plane with general homogeneous kernel

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Abstract

By using the idea of introducing parameters and weight coefficients, a new reverse discrete Mulholland-type inequality in the whole plane with general homogeneous kernel is given, which is an extension of the reverse Mulholland inequality. The equivalent forms are obtained. The equivalent statements of the best possible constant factor related to several parameters and a few applied examples are presented.

MSC: 26D15

Keywords: Weight coefficient; Mulholland-type inequality; Equivalent form; Equivalent statement; Parameter; Reverse

1 Introduction

Assuming that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, we have the following well-known Hardy–Hilbert inequality with the best possible constant $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

With regards to the similar assumption to (1), we still have Mulholland's inequality with the same best possible constant factor as follows (cf. [1], Theorem 343, replacing $\frac{a_m}{m}$, $\frac{b_n}{n}$ by a_m, b_n):

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty} \frac{a_m^p}{m^{1-p}}\right)^{1/p}\left(\sum_{n=2}^{\infty} \frac{b_n^q}{n^{1-q}}\right)^{1/q}. \tag{2}$$

If $f(x), g(y) \geq 0$, $0 < \int_0^\infty f^p(x)\,dx < \infty$ and $0 < \int_0^\infty g^q(y)\,dy < \infty$, then we have the integral analogous to (1), called Hardy–Hilbert's integral inequality, as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)\,dy\right)^{\frac{1}{q}}, \tag{3}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is still the best possible (cf. [1], Theorem 316).

In 1998, by introducing an independent parameter $\lambda > 0$, Yang [2, 3] gave an extension of (2) (for $p = q = 2$) with the kernel as $\frac{1}{(x+y)^\lambda}$ and a best possible constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ ($B(u, v)$ $(u, v > 0)$ is the beta function). Inequalities (1), (2) and (3) with their extensions and reverses play an important role in analysis and its applications (cf. [4–18]).

The following half-discrete Hilbert-type inequality was presented in 1934 (cf. [1], Theorem 351): If $K(x)$ $(x > 0)$ is decreasing, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi(s) = \int_0^\infty K(x)x^{s-1}\,dx < \infty$, then, for $a_n \geq 0$, $0 < \sum_{n=1}^\infty a_n^p < \infty$,

$$\int_0^\infty x^{p-2}\left(\sum_{n=1}^\infty K(nx)a_n\right)^p dx < \phi^p\left(\frac{1}{q}\right)\sum_{n=1}^\infty a_n^p. \tag{4}$$

Some new extensions of (4) with the reverses were obtained by [19–24] in recent years.

In 2016, by using the technique of real analysis and the weight coefficients, Hong et al. [25] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to several parameters. Other similar results about the extensions and the reverses of (1)–(4) were given by [26–46].

In this paper, following [25], by means of the idea of introducing parameters and the weight coefficients, a reverse discrete Mulholland-type in the whole plane is given as follows: for $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$,

$$\begin{aligned}&\sum_{|n|=3}^\infty \sum_{|m|=3}^\infty \frac{a_m b_n}{\ln|mn|}\\ &\quad > \frac{2\pi}{\sin(\pi/r)}\left[\sum_{|m|=3}^\infty \left(1 - \tilde{\theta}_1\left(\frac{1}{s}, m\right)\right)\frac{\ln^{\frac{p}{s}-1}|m|}{|m|^{1-p}}a_m^p\right]^{\frac{1}{p}}\left(\sum_{|n|=3}^\infty \frac{\ln^{\frac{q}{r}-1}|n|}{|n|^{1-q}}b_n^q\right)^{\frac{1}{q}},\end{aligned} \tag{5}$$

which is an extension of the reverse of (2). The general forms as well as the equivalent forms are obtained. The equivalent statements of the best possible constant factor related to several parameters are presented, and a few applied examples are considered.

2 Some lemmas

In what follows, we suppose that $0 < p < 1$ $(q < 0)$, $\frac{1}{p} + \frac{1}{q} = 1$, $-\frac{1}{2} \leq \alpha, \beta \leq \frac{1}{2}$, $\lambda, \lambda_1, \lambda_2 \in R = (-\infty, \infty)$, $c := \lambda - \lambda_1 - \lambda_2$, $k_\lambda(x, y)$ (≥ 0) is a homogeneous function of degree $-\lambda$, satisfying

$$k_\lambda(ux, uy) = u^{-\lambda}k_\lambda(x, y) \quad (u, x, y > 0),$$

and $k_\lambda(x, y)x^{\lambda_1 - 1}$ (resp. $k_\lambda(x, y)y^{\lambda_2 - 1}$) is strictly decreasing with respect to $x > 0$ (resp. $y > 0$), such that

$$k_\lambda(\gamma) := \int_0^\infty k_\lambda(1, u)u^{\gamma-1}\,du \in R_+ = (0, \infty) \quad (\gamma = \lambda_2, \lambda - \lambda_1). \tag{6}$$

We still assume that

$$\theta_\lambda(\lambda_2, m) := \frac{1}{k_\lambda(\lambda_2)} \int_0^{\frac{2}{\ln(|m|+\alpha m)}} k_\lambda(1,u) u^{\lambda_2 - 1}\, du \in (0,1),$$

$a_m, b_n \geq 0$ ($|m|, |n| \in N \backslash \{1,2\} = \{3,4,\ldots\}$), satisfying

$$0 < \sum_{|m|=3}^{\infty} \frac{\ln^{p(1-\lambda_1)-c-1}(|m|+\alpha m)}{(|m|+\alpha m)^{1-p}} a_m^p < \infty \quad \text{and}$$

$$0 < \sum_{|n|=3}^{\infty} \frac{\ln^{q(1-\lambda_2)-c-1}(|n|+\beta n)}{(|n|+\beta n)^{1-q}} b_n^q < \infty,$$

where $\sum_{|j|=3}^{\infty} \cdots = \sum_{j=-3}^{-\infty} \cdots + \sum_{j=3}^{\infty} \cdots$ $(j = m, n)$.

Lemma 1 *For* $\eta > 0$, *we have the following inequalities*:

$$\frac{2^{1-\eta}}{\eta(1-\beta^2)} < \sum_{|n|=3}^{\infty} \frac{\ln^{-\eta-1}(|n|+\beta n)}{|n|+\beta n} < \frac{1}{\eta}\left(\eta O_1 + \frac{2}{1-\beta^2}\right). \tag{7}$$

Proof In view of the decreasing property of series, for $\frac{e}{1\pm\beta} < 6$, $3 < \frac{e^2}{1\pm\beta}$, we have

$$\begin{aligned}
&\sum_{|n|=3}^{\infty} \frac{\ln^{-\eta-1}(|n|+\beta n)}{|n|+\beta n} \\
&\quad = \sum_{n=-3}^{-\infty} \frac{\ln^{-\eta-1}[(1-\beta)(-n)]}{(1-\beta)(-n)} + \sum_{n=3}^{\infty} \frac{\ln^{-\eta-1}[(1+\beta)n]}{(1+\beta)n} \\
&\quad = O_1 + \sum_{n=7}^{\infty} \frac{\ln^{-\eta-1}[(1-\beta)n]}{(1-\beta)n} + \sum_{n=7}^{\infty} \frac{\ln^{-\eta-1}[(1+\beta)n]}{(1+\beta)n} \\
&\quad < O_1 + \int_{\frac{e}{1-\beta}}^{\infty} \frac{\ln^{-\eta-1}[(1-\beta)y]}{(1-\beta)y}\, dy + \int_{\frac{e}{1+\beta}}^{\infty} \frac{\ln^{-\eta-1}[(1+\beta)y]}{(1+\beta)y}\, dy \\
&\quad = O_1 + \frac{1}{\eta(1-\beta)} + \frac{1}{\eta(1+\beta)} = \frac{1}{\eta}\left(\eta O_1 + \frac{2}{1-\beta^2}\right), \\
&\sum_{|n|=3}^{\infty} \frac{\ln^{-\eta-1}(|n|+\beta n)}{|n|+\beta n} \\
&\quad = \sum_{n=3}^{\infty} \frac{\ln^{-\eta-1}[(1-\beta)n]}{(1-\beta)n} + \sum_{n=3}^{\infty} \frac{\ln^{-\eta-1}[(1+\beta)n]}{(1+\beta)n} \\
&\quad > \int_{\frac{e^2}{1-\beta}}^{\infty} \frac{\ln^{-\eta-1}[(1-\beta)y]}{(1-\beta)y}\, dy + \int_{\frac{e^2}{1+\beta}}^{\infty} \frac{\ln^{-\eta-1}[(1+\beta)y]}{(1+\beta)y}\, dy \\
&\quad = \frac{2^{-\eta}}{\eta(1-\beta)} + \frac{2^{-\eta}}{\eta(1+\beta)} = \frac{2^{1-\eta}}{\eta(1-\beta^2)}.
\end{aligned}$$

Hence, we have (7).

