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Remarks on a recent paper titled: "On the split common fixed point problem for strict pseudocontractive and asymptotically nonexpansive mappings in Banach spaces"

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Abstract

In a recently published theorem on the split common fixed point problem for strict pseudocontractive and asymptotically nonexpansive mappings, Tang *et al.* (J. Inequal. Appl. 2015:305, 2015) studied a uniformly convex and 2-uniformly smooth real Banach space with the Opial property and best smoothness constant κ satisfying the condition $0 < \kappa < \frac{1}{\sqrt{2}}$, as a real Banach space more general than Hilbert spaces.

A well-known example of a uniformly convex and 2-uniformly smooth real Banach space with the Opial property is $E = l_p$, $2 \leq p < \infty$. It is shown in this paper that, if κ is the best smoothness constant of E and satisfies the condition $0 < \kappa \leq \frac{1}{\sqrt{2}}$, then E is necessarily l_2, a real Hilbert space. Furthermore, some important remarks concerning the proof of this theorem are presented.

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1 Introduction

Let H_1 and H_2 be two real Hilbert spaces, C and Q be nonempty closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear mapping. The *split feasibility problem* (SFP) is the following:

$$\text{find} \quad x^* \in C \quad \text{such that } Ax^* \in Q.$$

Let $T : C \to C$ and $S : Q \to Q$ be two mappings with $F(T) := \{x \in C : Tx = x\} \neq \emptyset$, $F(S) := \{x \in Q : Sx = x\} \neq \emptyset$. Then the *split common fixed point problem* (SCFPP) for mappings T and S is to find a point

$$q^* \in F(T) \quad \text{such that } Aq^* \in F(S).$$

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We shall denote the set of solutions of the SCFPP for mappings T and S by Ω, that is,

$$\Omega := \left\{ x^* \in F(T) \text{ such that } Ax^* \in F(S) \right\}.$$

The SFP was first introduced by Censor and Elfving [3] in finite dimensional real Hilbert spaces for modeling inverse problems which arise from phase retrievals and medical image reconstruction [2]. The split common fixed point problem in Hilbert spaces was introduced by Moudafi [9] in 2010. It is now well known that the SFP and SCFPP have applications in very important real life problems. Consequently, these problems have attracted the interest of many researchers.

These problems and their generalizations have been studied in real Hilbert spaces and iterative methods for approximating their solutions, assuming existence, in this setting abound in the literature (see, for example, Tang *et al.* [12], and the references therein).

Studies of SFP and SCFPP in real Banach spaces more general than Hilbert spaces are very scanty in the literature. Part of the reason for this may be that most of the tools used in real Hilbert spaces for studying them are confined to real Hilbert spaces. Consequently, new tools have to be developed for solving these problems in real Banach spaces more general than Hilbert spaces. Some of the earliest successes in this direction are the results of Takahashi [10] and Takahashi and Yao [11]. In 2015, Tang *et al.* [12] studied the SFP and SCFPP in real Banach spaces more general than Hilbert spaces and by using hybrid methods and Halpern-type methods, they proved strong and weak convergence of the sequence generated by their algorithm to solutions of these problems. For other results obtained in studying the SFP, the SCFPP and some of their generalizations, in real Banach spaces more general than Hilbert spaces, the reader may see any of the following recent papers [5, 6], and the references therein.

In 2014, Cui and Wang [7] studied the SCFPP of *τ-quasi-strictly pseudocontractive mappings* in the setting of Hilbert space. Motivated and inspired by this study and the research going on in the domain of split feasibility problems and SCFPP, Tang *et al.* [12] studied the SCFPP for a τ-quasi-strictly pseudocontractive mapping and an asymptotically nonexpansive mapping in the setting of two real Banach space which they assumed are more general than Hilbert spaces. Their setting is the following:

1. E_1 is a uniformly convex and 2-uniformly smooth real Banach space which has the Opial property and the best smoothness constant κ satisfying $0 < \kappa < \frac{1}{\sqrt{2}}$.
2. E_2 is a real Banach space.
3. $A : E_1 \to E_2$ is a bounded linear mapping and A^* is the adjoint of A.
4. $S : E_1 \to E_1$ is an l_n-asymptotically nonexpansive mapping with $\{l_n\} \subset [1, \infty)$, and $l_n \to 1$, as $n \to \infty$. $T : E_2 \to E_2$ is a τ-quasi-strict pseudocontractive mapping with $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$, and T is demiclosed at zero.

They proved the following theorem.

Theorem 1.1 (Theorem 3.1 of [12]) *Let E_1, E_2, A, S, T and $\{l_n\}$ be the same as above. For each $x_1 \in E_1$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} z_n = x_n + \gamma J_1^{-1} A^* J_2 (T - I) A x_n, \\ x_{n+1} = (1 - \alpha_n) z_n + \alpha_n S^n z_n, \qquad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ with $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, γ is a positive constant satisfying

$$0 < \gamma < \min\left\{ \frac{1 - 2\kappa^2}{\|A\|^2}, \frac{1 - \tau}{\|A\|^2} \right\},$$

$\{l_n\}$ is a sequence in $[1, \infty)$ with $L := \sup_{n\geq 1}\{l_n\}$ and $\sum_{n=1}^{\infty}(l_n - 1) < \infty$.

(I) *If $\Gamma := \{p \in F(S) : Ap \in F(T)\} \neq \emptyset$ (the set of solutions of the SCFPP is nonempty), then the sequence $\{x_n\}$ converges weakly to a point $x^* \in \Gamma$.*

(II) *In addition, if $\Gamma \neq \emptyset$, and S is semi-compact, then $\{x_n\}$ converges strongly to a point $x^* \in \Gamma$.*

We first make the following remarks concerning the proof of this theorem in Tang *et al.* [12].

