Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

An accelerated viscosity forward-backward splitting algorithm with the linesearch process for convex minimization problems

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Abstract

In this paper, we consider and investigate a convex minimization problem of the sum of two convex functions in a Hilbert space. The forward-backward splitting algorithm is one of the popular optimization methods for approximating a minimizer of the function; however, the stepsize of this algorithm depends on the Lipschitz constant of the gradient of the function, which is not an easy work to find in general practice. By using a new modification of the linesearches of Cruz and Nghia [Optim. Methods Softw. 31:1209–1238, 2016] and Kankam et al. [Math. Methods Appl. Sci. 42:1352–1362, 2019] and an inertial technique, we introduce an accelerated viscosity-type algorithm without any Lipschitz continuity assumption on the gradient. A strong convergence result of the proposed algorithm is established under some control conditions. As applications, we apply our algorithm to solving image and signal recovery problems. Numerical experiments show that our method has a higher efficiency than the well-known methods in the literature.

MSC: 65K05; 90C25; 90C30

Keywords: Convex minimization problems; Forward-backward splitting; Linesearch; Inertial techniques; Viscosity approximation; Strong convergence

1 Introduction

The convex minimization problem is one of the important problems in mathematical optimization. It has been widely studied because its applications are desirable and can be used in many branches of science and in various real-world applications such as in image and signal processing, data classification and regression problems, etc., see [3, 5, 8, 10, 12, 13] and the references therein. Various optimization methods for solving the convex minimization problem have been introduced and developed by many researchers, see [1, 3–5, 7–9, 11, 14, 16–19, 23, 26, 28] for instance. In this work, we are interested in studying an unconstrained convex minimization problem of the sum of the following form:

$$\underset{x\in\mathcal{X}}{\operatorname{minimize}}\, h_1(x) + h_2(x), \tag{1}$$

where $\mathcal{X}$ is a Hilbert space, $h_1 : \mathcal{X} \to \mathbb{R}$ is a convex and differentiable function, and $h_2 : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ is a proper, lower semi-continuous, and convex function.

It is known that if a minimizer p^* of $h_1 + h_2$ exists, then p^* is a fixed point of the forward-backward operator

$$FB_\alpha := \underbrace{\operatorname{prox}_{\alpha h_2}}_{\text{backward step}} \underbrace{(I_d - \alpha \nabla h_1)}_{\text{forward step}},$$

where $\alpha > 0$, $\operatorname{prox}_{h_2}$ is the proximity operator of h_2, and ∇h_1 stands for the gradient of h_1, that is, $p^* = FB_\alpha(p^*)$. If ∇h_1 is Lipschitz continuous with a coefficient $L > 0$ and $\alpha \in (0, 2/L)$, then the forward-backward operator FB_α is nonexpansive. In this case, we can employ fixed point approximation methods for the class of nonexpansive operators to solve (1). One of the popular methods is known as the *forward-backward splitting* (FBS) algorithm [8, 18].

Method FBS Let $x_1 \in \mathcal{X}$. For $k \geq 1$, let

$$x_{k+1} = \operatorname{prox}_{\alpha_k h_2}\bigl(x_k - \alpha_k \nabla h_1(x_k)\bigr),$$

where $0 < \alpha_k < 2/L$.

This method includes the proximal point algorithm [19, 26], the gradient method [4, 11], and the *CQ* algorithm [6] as special cases. It can be seen from Method FBS that we need to assume the Lipschitz continuity condition on the gradient of h_1, and the stepsize α_k depends on the Lipschitz constant L. However, finding such a Lipschitz constant is not an easy task in general practice. This leads to the natural question:

Question: How can we construct an algorithm whose stepsize does not depend on any Lipschitz constant of the gradient for solving Problem (1)?

In the sequel, we set the standing hypotheses on Problem (1) as follows:

(AI) $h_1 : \mathcal{X} \to \mathbb{R}$ is a convex and differentiable function and the gradient ∇h_1 is uniformly continuous on $\mathcal{X}$;

(AII) $h_2 : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ is a proper, lower semi-continuous, and convex function.

We see that the second part of (AI) is a weaker condition than the Lipschitz continuity condition on ∇h_1.

In 2016, Cruz and Nghia [9] suggested one of the ways to select the stepsize α_k which is independent of the Lipschitz constant L by using the following linesearch process.

Linesearch A: Fix $x \in \mathcal{X}$, $\sigma > 0$, $\delta > 0$, and $\theta \in (0,1)$

Input $\alpha = \sigma$.
  While $\alpha \|\nabla h_1(FB_\alpha(x)) - \nabla h_1(x)\| > \delta \|FB_\alpha(x) - x\|$, **do**
    $\alpha = \theta\alpha$.
  End
Output α.

It was proved that Linesearch A is well defined, this means that it stops after finitely many steps, see [9, Lemma 3.1] and [32, Theorem 3.4(a)]. Linesearch A is a special case of the linesearch proposed in [32] for inclusion problems. Cruz and Nghia [9] employed the forward-backward splitting method where the stepsize α_k is generated by Linesearch A.

Method 1 Let $x_1 \in \mathcal{X}$, $\sigma > 0$, $\delta \in (0, 1/2)$, and $\theta \in (0, 1)$. For $k \geq 1$, let

$$x_{k+1} = \operatorname{prox}_{\alpha_k h_2}\bigl(x_k - \alpha_k \nabla h_1(x_k)\bigr),$$

where $\alpha_k :=$ Linesearch A$(x_k, \sigma, \theta, \delta)$.