The lemma is proved. □

Definition 1 We set

$$K(m,n) := k_\lambda\big(\ln(|m| + \alpha m), \ln(|n| + \beta n)\big) \quad \big(|m|, |n| \in N\backslash\{1,2\}\big)$$

and define the following weight coefficients:

$$\omega(\lambda_2, m) := \ln^{\lambda-\lambda_2}\big(|m| + \alpha m\big) \sum_{|n|=3}^{\infty} K(m,n) \frac{\ln^{\lambda_2-1}(|n| + \beta n)}{|n| + \beta n} \quad \big(|m| \in N\backslash\{1,2\}\big), \tag{8}$$

$$\varpi(\lambda_1, n) := \ln^{\lambda-\lambda_1}\big(|n| + \beta n\big) \sum_{|m|=3}^{\infty} K(m,n) \frac{\ln^{\lambda_1-1}(|m| + \alpha m)}{|m| + \alpha m} \quad \big(|n| \in N\backslash\{1,2\}\big). \tag{9}$$

Lemma 2 *The following inequalities are valid*:

$$0 < \frac{2k_\lambda(\lambda_2)}{1-\beta^2}\big(1 - \theta_\lambda(\lambda_2, m)\big) < \omega(\lambda_2, m) < \frac{2k_\lambda(\lambda_2)}{1-\beta^2} \quad \big(|m| \in N\backslash\{1,2\}\big). \tag{10}$$

Proof For $|m| \in N\backslash\{1,2\}$, we set

$$K^{(1)}(m,y) := k_\lambda\big(\ln(|m| + \alpha m), \ln[(1-\beta)(-y)]\big), \quad y < -\frac{1}{1-\beta},$$
$$K^{(2)}(m,y) := k_\lambda\big(\ln(|m| + \alpha m), \ln[(1+\beta)y]\big), \quad y > \frac{1}{1+\beta},$$

where, from for $y > \frac{1}{1-\beta}$, $K^{(1)}(m,-y) = k_\lambda(\ln(|m| + \alpha m), \ln[(1-\beta)y])$. We find

$$\begin{aligned}
\omega(\lambda_2, m) &= \ln^{\lambda-\lambda_2}\big(|m| + \alpha m\big)\Bigg\{\sum_{n=-3}^{-\infty} K^{(1)}(m,n) \frac{\ln^{\lambda_2-1}[(1-\beta)(-n)]}{(1-\beta)(-n)} \\
&\quad + \sum_{n=3}^{\infty} K^{(2)}(m,n) \frac{\ln^{\lambda_2-1}[(1+\beta)n]}{(1+\beta)n}\Bigg\} \\
&= \ln^{\lambda-\lambda_2}\big(|m| + \alpha m\big)\Bigg\{\sum_{n=3}^{\infty} K^{(1)}(m,-n) \frac{\ln^{\lambda_2-1}[(1-\beta)n]}{(1-\beta)n} \\
&\quad + \sum_{n=3}^{\infty} K^{(2)}(m,n) \frac{\ln^{\lambda_2-1}[(1+\beta)n]}{(1+\beta)n}\Bigg\}.
\end{aligned}$$

It is evident that, for fixed $|m| \in N\backslash\{1,2\}$, by the assumptions, both

$$K^{(1)}(m,-y)\frac{\ln^{\lambda_2-1}[(1-\beta)y]}{(1-\beta)y} = \frac{1}{(1-\beta)y} k_\lambda\big(\ln(|m| + \alpha m), \ln[(1-\beta)y]\big) \ln^{\lambda_2-1}[(1-\beta)y]$$

and $K^{(1)}(m,y)\frac{\ln^{\lambda_2-1}[(1+\beta)y]}{(1+\beta)y}$ are strictly decreasing with respect to $y > 2$. By the decreasing property of series, we have

$$\begin{aligned}
\omega(\lambda_2, m) &< \ln^{\lambda-\lambda_2}\big(|m| + \alpha m\big)\Bigg\{\int_2^{\infty} k_\lambda(\ln(|m| + \alpha m, \ln[(1-\beta)y]) \frac{\ln^{\lambda_2-1}[(1-\beta)y]}{(1-\beta)y}\, dy \\
&\quad + \int_2^{\infty} k_\lambda(\ln(|m| + \alpha m, \ln[(1+\beta)y]) \frac{\ln^{\lambda_2-1}[(1+\beta)y]}{(1+\beta)y}\, dy\Bigg\},
\end{aligned}$$

$$\omega(\lambda_2,m) > \ln^{\lambda-\lambda_2}\big(|m|+\alpha m\big)\left\{\int_3^\infty k_\lambda\big(\ln\big(|m|+\alpha m,\ln\big[(1-\beta)y\big]\big)\frac{\ln^{\lambda_2-1}[(1-\beta)y]}{(1-\beta)y}\,dy\right.$$
$$\left. + \int_3^\infty k_\lambda\big(\ln\big(|m|+\alpha m,\ln\big[(1+\beta)y\big]\big)\frac{\ln^{\lambda_2-1}[(1+\beta)y]}{(1+\beta)y}\,dy\right\}.$$

Setting $u = \frac{\ln[(1-\beta)y]}{\ln[|m|+\alpha m]}$ (resp. $u = \frac{\ln[(1+\beta)y]}{\ln(|m|+\alpha m)}$) in the above first (resp. second) integrals, since $2(1\pm\beta)\geq 1$ and $3(1\pm\beta) < e^2$ ($\beta\in[-\frac{1}{2},\frac{1}{2}]$), we obtain

$$\omega(\lambda_2,m) < \big[(1-\beta)^{-1}+(1+\beta)^{-1}\big]\int_0^\infty k_\lambda(1,u)u^{\lambda_2-1}\,du = \frac{2k_\lambda(\lambda_2)}{1-\beta^2},$$
$$\begin{aligned}\omega(\lambda_2,m) &> \frac{1}{1-\beta}\int_{\frac{\ln[3(1-\beta)]}{\ln(|m|+\alpha m)}}^\infty k_\lambda(1,u)u^{\lambda_2-1}\,du + \frac{1}{1+\beta}\int_{\frac{\ln[3(1+\beta)]}{\ln(|m|+\alpha m)}}^\infty k_\lambda(1,u)u^{\lambda_2-1}\,du\\ &\geq \frac{2}{1-\beta^2}\int_{\frac{2}{\ln(|m|+\alpha m)}}^\infty k_\lambda(1,u)u^{\lambda_2-1}\,du\\ &= \frac{2k_\lambda(\lambda_2)}{1-\beta^2}\big(1-\theta_\lambda(\lambda_2,m)\big) > 0.\end{aligned}$$

Hence, we have (11).