Remark 1 Definition 2.3(ii) of the τ-strict pseudocontractive mapping for $T : C \to C, C \subset E$, where E is a real Banach space, given by inequality (2.2), is an error. This definition is restricted to real Hilbert spaces. The definition for τ-strict pseudocontractive mapping in real Banach spaces, E, more general than Hilbert spaces, in the sense of Browder and Pertishyn, is the following: $T : C \to C, C \subset E$, is said to be τ-strict pseudocontractive if, $\forall x, y \in C$, there exist $j(x - y) \in J(x - y)$ and $\tau \in (0, 1)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \tau \left\| (I - T)x - (I - T)y \right\|^2, \tag{1.1}$$

where $J : E \to 2^{E^*}$ is the normalized duality mapping on E. It is easy to show that in a real *Hilbert space*, inequality (1.1) is equivalent to inequality (2.2) in Tang *et al.* [12], and it is well known that in real Banach spaces more general than Hilbert spaces, inequality (2.2) in Tang *et al.* [12] is not equivalent to inequality (1.1). Consequently, the use of inequality (2.2) in the proof of Theorem 3.1 in Tang *et al.* [12] invalidates the argument of the proof.

Remark 2 The use of Lemma 2.1 by Tang *et al.* [12] in the proof of Theorem 1.1 is an error. As correctly stated in Lemma 2.1 (see Tang *et al.* [12] for the statement of Lemma 2.1) the inequality of the lemma can be used *provided that the vectors x and y are bounded*. They used this inequality in the proof without establishing the boundedness of the vectors $(z_n - p)$ and $(S^n z_n - p)$. This erroneous application of the lemma led to inequality (3.5) which showed that $\lim_{n\to\infty} \|x_n - p\|$ exists and consequently that $\{x_n\}$ is bounded, which plays a crucial role in what followed. This use of Lemma 2.1 also invalidates the argument in the proof of Theorem 3.1.

Remark 3 We begin by recalling the following definition.

Definition 1.2 Let E be a normed space with $\dim(E) \geq 2$. The *modulus of smoothness of E* is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup\left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau \right\}$$

$$= \sup\left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = 1; \|y\| = 1 \right\}.$$

The normed space E is called *uniformly smooth* if and only if $\lim_{\tau \to 0^+} \frac{\rho_E(\tau)}{\tau} = 0$. For $\alpha > 1$, a normed space E is said to be α-*uniformly smooth* if there exists a constant $c > 0$ such that $\rho_E(\tau) \le c\tau^\alpha$, $\tau > 0$.

It is well known that L_p, l_p and the Sobolev spaces $W_p^m(\Omega)$, $1 < p < \infty$, are all p-uniformly smooth and that the following estimates hold:

$$\rho_{L_p}(\tau) = \rho_{l_p}(\tau) = \rho_{W_p^m(\Omega)}(\tau) = \begin{cases} (1 + \tau^p)^{\frac{1}{p}} - 1 \le \frac{1}{p}\tau^p, & 1 < p < 2; \\ \frac{p-1}{2}\tau^2 + o(\tau^2) \le \frac{(p-1)}{2}\tau^2, & p \ge 2; \end{cases} \tag{1.2}$$

where $\tau \ge 0$ (see, e.g., Lindenstrauss and Tzafriri, [8]; see also Chidume [4], page 44). From (1.2), it is clear that, if $1 < p < 2$, then $E = L_p, l_p$ or $W_p^m(\Omega)$ is *not* 2-uniformly smooth.

Consider now $E = L_p, l_p$ or $W_p^m(\Omega)$ for $p \in [2, \infty)$. From (1.2), these spaces are 2-uniformly smooth. Furthermore, for these spaces, the following inequality holds:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x) \rangle + (p - 1)\|y\|^2, \quad \forall x, y \in E, \tag{1.3}$$

and $(p - 1)$ is the best smoothness constant (see, e.g., Bynun [1], Xu [13]; see also Chidume [4], page 44). Comparing inequality (1.3) with the following inequality in Tang *et al.* [12]:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x) \rangle + 2\|\kappa y\|^2, \quad \forall x, y \in E,$$

we find that $2\kappa^2 = p - 1$ so that $\kappa = \frac{\sqrt{p-1}}{\sqrt{2}}$. The condition that $0 < \kappa \le \frac{1}{\sqrt{2}}$ now implies that $p \le 2$. But if $p < 2$, then, from Eq. (1.2), E is *not* 2-uniformly smooth. So, the only possibility is that $p = 2$, i.e., $E = l_2$, a real Hilbert space.

2 Conclusion

The space E_1 described in the paper of Tang *et al.* [12] as a real Banach space more general than real Hilbert space is, in fact, necessarily a real Hilbert space. Furthermore, there are *serious* errors in the proof of Theorem 3.1 in Tang *et al.* [12]. Consequently, the following important problem which Tang *et al.* [12] tried to solve is still open.

3 Open problem

It is of interest to define a space E_1 that is a real Banach space more general than real Hilbert spaces, E_2 as defined in the paper of Tang *et al.* [12] and an iterative algorithm for solving split common fixed point problem involving a quasi-strict pseudocontractive mapping and an asymptotically nonexpansive mapping such that the sequence generated by the algorithm converges strongly to a solution of the problem.

Authors' contributions

The author read and approved the final manuscript.

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