The lemma is proved. □

Note In the same way, we still have the following inequality:

$$\varpi(\lambda_1,n) < \frac{2}{1-\alpha^2}k_\lambda(\lambda-\lambda_1) \quad \big(|n|\in N\backslash\{1,2\}\big). \tag{11}$$

Lemma 3 *The following reverse Mulholland-type inequality in the whole plane is valid*:

$$\begin{aligned} I &:= \sum_{|n|=3}^\infty\sum_{|m|=3}^\infty K(m,n)a_m b_n\\ &> \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\sum_{|m|=3}^\infty\big(1-\theta_\lambda(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-c-1}(|m|+\alpha m)}{(|m|+\alpha m)^{1-p}}a_m^p\right]^{\frac{1}{p}}\\ &\quad\times\left[\sum_{|n|=3}^\infty\frac{\ln^{q(1-\lambda_2)-c-1}(|n|+\beta n)}{(|n|+\beta n)^{1-q}}b_n^q\right]^{\frac{1}{q}}. \end{aligned}\tag{12}$$

Proof By the reverse Hölder inequality with weight (cf. [47]), we obtain

$$\begin{aligned} I &= \sum_{|n|=3}^\infty\sum_{|m|=3}^\infty K(m,n)\left[\frac{(|m|+\alpha m)^{1/q}\ln^{(\lambda_2-1)/p}(|n|+\beta n)}{(|n|+\beta n)^{1/p}\ln^{(\lambda_1-1)/q}(|m|+\alpha m)}a_m\right]\\ &\quad\times\left[\frac{(|n|+\beta n)^{1/p}\ln^{(\lambda_1-1)/q}(|m|+\alpha m)}{(|m|+\alpha m)^{1/q}\ln^{(\lambda_2-1)/p}(|n|+\beta n)}b_n\right]\\ &\geq\left[\sum_{|m|=3}^\infty\sum_{|n|=3}^\infty K(m,n)\frac{(|m|+\alpha m)^{p-1}\ln^{\lambda_2-1}(|n|+\beta n)}{(|n|+\beta n)\ln^{(\lambda_1-1)(p-1)}(|m+\alpha m)}a_m^p\right]^{\frac{1}{p}} \end{aligned}$$

$$\times \left[\sum_{|n|=3}^{\infty} \sum_{|m|=3}^{\infty} K(m,n) \frac{(|n|+\beta n)^{q-1} \ln^{\lambda_1-1}(|m|+\alpha m)}{(|m|+\alpha m) \ln^{(\lambda_2-1)(q-1)}(|n|+\beta n)} b_n^q \right]^{\frac{1}{q}}$$

$$= \left[\sum_{|m|=3}^{\infty} \omega(\lambda_2, m) \frac{\ln^{p(1-\lambda_1)-c-1}(|m|+\alpha m)}{(|m|+\alpha m)^{1-p}} a_m^p \right]^{\frac{1}{p}}$$

$$\times \left[\sum_{|n|=3}^{\infty} \varpi(\lambda_1, n) \frac{\ln^{q(1-\lambda_2)-c-1}(|n|+\beta n)}{(|n|+\beta n)^{1-q}} b_n^q \right]^{\frac{1}{q}}.$$

Then, by (10) and (11), for $0 < p < 1$, $q < 0$, we have (12).

The lemma is proved. □

Remark 1 (i) By (12), for $\lambda_1 + \lambda_2 = \lambda$ (or $c = 0$), we find

$$0 < \sum_{|m|=3}^{\infty} \frac{\ln^{p(1-\lambda_1)-1}(|m|+\alpha m)}{(|m|+\alpha m)^{1-p}} a_m^p < \infty,$$

$$0 < \sum_{|n|=3}^{\infty} \frac{\ln^{q(1-\lambda_2)-1}(|n|+\beta n)}{(|n|+\beta n)^{1-q}} b_n^q < \infty,$$

and the following Mulholland-type inequality in the whole plane:

$$\sum_{|n|=3}^{\infty} \sum_{|m|=3}^{\infty} k_\lambda\bigl(\ln(|m|+\alpha m), \ln(|n|+\beta n)\bigr) a_m b_n$$

$$> \frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \left[\sum_{|m|=3}^{\infty} \bigl(1-\theta_\lambda(\lambda_2,m)\bigr) \frac{\ln^{p(1-\lambda_1)-1}(|m|+\alpha m)}{(|m|+\alpha m)^{1-p}} a_m^p \right]^{\frac{1}{p}}$$

$$\times \left[\sum_{|n|=3}^{\infty} \frac{\ln^{q(1-\lambda_2)-1}(|n|+\beta n)}{(|n|+\beta n)^{1-q}} b_n^q \right]^{\frac{1}{q}}. \tag{13}$$

In particular, for $\alpha = \beta = 0$, $\lambda = 1$, $k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \frac{1}{r}$, $\lambda_2 = \frac{1}{s}$ ($r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$),

$$\tilde{\theta}_1\left(\frac{1}{s}, m\right) := \frac{\sin(\pi/r)}{\pi} \int_0^{\frac{2}{\ln|m|}} \frac{1}{1+u} u^{-\frac{1}{r}}\, du \in (0,1),$$

(14) reduces to (5); for $\alpha = \beta = 0$, $a_{-m} = a_m$, $b_{-n} = b_n$ ($m, n \in N \backslash \{1,2\}$) in (14), we have

$$\vartheta_\lambda(\lambda_2, m) := \frac{1}{k_\lambda(\lambda_2)} \int_0^{\frac{2}{\ln m}} k_\lambda(1,u) u^{\lambda_2-1}\, du \in (0,1) \quad \bigl(m \in \{3,4,\ldots\}\bigr),$$

$$\sum_{n=3}^{\infty} \sum_{m=3}^{\infty} k_\lambda(\ln m, \ln n) a_m b_n$$

$$> k_\lambda(\lambda_2) \left[\sum_{m=3}^{\infty} \bigl(1-\vartheta_\lambda(\lambda_2,m)\bigr) \frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=3}^{\infty} \frac{\ln^{q(1-\lambda_2)-1}}{n^{1-q}} b_n^q \right]^{\frac{1}{q}}. \tag{14}$$

(ii) For $\alpha = \beta = \pm\frac{1}{2}$ in (13), we have the following Mulholland-type inequality in the whole plane:

$$\begin{aligned}
&\sum_{|n|=3}^{\infty}\sum_{|m|=3}^{\infty} k_\lambda\left(\ln\left(|m| \pm \frac{1}{2}m\right), \ln\left(|n| \pm \frac{1}{2}n\right)\right)a_m b_n \\
&\quad > \frac{8}{3}k_\lambda(\lambda_2)\left[\sum_{|m|=3}^{\infty}\left(1-\hat{\theta}_\lambda(\lambda_2,m)\right)\frac{\ln^{p(1-\lambda_1)-1}(|m| \pm \frac{1}{2}m)}{(|m| \pm \frac{1}{2}m)^{1-p}}a_m^p\right]^{\frac{1}{p}} \\
&\quad\quad \times \left[\sum_{|n|=3}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}(|n| \pm \frac{1}{2}n)}{(|n| \pm \frac{1}{2}n)^{1-q}}b_n^q\right]^{\frac{1}{q}}, \qquad (15)
\end{aligned}$$

where $\hat{\theta}_\lambda(\lambda_2,m) := \frac{1}{k_\lambda(\lambda_2)}\int_0^{\frac{2}{\ln(|m| \pm \frac{1}{2}m)}} k_\lambda(1,u)u^{\lambda_2-1}\,du \in (0,1)$.

Lemma 4 *If there exists a constant $\delta_0 > 0$, such that $k_\lambda(\lambda_2 \pm \delta_0) < \infty$, then, for any $0 < \delta < \delta_0$, we have $k_\lambda(\lambda_2 \pm \delta) < \infty$, and*

$$k_\lambda(\lambda_2 \pm \delta) \to k_\lambda(\lambda_2) \quad \left(\delta \to 0^+\right). \qquad (16)$$

Proof For any $0 < \delta < \delta_0$, we have

$$\begin{aligned}
k_\lambda(\lambda_2 \pm \delta) &= \int_0^1 k_\lambda(1,u)u^{\lambda_2\pm\delta-1}\,du + \int_1^\infty k_\lambda(1,u)u^{\lambda_2\pm\delta-1}\,du \\
&\le \int_0^1 k_\lambda(1,u)u^{\lambda_2-\delta_0-1}\,du + \int_1^\infty k_\lambda(1,u)u^{\lambda_2+\delta_0-1}\,du \\
&\le k_\lambda(\lambda_2-\delta_0) + k_\lambda(\lambda_2+\delta_0) < \infty.
\end{aligned}$$

We find

$$\begin{aligned}
&0 \le k_\lambda(1,u)u^{\lambda_2\pm\delta-1} \le F(u) := k_\lambda(1,u)u^{\lambda_2-\delta_0-1} \quad \left(u \in (0,1)\right), \\
&0 \le k_\lambda(1,u)u^{\lambda_2\pm\delta-1} \le F(u) := k_\lambda(1,u)u^{\lambda_2+\delta_0-1} \quad \left(u \in [1,\infty)\right),
\end{aligned}$$

and then

$$\begin{aligned}
\int_0^\infty F(u)\,du &= \int_0^1 F(u)\,du + \int_1^\infty F(u)\,du \\
&\le k_\lambda(\lambda_2-\delta_0) + k_\lambda(\lambda_2+\delta_0) < \infty.
\end{aligned}$$

By Lebesgue dominated convergence theorem (cf. [48]), it follows that

$$\int_0^\infty k_\lambda(1,u)u^{\lambda_2\pm\delta-1}\,du \to \int_0^\infty k_\lambda(1,u)u^{\lambda_2-1}\,du \quad \left(\delta \to 0^+\right).$$

The lemma is proved. □

Lemma 5 *If there exist constants $\sigma_0, \delta_0 > 0$, such that $\theta_\lambda(\lambda_2,m) = O(\frac{1}{\ln^{\sigma_0}(|m|+\alpha m)})$ and $k_\lambda(\lambda_2 \pm \delta_0) < \infty$, then the constant factor $\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ in (13) is the best possible.*

Proof For any $0 < \varepsilon < p\delta_0$, we set

$$\tilde{a}_m := \frac{\ln^{(\lambda_1 - \frac{\varepsilon}{p}) - 1}(|m| + \alpha m)}{|m| + \alpha m}, \qquad \tilde{b}_n := \frac{\ln^{(\lambda_2 - \frac{\varepsilon}{q}) - 1}(|n| + \beta n)}{|n| + \beta n} \quad \big(|m|, |n| \in N \backslash \{1, 2\}\big).$$

If there exists a constant $M(\geq \frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}})$, such that (13) is valid when replacing $\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ by M, then in particular, we have

$$\begin{aligned}
\tilde{I} &:= \sum_{|n|=3}^{\infty} \sum_{|m|=3}^{\infty} K(m,n) \tilde{a}_m \tilde{b}_n \\
&> M \left[\sum_{|m|=3}^{\infty} \big(1 - \theta_\lambda(\lambda_2, m)\big) \frac{\ln^{p(1-\lambda_1)-1}(|m| + \alpha m)}{(|m| + \alpha m)^{1-p}} \tilde{a}_m^p \right]^{\frac{1}{p}} \\
&\quad \times \left[\sum_{|n|=3}^{\infty} \frac{\ln^{q(1-\lambda_2)-1}(|n| + \beta n)}{(|n| + \beta n)^{1-q}} \tilde{b}_n^q \right]^{\frac{1}{q}}.
\end{aligned}$$

By the assumption, (10) and Lemma 1, we obtain

$$\begin{aligned}
\tilde{I} &> M \left[\sum_{|m|=3}^{\infty} \left(1 - O\left(\frac{1}{\ln^{\sigma_0}(|m| + \alpha m)}\right)\right) \frac{\ln^{-\varepsilon-1}(|m| + \alpha m)}{|m| + \alpha m} \right]^{\frac{1}{p}} \left[\sum_{|n|=3}^{\infty} \frac{\ln^{-\varepsilon-1}(|n| + \beta n)}{|n| + \beta n} \right]^{\frac{1}{q}} \\
&= M \left[\sum_{|m|=3}^{\infty} \frac{\ln^{-\varepsilon-1}(|m| + \alpha m)}{|m| + \alpha m} - \sum_{|m|=3}^{\infty} O\left(\frac{\ln^{-(\sigma_0+\varepsilon)-1}(|m| + \alpha m)}{|m| + \alpha m}\right) \right]^{\frac{1}{p}} \\
&\quad \times \left[\sum_{|n|=3}^{\infty} \frac{\ln^{-\varepsilon-1}(|n| + \beta n)}{|n| + \beta n} \right]^{\frac{1}{q}} \\
&> \frac{M}{\varepsilon} \left(\frac{2^{1-\varepsilon}}{1 - \alpha^2} - \varepsilon O(1) \right)^{\frac{1}{p}} \left(\varepsilon \tilde{O}_1 + \frac{2}{1 - \beta^2} \right)^{\frac{1}{q}}.
\end{aligned}$$

By (11) (for $(\lambda_1 - \frac{\varepsilon}{p}) + (\lambda_2 + \frac{\varepsilon}{p}) = \lambda$) and Lemma 1, since

$$k_\lambda(x,y) x^{\lambda_1 - \frac{\varepsilon}{p} - 1} = \big(k_\lambda(x,y) x^{\lambda_1 - 1}\big) x^{-\frac{\varepsilon}{p}}$$

is also strictly decreasing with respect to $x > 0$, we obtain

$$\begin{aligned}
\tilde{I} &= \sum_{|n|=3}^{\infty} \sum_{|m|=3}^{\infty} K(m,n) \frac{\ln^{(\lambda_1 - \frac{\varepsilon}{p})-1}(|m| + \alpha m)}{|m| + \alpha m} \frac{\ln^{(\lambda_2 - \frac{\varepsilon}{q})-1}(|n| + \beta n)}{|n| + \beta n} \\
&= \sum_{|n|=3}^{\infty} \varpi \left(\lambda_1 - \frac{\varepsilon}{p}, n \right) \frac{\ln^{-\varepsilon-1}(|n| + \beta n)}{|n| + \beta n} < \frac{2k_\lambda(\lambda_2 + \frac{\varepsilon}{p})}{1 - \alpha^2} \sum_{|n|=3}^{\infty} \frac{\ln^{-\varepsilon-1}(|n| + \beta n)}{|n| + \beta n} \\
&< \frac{2k_\lambda(\lambda_2 + \frac{\varepsilon}{p})}{1 - \alpha^2} \left[O_1 + \frac{2}{\varepsilon(1 - \beta^2)} \right].
\end{aligned}$$

In view of the above results, we have

$$\frac{2k_\lambda(\lambda_2+\frac{\varepsilon}{p})}{1-\alpha^2}\left(\varepsilon O_1+\frac{2}{1-\beta^2}\right)>\varepsilon\tilde{I}>M\left(\frac{2^{1-\varepsilon}}{1-\alpha^2}-\varepsilon O(1)\right)^{\frac{1}{p}}\left(\varepsilon\tilde{O}_1+\frac{2}{1-\beta^2}\right)^{\frac{1}{q}}.$$

For $\varepsilon\to 0^+$, by Lemma 4, we find $k_\lambda(\lambda_2+\frac{\varepsilon}{p})\to k_\lambda(\lambda_2)$ and then

$$\frac{4k_\lambda(\lambda_2)}{(1-\alpha^2)(1-\beta^2)}\geq\frac{2M}{(1-\alpha^2)^{1/p}(1-\beta^2)^{1/q}},$$

namely, $\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\geq M$, which means that $M=\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ is the best possible constant factor of (13).

The lemma is proved. □

Remark 2

(i) Following the assumption of Lemma 5, the constant factors in (14) and (15) are also the best possible.

(ii) If there exists a constant $\delta_0>0$, such that $k_\lambda(\lambda_2\pm\delta_0)<\infty$, setting

$$\hat{\lambda}_1:=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}=\lambda_1+\frac{c}{p},\qquad\hat{\lambda}_2:=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\lambda_2+\frac{c}{q},$$

then we find $\hat{\lambda}_1+\hat{\lambda}_2=\lambda_1+\frac{c}{p}+\lambda_2+\frac{c}{q}=\lambda$, and for $c\in(-|q|\delta_0,|q|\delta_0)$, by Lemma 4 and the reverse Hölder inequality (cf. [47]), we obtain

$$\begin{aligned}\infty&>k_\lambda(\hat{\lambda}_2)\\&=k_\lambda\left(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q}\right)\\&=\int_0^\infty k_\lambda(1,u)u^{\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q}-1}\,du\\&=\int_0^\infty k_\lambda(1,u)\big(u^{\frac{\lambda_2-1}{p}}\big)\big(u^{\frac{\lambda-\lambda_1-1}{q}}\big)\,du\\&\geq\left(\int_0^\infty k_\lambda(1,u)u^{\lambda_2-1}\,du\right)^{\frac{1}{p}}\left(\int_0^\infty k_\lambda(1,u)u^{\lambda-\lambda_1-1}\,du\right)^{\frac{1}{q}}\\&=k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)>0.\end{aligned}\tag{17}$$

We can reduce (12) to the following:

$$\begin{aligned}I&>\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\sum_{|m|=3}^\infty\big(1-\theta_\lambda(\lambda_2,m)\big)\frac{\ln^{p(1-\hat{\lambda}_1)-1}(|m|+\alpha m)}{(|m|+\alpha m)^{1-p}}a_m^p\right]^{\frac{1}{p}}\\&\quad\times\left[\sum_{|n|=3}^\infty\frac{\ln^{q(1-\hat{\lambda}_2)-1}(|n|+\beta n)}{(|n|+\beta n)^{1-q}}b_n^q\right]^{\frac{1}{q}}.\end{aligned}\tag{18}$$

Lemma 6 *If there exists a constant* $\delta_0 > 0$, *such that* $k_\lambda(\lambda_2 \pm \delta_0) < \infty$, $c = \lambda - \lambda_1 - \lambda_2 \in (-|q|\delta_0, |q|\delta_0)$, *and the constant factor* $\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *in* (12) *is the best possible, then we have* $\lambda_1 + \lambda_2 = \lambda$ (*or* $c = 0$).

Proof If the constant factor $\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ in (12) is the best possible, then, by (17) and (13) (for $\lambda_i = \hat{\lambda}_i$ $(i = 1,2)$), we have the following inequality:

$$\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \geq \frac{2k_\lambda(\hat{\lambda}_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \quad (\in R_+),$$

namely, $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1) \geq k_\lambda(\hat{\lambda}_2)$, from which it follows that (17) keeps the form of equality.

We observe that (17) keeps the form of equality if and only if there exist constants A and B, such that they are not both zero and (cf. [47])

$$Au^{\lambda_2-1} = Bu^{\lambda-\lambda_1-1} \quad a.e. \text{ in } R_+.$$

Assuming that $A \neq 0$, it follows that $u^{\lambda_2+\lambda_1-\lambda} = \frac{B}{A}$ *a.e.* in R_+, and then $\lambda_2 + \lambda_1 - \lambda = 0$, namely, $\lambda_1 + \lambda_2 = \lambda$.

The lemma is proved. □

3 Main results

Theorem 1 *Inequality* (12) *is equivalent to the following reverse Mulholland-type inequalities in the whole plane*:

$$\begin{aligned} J_1 &:= \left[\sum_{|n|=3}^{\infty} \frac{\ln^{p(\lambda_2+c)-c-1}(|n|+\beta n)}{|n|+\beta n}\left(\sum_{|m|=3}^{\infty} K(m,n)a_m\right)^p\right]^{\frac{1}{p}} \\ &> \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\sum_{|m|=3}^{\infty}(1-\theta_\lambda(\lambda_2,m))\frac{\ln^{p(1-\lambda_1)-c-1}(|m|+\alpha m)}{(|m|+\alpha m)^{1-p}}a_m^p\right]^{\frac{1}{p}}, \end{aligned} \tag{19}$$

$$\begin{aligned} J_2 &:= \left[\sum_{|m|=3}^{\infty} \frac{\ln^{q(\lambda_1+c)-c-1}(|m|+\alpha m)}{(|m|+\alpha m)(1-\theta_\lambda(\lambda_2,m))^{q-1}}\left(\sum_{|n|=3}^{\infty} K(m,n)b_n\right)^q\right]^{\frac{1}{q}} \\ &> \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\sum_{|n|=3}^{\infty}\frac{\ln^{q(1-\lambda_2)-c-1}(|n|+\beta n)}{(|n|+\beta n)^{1-q}}b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{20}$$

Proof Suppose that (19) is valid. By the reverse Hölder inequality (cf. [47]), we find

$$\begin{aligned} I &= \sum_{|n|=3}^{\infty}\left[\frac{\ln^{\frac{-1}{p}+\lambda_2+\frac{c}{q}}(|n|+\beta n)}{(|n|+\beta n)^{1/p}}\sum_{|m|=3}^{\infty}K(m,n)a_m\right]\left[\frac{\ln^{\frac{1}{p}-\lambda_2-\frac{c}{q}}(|n|+\beta n)}{(|n|+\beta n)^{-1/p}}b_n\right] \\ &\geq J_1 \cdot \left[\sum_{|n|=3}^{\infty}\frac{\ln^{q(1-\lambda_2)-c-1}(|n|+\beta n)}{(|n|+\beta n)^{1-q}}b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{21}$$

Then, by (19), we obtain (13).

On the other hand, assuming that (13) is valid, we set

$$b_n := \frac{\ln^{p(\lambda_2+c)-c-1}(|n|+\beta n)}{|n|+\beta n}\left(\sum_{|m|=3}^{\infty} K(m,n)a_m\right)^{p-1}, \quad |n| \in N\backslash\{1,2\}.$$

Then we have

$$J_1^p = \sum_{|n|=3}^{\infty} \frac{\ln^{q(1-\lambda_2)-c-1}(|n|+\beta n)}{(|n|+\beta n)^{1-q}} b_n^q = I. \tag{22}$$

If $J_1 = 0$, then (19) is naturally valid; if $J_1 = \infty$, then it is impossible that makes (19) valid, namely, $J_1 < \infty$. Suppose that $0 < J_1 < \infty$. By (13), it follows that

$$\begin{aligned}
&\sum_{|n|=3}^{\infty} \frac{\ln^{q(1-\lambda_2)-c-1}(|n|+\beta n)}{(|n|+\beta n)^{1-q}} b_n^q \\
&\quad = J_1^p = I \\
&\quad > \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\sum_{|m|=3}^{\infty}\big(1-\theta_\lambda(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-c-1}(|m|+\alpha m)}{(|m|+\alpha m)^{1-p}} a_m^p\right]^{\frac{1}{p}} J_1^{p-1}, \\
&J_1 = \left[\sum_{|n|=3}^{\infty} \frac{\ln^{q(1-\lambda_2)-c-1}(|n|+\beta n)}{(|n|+\beta n)^{1-q}} b_n^q\right]^{\frac{1}{p}} \\
&\quad > \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\sum_{|m|=3}^{\infty}\big(1-\theta_\lambda(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-c-1}(|m|+\alpha m)}{(|m|+\alpha m)^{1-p}} a_m^p\right]^{\frac{1}{p}},
\end{aligned}$$

namely, (19) follows, which is equivalent to (12).

Suppose that (20) is valid. By the reverse Hölder inequality (cf. [47]), we find

$$\begin{aligned}
I &= \sum_{|m|=3}^{\infty}\left[\big(1-\theta_\lambda(\lambda_2,m)\big)^{\frac{1}{p}}\frac{\ln^{\frac{1}{q}-\lambda_1-\frac{c}{p}}(|m|+\alpha m)}{(|m|+\alpha m)^{-1/q}} a_m\right] \\
&\quad \times \left[\frac{\ln^{\frac{-1}{q}+\lambda_1+\frac{c}{p}}(|m|+\alpha m)}{(|m|+\alpha m)^{1/q}(1-\theta_\lambda(\lambda_2,m))^{1/p}}\sum_{|n|=3}^{\infty} K(m,n)b_n\right] \\
&\geq \left[\sum_{|m|=3}^{\infty}\big(1-\theta_\lambda(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-c-1}(|m|+\alpha m)}{(|m|+\alpha m)^{1-p}} a_m^p\right]^{\frac{1}{p}} J_2.
\end{aligned} \tag{23}$$

Then, by (20), we obtain (12).

On the other hand, assuming that (12) is valid, we set

$$a_m := \frac{\ln^{q(\lambda_1+c)-c-1}(|m|+\alpha m)}{(|m|+\alpha m)(1-\theta_\lambda(\lambda_2,m))^{q-1}}\left(\sum_{|m|=3}^{\infty} K(m,n)a_m\right)^{q-1}, \quad |m| \in N\backslash\{1,2\}.$$

Then we have

$$J_2^q = \sum_{|m|=3}^{\infty} \left(1 - \theta_\lambda(\lambda_2, m)\right) \frac{\ln^{p(1-\lambda_1)-c-1}(|m| + \alpha m)}{(|m| + \alpha m)^{1-p}} a_m^p = I. \tag{24}$$

If $J_2 = 0$, then (20) is naturally valid; if $J_2 = \infty$, then it is impossible that makes (20) valid, namely, $J_2 < \infty$. Suppose that $0 < J_2 < \infty$. By (12), it follows that

$$\begin{aligned}
&\sum_{|m|=3}^{\infty} \left(1 - \theta_\lambda(\lambda_2, m)\right) \frac{\ln^{p(1-\lambda_1)-c-1}(|m| + \alpha m)}{(|m| + \alpha m)^{1-p}} a_m^p \\
&\quad = J_2^q = I \\
&\quad > \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} J_2^{q-1} \cdot \left[\sum_{|n|=3}^{\infty} \frac{\ln^{q(1-\lambda_2)-c-1}(|n| + \beta n)}{(|n| + \beta n)^{1-q}} b_n^q\right]^{\frac{1}{q}}, \\
&J_2 = \left\{\sum_{|m|=3}^{\infty} \left(1 - \theta_\lambda(\lambda_2, m)\right) \frac{\ln^{p(1-\lambda_1)-c-1}(|m| + \alpha m)}{(|m| + \alpha m)^{1-p}} a_m^p\right\}^{\frac{1}{q}} \\
&\quad > \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \left[\sum_{|n|=3}^{\infty} \frac{\ln^{q(1-\lambda_2)-c-1}(|n| + \beta n)}{(|n| + \beta n)^{1-q}} b_n^q\right]^{\frac{1}{q}},
\end{aligned}$$

namely, (20) follows, which is equivalent to (12).

Hence, inequalities (12), (19) and (20) are equivalent.

The theorem is proved. □

Theorem 2 *Suppose that there exists a constant* $\delta_0 > 0$, *such that* $k_\lambda(\lambda_2 \pm \delta_0) < \infty$. *The following statements* (i), (ii), (iii), (iv), (v) *and* (vi) *are equivalent*:

(i) *Both* $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1)$ *and* $k_\lambda(\frac{\lambda_2}{p} + \frac{\lambda-\lambda_1}{q})$ *are independent of* p, q;

(ii)

$$k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1) = k_\lambda\left(\frac{\lambda_2}{p} + \frac{\lambda - \lambda_1}{q}\right);$$

(iii) *if* $c \in (-|q|\delta_0, |q|\delta_0)$, *then* $\lambda_1 + \lambda_2 = \lambda$ (*or* $c = 0$);

(iv) *if there exists a constant* $\sigma_0 > 0$, *such that* $\theta_\lambda(\lambda_2, m) = O(\frac{1}{\ln^{\sigma_0}(|m|+\alpha m)})$, *then* $\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *is the best possible constant factor of* (12);

(v) $\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *is the best possible constant factor of* (19);

(vi) $\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *is the best possible constant factor of* (20).

If the statement (iii) follows, namely, $\lambda_1 + \lambda_2 = \lambda$ (or $c = 0$), there exist constants $\sigma_0 > 0$, such that $\theta_\lambda(\lambda_2, m) = O(\frac{1}{\ln^{\sigma_0}(|m|+\alpha m)})$, then we have the following equivalent inequalities equivalent to (13) with the best possible constant factor $\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$:

$$\left[\sum_{|n|=3}^{\infty}\frac{\ln^{p\lambda_2-1}(|n|+\beta n)}{|n|+\beta n}\left(\sum_{|m|=3}^{\infty}k_\lambda\big(\ln(|m|+\alpha m),\ln(|n|+\beta n)\big)a_m\right)^p\right]^{\frac{1}{p}}$$

$$>\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\sum_{|m|=3}^{\infty}\big(1-\theta_\lambda(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-1}(|m|+\alpha m)}{(|m|+\alpha m)^{1-p}}a_m^p\right]^{\frac{1}{p}}, \tag{25}$$

$$\left[\sum_{|m|=3}^{\infty}\frac{\ln^{q\lambda_1-1}(|m|+\alpha m)}{(|m|+\alpha m)(1-\theta_\lambda(\lambda_2,m))^{q-1}}\left(\sum_{|n|=3}^{\infty}k_\lambda\big(\ln(|m|+\alpha m),\ln(|n|+\beta n)\big)b_n\right)^q\right]^{\frac{1}{q}}$$

$$>\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\sum_{|n|=3}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}(|n|+\beta n)}{(|n|+\beta n)^{1-q}}b_n^q\right]^{\frac{1}{q}}. \tag{26}$$

In particular, for $\alpha=\beta=0$, $a_{-m}=a_m$, $b_{-n}=b_n$ $(m,n\in N\backslash\{1,2\})$ in (25) and (26), we have the following inequalities equivalent to (14) with the best possible constant factor $k_\lambda(\lambda_2)$:

$$\left[\sum_{n=3}^{\infty}\frac{\ln^{p\lambda_2-1}n}{n}\left(\sum_{m=3}^{\infty}k_\lambda(\ln m,\ln n)a_m\right)^p\right]^{\frac{1}{p}}$$

$$>k_\lambda(\lambda_2)\left[\sum_{m=3}^{\infty}\big(1-\vartheta_\lambda(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-1}m}{m^{1-p}}a_m^p\right]^{\frac{1}{p}}, \tag{27}$$

$$\left[\sum_{m=3}^{\infty}\frac{\ln^{q\lambda_1-1}m}{m(1-\vartheta_\lambda(\lambda_2,m))^{q-1}}\left(\sum_{n=3}^{\infty}k_\lambda(\ln m,\ln n)b_n\right)^q\right]^{\frac{1}{q}}$$

$$>k_\lambda(\lambda_2)\left[\sum_{n=3}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}n}{n^{1-q}}b_n^q\right]^{\frac{1}{q}}. \tag{28}$$

Proof (i) ⇒ (ii). Since $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)$ is independent of p, q, we find

$$k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)=\lim_{p\to1^-}\lim_{q\to-\infty}k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)=k_\lambda(\lambda_2).$$

Then, by Lemma 4, we have the following equality:

$$\begin{aligned}k_\lambda\left(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q}\right)&=k_\lambda\left(\lambda_2+\frac{c}{q}\right)\\&=\lim_{q\to-\infty}k_\lambda\left(\lambda_2+\frac{c}{q}\right)\\&=k_\lambda(\lambda_2)=k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1).\end{aligned}$$

(ii) ⇒ (iii). If $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)=k_\lambda(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})$, then (17) keeps the form of an equality. By the proof of Lemma 6, it follows that $\lambda_1+\lambda_2=\lambda$.

(iii) ⇒ (i). If $\lambda_1 + \lambda_2 = \lambda$, then we have

$$k_\lambda\left(\frac{\lambda_2}{p} + \frac{\lambda - \lambda_1}{q}\right) = k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1) = k_\lambda(\lambda_2).$$

Both $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1)$ and $k_\lambda(\frac{\lambda_2}{p} + \frac{\lambda-\lambda_1}{q})$ are independent of p, q.

Hence, we have (i) ⇔ (ii) ⇔ (iii).

(iii) ⇔ (iv). By Lemma 5 and Lemma 6, we obtain the conclusions.

(iv) ⇔ (v). If the constant factor in (12) is the best possible, then so is constant factor in (19). Otherwise, by (21), we would arrive at a contradiction that the constant factor in (12) is not the best possible. On the other hand, if the constant factor in (19) is the best possible, then so is constant factor in (12). Otherwise, by (22), we would reach a contradiction that the constant factor in (19) is not the best possible.

(iv) ⇔ (vi). If the constant factor in (12) is the best possible, then so is constant factor in (20). Otherwise, by (23), we would reach a contradiction that the constant factor in (12) is not the best possible. On the other hand, if the constant factor in (20) is the best possible, then so is constant factor in (12). Otherwise, by (24), we would reach a contradiction that the constant factor in (20) is not the best possible.

Therefore, the statements (i), (ii), (iii), (iv), (v) and (vi) are equivalent.

The theorem is proved. □

4 Some applied examples

Example 1 For $\lambda > 0$, $\lambda_i \in (0,\lambda) \cap (0,1]$ $(i = 1,2)$, setting $k_\lambda(x,y) = \frac{1}{(x+y)^\lambda}$ $(x,y > 0)$, then $k_\lambda(x,y)x^{\lambda_1-1}$ (resp. $k_\lambda(x,y)y^{\lambda_2-1}$) is strictly decreasing with respect to $x > 0$ (resp. $y > 0$), such that (cf. [49])

$$k_\lambda(\gamma) = \int_0^\infty \frac{u^{\gamma-1}}{(1+u)^\lambda}\,du = B(\gamma.\lambda - \gamma) \in R_+ \quad (\gamma = \lambda_2, \lambda - \lambda_1),$$

$$\begin{aligned} 0 &< \theta_\lambda(\lambda_2, m) \\ &= \frac{1}{B(\lambda - \lambda_2, \lambda_2)} \int_0^{\frac{2}{\ln(|m|+\alpha m)}} \frac{u^{\lambda_2-1}}{(1+u)^\lambda}\,du \\ &\le \frac{1}{B(\lambda - \lambda_2, \lambda_2)} \int_0^{\frac{2}{\ln(|m|+\alpha m)}} u^{\lambda_2-1}\,du \\ &= \frac{1}{\lambda_2 B(\lambda - \lambda_2, \lambda_2)} \left(\frac{2}{\ln(|m| + \alpha m)}\right)^{\lambda_2} \quad (\sigma_0 = \lambda_2 > 0). \end{aligned}$$

Substitution of $K(m,n) = \frac{1}{\ln^\lambda[(|m|+\alpha m)(|n|+\beta n)]}$ and

$$k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1) = \big(B(\lambda - \lambda_2, \lambda_2)\big)^{\frac{1}{p}}\big(B(\lambda_1, \lambda - \lambda_1)\big)^{\frac{1}{q}}$$

in Lemma 3 and Theorem 1, we have the equivalent inequalities (12), (19) and (20) with the particular kernel as well as the particular constant factor. We set $\delta_0 = \frac{1}{2}\min\{\lambda_2, \lambda - \lambda_2\} > 0$, satisfying

$$k_\lambda(\lambda_2 \pm \delta_0) = B(\lambda - \lambda_2 \mp \delta_0, \lambda_2 \pm \delta_0) \in R_+.$$

Then, by Theorem 2, $\lambda_1 + \lambda_2 = \lambda$ if and only if the constant factor

$$\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left(=\frac{2B(\lambda_1,\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\right)$$

in (12), (19) and (20) is the best possible.

Example 2 For $\lambda > 0$, $\lambda_i \in (0,\lambda)\cap(0,1]$ $(i = 1,2)$, setting $k_\lambda(x,y) = \frac{\ln(x/y)}{x^\lambda - y^\lambda}$ $(x,y > 0)$, then $k_\lambda(x,y)x^{\lambda_1-1}$ (resp. $k_\lambda(x,y)y^{\lambda_2-1}$) is strictly decreasing with respect to $x > 0$ (resp. $y > 0$), such that

$$k_\lambda(\gamma) = \int_0^\infty \frac{u^{\gamma-1}\ln u}{u^\lambda - 1}\,du = \left[\frac{\pi}{\lambda\sin(\pi\gamma/\lambda)}\right]^2 \in R_+ \quad (\gamma = \lambda_2, \lambda-\lambda_1).$$

For fixed m, since $f(u) := \frac{u^{\lambda_2/2}\ln u}{u^\lambda - 1}$ is continuous at $[0,2]$ ($f(0) := 0, f(1) := \frac{1}{\lambda}$), we have

$$\begin{aligned}
0 &< \theta_\lambda(\lambda_2,m) \\
&= \left[\frac{\lambda\sin(\pi\lambda_2/\lambda)}{\pi}\right]^2 \int_0^{\frac{2}{\ln(|m|+\alpha m)}} \frac{u^{\lambda_2-1}\ln u}{u^\lambda-1}\,du \\
&= \left[\frac{\lambda\sin(\pi\lambda_2/\lambda)}{\pi}\right]^2 \int_0^{\frac{2}{\ln(|m|+\alpha m)}} \left(\frac{u^{\lambda/2}\ln u}{u^\lambda-1}\right)u^{\frac{\lambda_2}{2}-1}\,du \\
&\le \left[\frac{\lambda\sin(\pi\lambda_2/\lambda)}{\pi}\right]^2 M\int_0^{\frac{2}{\ln(|m|+\alpha m)}} u^{\frac{\lambda_2}{2}-1}\,du \\
&= \left[\frac{\lambda\sin(\pi\lambda_2/\lambda)}{\pi}\right]^2 \frac{2M}{\lambda_2}\left(\frac{2}{\ln(|m|+\alpha m)}\right)^{\frac{\lambda_2}{2}} \quad \left(\sigma_0 = \frac{\lambda_2}{2} > 0\right).
\end{aligned}$$

Substitution of $K(m,n) = \frac{\ln[\ln(|m|+\alpha m)/\ln(|n|+\beta n)]}{\ln^\lambda(|m|+\alpha m)-\ln^\lambda(|n|+\beta n)}$ and

$$k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1) = \frac{1}{\lambda^2}\left[\frac{\pi}{\sin(\pi\lambda_2/\lambda)}\right]^{\frac{2}{p}}\left[\frac{\pi}{\sin(\pi\lambda_1/\lambda)}\right]^{\frac{2}{q}}$$

in Lemma 3 and Theorem 1, we have equivalent reverse inequalities (12), (19) and (20) with the particular kernel as well as the particular constant factor. We set $\delta_0 = \frac{1}{2}\min\{\lambda_2, \lambda-\lambda_2\} > 0$, satisfying

$$k_\lambda(\lambda_2 \pm \delta_0) = \left[\frac{\pi}{\lambda\sin\pi(\lambda_2\pm\delta_0)/\lambda}\right]^2 \in R_+.$$

Then, by Theorem 2, $\lambda_1 + \lambda_2 = \lambda$ if and only if the constant factor

$$\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left(=\frac{2[\frac{\pi}{\lambda\sin(\pi\lambda_2/\lambda)}]^2}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\right)$$

in (12), (19) and (20) is the best possible.

Example 3 For $0 < \eta + \lambda_i < 1$ $(i = 1,2)$, $\lambda + 2\eta > \min_{i=1,2}\{0, \eta + \lambda_i\}$, setting $k_\lambda(x,y) = \frac{(\min\{x,y\})^\eta}{(\max\{x,y\})^{\lambda+\eta}}$ $(x, y > 0)$, then

$$k_\lambda(x,y)x^{\lambda_1-1} = \frac{(\min\{x,y\})^\eta x^{\lambda_1-1}}{(\max\{x,y\})^{\lambda+\eta}} = \begin{cases} x^{\eta+\lambda_1-1}, & 0 < x < y, \\ \frac{y^\eta}{x^{\lambda+\eta-\lambda_1+1}}, & x \geq y \end{cases}$$

(resp. $k_\lambda(x,y)y^{\lambda_2-1}$) is strictly decreasing with respect to $x > 0$ (resp. $y > 0$), such that

$$\begin{aligned} k_\lambda(\gamma) &= \int_0^\infty \frac{(\min\{1,u\})^\eta u^{\gamma-1}}{(\max\{1,u\})^{\lambda+\eta}}\,du \\ &= \int_0^1 u^{\eta+\gamma-1}\,du + \int_1^\infty \frac{u^{\gamma-1}}{u^{\lambda+\eta}}\,du \\ &= \frac{\lambda+2\eta}{(\eta+\gamma)(\lambda+\eta-\gamma)} \in R_+ \quad (\gamma = \lambda_2, \lambda-\lambda_1), \end{aligned}$$

$$\begin{aligned} 0 < \theta_\lambda(\lambda_2, m) &= \frac{(\lambda_2+\eta)(\lambda-\lambda_2+\eta)}{\lambda+\eta} \int_0^{\frac{2}{\ln(|m|+\alpha m)}} \frac{(\min\{1,u\})^\eta u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\eta}}\,du \\ &= \frac{(\lambda_2+\eta)(\lambda-\lambda_2+\eta)}{\lambda+\eta} \int_0^{\frac{2}{\ln(|m|+\alpha m)}} u^{\eta+\lambda_2-1}\,du \\ &= \frac{\lambda-\lambda_2+\eta}{\lambda+\eta}\left(\frac{2}{\ln(|m|+\alpha m)}\right)^{\eta+\lambda_2} \quad \bigl(\ln\bigl(|m|+\alpha m\bigr) > 2; \sigma_0 = \eta+\lambda_2 > 0\bigr). \end{aligned}$$

Substitution of $K(m,n) = \frac{(\min\{\ln(|m|+\alpha m), \ln(|n|+\beta n)\})^\eta}{(\max\{\ln(|m|+\alpha m), \ln(|n|+\beta n)\})^{\lambda+\eta}}$ and

$$k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1) = \left[\frac{\lambda+2\eta}{(\lambda_2+\eta)(\lambda-\lambda_2+\eta)}\right]^{\frac{2}{p}}\left[\frac{\lambda+2\eta}{(\lambda_1+\eta)(\lambda-\lambda_1+\eta)}\right]^{\frac{2}{q}}$$

in Lemma 3 and Theorem 1, we have equivalent reverse inequalities (12), (19) and (20) with the particular kernel as well as the particular constant factor. We set $\delta_0 = \frac{1}{2}\min\{\eta + \lambda_2, \eta + \lambda - \lambda_2\} > 0$, satisfying

$$k_\lambda(\lambda_2 \pm \delta_0) = \frac{\lambda+2\eta}{(\eta+\lambda_2\pm\delta_0)(\eta+\lambda-\lambda_2\mp\delta_0)} \in R_+.$$

Then, by Theorem 2, $\lambda_1 + \lambda_2 = \lambda$ if and only if the constant factor

$$\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left(=\frac{2}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\frac{\lambda+2\eta}{(\eta+\lambda_2)(\eta+\lambda-\lambda_2)}\right)$$

in (12), (19) and (20) is the best possible.

Example 4

(i) In view of the following expression for the cotangent function (cf. [50]):

$$\cot x = \frac{1}{x} + \sum_{k=1}^\infty \left(\frac{1}{x-\pi k} + \frac{1}{x+\pi k}\right) \quad \bigl(x \in (0,\pi)\bigr),$$

for $b \in (0,1)$, by the Lebesgue term by term theorem (cf. [44]), we obtain

$$\begin{aligned} A_b &:= \int_0^\infty \frac{u^{b-1}}{1-u}\,du = \int_0^1 \frac{u^{b-1}}{1-u}\,du + \int_1^\infty \frac{u^{b-1}}{1-u}\,du \\ &= \int_0^1 \frac{u^{b-1}}{1-u}\,du - \int_0^1 \frac{v^{-b}}{1-v}\,dv = \int_0^1 \frac{u^{b-1}-u^{-b}}{1-u}\,du \\ &= \int_0^1 \sum_{k=0}^\infty \left(u^{k+b-1} - u^{k-b}\right) du = \sum_{k=0}^\infty \int_0^1 \left(u^{k+b-1} - u^{k-b}\right) du \\ &= \sum_{k=0}^\infty \left(\frac{1}{k+b} - \frac{1}{k+1-b}\right) = \pi \left[\frac{1}{\pi b} + \sum_{k=1}^\infty \left(\frac{1}{\pi b - \pi k} + \frac{1}{\pi b + \pi k}\right)\right] \\ &= \pi \cot \pi b \in R := (-\infty, \infty). \end{aligned}$$

(ii) For $\lambda, \eta > 0$, we set the homogeneous function of degree $-\lambda$ as follows:

$$k_\lambda(x,y) := \frac{x^\eta - y^\eta}{x^{\lambda+\eta} - y^{\lambda+\eta}} \quad (x,y>0),$$

satisfying $k_\lambda(v,v) := \frac{\eta}{(\lambda+\eta)v^\lambda}$ $(v>0)$. It follows that $k_\lambda(x,y)$ is a positive and continuous function with respect to $x,y>0$. For $x \neq y$, we find

$$\frac{\partial}{\partial x} k_\lambda(x,y) = -x^{\eta-1}\left(x^{\lambda+\eta} - y^{\lambda+\eta}\right)^{-2} \varphi(x,y),$$

where we set the following differentiable function:

$$\varphi(x,y) := \lambda x^{\lambda+\eta} - (\lambda+\eta) y^\eta x^\lambda + \eta y^{\lambda+\eta} \quad (x,y>0).$$

We find that, for $0 < x < y$,

$$\frac{\partial}{\partial x} \varphi(x,y) = \lambda(\lambda+\eta) x^{\lambda-1}\left(x^\eta - y^\eta\right) < 0;$$

for $x > y$, $\frac{\partial}{\partial x}\varphi(x,y) > 0$. It follows that $\varphi(x,y)$ is strictly decreasing (resp. increasing) with respect to $x < y$ (resp. $x > y$). Since $\varphi(y,y) = \min_{x>0} \varphi(x,y) = 0$ $(y>0)$, we have $\varphi(x,y) > 0$ $(x \neq y)$, namely, $\frac{\partial}{\partial x} k_\lambda(x,y) < 0$ $(x \neq y)$. Therefore, in view of $k_\lambda(x,y)$ is continuous at $x = y$, we confirm that $k_\lambda(x,y)$ $(y>0)$ is strictly decreasing with respect to $x > 0$. In the same way, we can show that $k_\lambda(x,y)$ $(x>0)$ is also strictly decreasing with respect to $y > 0$. Hence, for $\lambda_i \in (0,\lambda) \cap (0,1]$ $(i=1,2)$, $k_\lambda(x,y)x^{\lambda_1-1}$ (resp. $k_\lambda(x,y)y^{\lambda_2-1}$) is strictly decreasing with respect to $x>0$ (resp. $y>0$).

(iii) Since $k_\lambda(x,y) > 0$, by (i), we obtain

$$\begin{aligned} k_{\lambda,\eta}(\gamma) &:= \int_0^\infty k_\lambda(1,u) u^{\gamma-1}\,du = \int_0^\infty \frac{1-u^\eta}{1-u^{\lambda+\eta}} u^{\gamma-1}\,du \\ &\overset{v=u^{\lambda+\eta}}{=} \frac{1}{\lambda+\eta}\left(\int_0^\infty \frac{v^{\frac{\gamma}{\lambda+\eta}-1}}{1-v}\,dv - \int_0^\infty \frac{v^{\frac{\gamma+\eta}{\lambda+\eta}-1}}{1-v}\,dv\right) \\ &= \frac{\pi}{\lambda+\eta}\left[\cot\left(\frac{\pi\gamma}{\lambda+\eta}\right) - \cot\left(\frac{\pi(\gamma+\eta)}{\lambda+\eta}\right)\right] \end{aligned}$$

$$= \frac{\pi}{\lambda+\eta}\left[\cot\left(\frac{\pi\gamma}{\lambda+\eta}\right) + \cot\left(\frac{\pi(\lambda-\gamma)}{\lambda+\eta}\right)\right] \in R_+ \quad (\lambda = \lambda_2, \lambda-\lambda_1),$$

$$0 < \theta_\lambda(\lambda_2, m)$$
$$= \frac{1}{k_{\lambda,\eta}(\lambda_2)}\int_0^{\frac{2}{\ln(|m|+\alpha m)}} \frac{(1-u^\eta)u^{\lambda_2-1}}{1-u^{\lambda+\eta}}\,du \le \frac{1}{k_{\lambda,\eta}(\lambda_2)}\int_0^{\frac{2}{\ln(|m|+\alpha m)}} u^{\lambda_2-1}\,du$$
$$= \frac{1}{\lambda_2 k_{\lambda,\eta}(\lambda_2)}\left(\frac{2}{\ln(|m|+\alpha m)}\right)^{\lambda_2} \quad (\sigma_0 = \lambda_2 > 0).$$

On substitution of $K(m,n) = \frac{\ln^\eta(|m|+\alpha m)-\ln^\eta(|n|+\beta n)}{\ln^{\lambda+\eta}(|m|+\alpha m)-\ln^{\lambda+\eta}(|n|+\beta n)}$ and

$$k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1) = k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda-\lambda_1)$$

in Lemma 3 and Theorem 1, we have the equivalent inequalities (12), (19) and (20) with the particular kernel as well as the particular constant factor. We set $\delta_0 = \frac{1}{2}\min\{\lambda_2, \lambda-\lambda_2\} > 0$, satisfying $k_{\lambda,\eta}(\lambda_2 \pm \delta_0) \in R_+$. Then, by Theorem 2, $\lambda_1 + \lambda_2 = \lambda$ if and only if the constant factor

$$\frac{2k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left(= \frac{2k_{\lambda,\eta}(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\right)$$

in (13), (19) and (20) is the best possible.

5 Conclusions

In this paper, by means of the idea of introducing parameters and the weight coefficients, a new reverse discrete Mulholland-type inequality in the whole plane is obtained in Lemma 3, which is an extension of the reverse of (2). The equivalent forms are given in Theorem 1. The equivalent statements of the best possible constant factor related to several parameters are considered in Theorem 2. Some particular inequalities are presented in Theorem 2 and Remark 1. Some applied examples are given in Example 1–4. The lemmas and theorems provide an extensive account of this type of inequalities.

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Availability of data and materials
We declare that the data and material in this paper can be used publicly.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. RL and XH participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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