In optimization theory, to speed up the convergence of iterative procedures, many mathematicians often use the inertial-type extrapolation [15, 22, 24] by supplementing the technical term $\beta_k(x_k - x_{k-1})$. We call the parameter β_k an *inertial* parameter, which controls the momentum $x_k - x_{k-1}$. Based on Method 1, Cruz and Nghia [9] also proposed an accelerated algorithm with an inertial technical term as follows.

Method 2 Let $x_0 = x_1 \in \mathcal{X}$, $\alpha_0 = \sigma > 0$, $\delta \in (0, 1/2)$, $\theta \in (0, 1)$, and $t_1 = 1$. For $k \geq 1$, let

$$t_{k+1} = \frac{1 + \sqrt{1 + 4t_k^2}}{2}, \qquad \beta_k = \frac{t_k - 1}{t_{k+1}},$$

$$y_k = x_k + \beta_k(x_k - x_{k-1}),$$

$$x_{k+1} = \operatorname{prox}_{\alpha_k h_2}\bigl(y_k - \alpha_k \nabla h_1(y_k)\bigr),$$

where $\alpha_k :=$ Linesearch A$(y_k, \alpha_{k-1}, \theta, \delta)$.

The technique of selecting β_k in Method 2 was first defined in the fast iterative shrinkage-thresholding algorithm (FISTA) by Beck and Teboulle [3].

In 2019, Kankam et al. [16] introduced a modification of Linesearch A as follows.

Linesearch B: Fix $x \in \mathcal{X}$, $\sigma > 0$, $\delta > 0$, and $\theta \in (0, 1)$

Input $\alpha = \sigma$.

While $\alpha \max\{\|\nabla h_1(FB_\alpha^2(x)) - \nabla h_1(FB_\alpha(x))\|, \|\nabla h_1(FB_\alpha(x)) - \nabla h_1(x)\|\}$
$> \delta(\|FB_\alpha^2(x) - FB_\alpha(x)\| + \|FB_\alpha(x) - x\|)$, **do**
$\alpha = \theta\alpha$.

End

Output α,

where $FB_\alpha^2(x) := FB_\alpha(FB_\alpha(x))$.

Using Linesearch B, they proposed the following double forward-backward splitting algorithm.

Method 3 Let $x_1 \in \mathcal{X}$, $\sigma > 0$, $\delta \in (0, 1/8)$, and $\theta \in (0, 1)$. For $k \geq 1$, let

$$y_k = \operatorname{prox}_{\alpha_k h_2}\bigl(x_k - \alpha_k \nabla h_1(x_k)\bigr),$$

$$x_{k+1} = \operatorname{prox}_{\alpha_k h_2}\bigl(y_k - \alpha_k \nabla h_1(y_k)\bigr),$$

where $\alpha_k :=$ Linesearch B$(x_k, \sigma, \theta, \delta)$.

We note that Methods 1–3 with some mild conditions guarantee only weak convergence results for Problem (1); however, strong convergence gives more desirable theoretical result. To get strong convergence, we focus on the forward-backward splitting algorithm based on the *viscosity approximation* method [21, 34] as follows.

Method 4 Let $x_1 \in \mathcal{X}$. For $k \geq 1$, let

$$x_{k+1} = \gamma_k f(x_k) + (1 - \gamma_k) \operatorname{prox}_{\alpha_k h_2}\bigl(x_k - \alpha_k \nabla h_1(x_k)\bigr),$$

where $f : \mathcal{X} \to \mathcal{X}$ is a contraction, $\gamma_k \in (0,1)$ and $\alpha_k > 0$.

In this work, inspired and motivated by the results of Cruz and Nghia [9] and Kankam et al. [16] and the above-mentioned research, we aim to improve Linesearches A and B and introduce a new accelerated algorithm using our proposed linesearch for strong convergence on a convex minimization problem of the sum of two convex functions in a Hilbert space. This paper is organized as follows. The notation, basic definitions, and some useful lemmas for proving our main result are given in Sect. 2. Our main result is in Sect. 3. In this section, we introduce a new modification of Linesearches A and B and present a double forward-backward algorithm based on the viscosity approximation method by using an inertial technique for solving Problem (1) with Assumptions (AI) and (AII). Subsequently, we prove a strong convergence theorem of the proposed method under some suitable control conditions. In Sect. 4, we apply the convex minimization problem to image and signal recovery problems. We analyze and illustrate the convergence behavior of our method, and also compare its efficiency with Methods 1–4.

2 Basic definitions and lemmas

The mathematical symbols adopted throughout this article are as follows. $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$ are the set of real numbers, the set of nonnegative real numbers, and the set of positive real numbers, respectively, and $\mathbb{N}$ stands for the set of positive integers. We suppose that $\mathcal{X}$ is a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Let I_d denote the identity operator on $\mathcal{X}$. Weak and strong convergence of a sequence $\{x_k\} \subset \mathcal{X}$ to $p \in \mathcal{X}$ are denoted by $x_k \rightharpoonup p$ and $x_k \to p$, respectively.

Let E be a nonempty closed convex subset of $\mathcal{X}$. An operator $A : E \to \mathcal{X}$ is said to be *Lipschitz continuous* if there exists $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in E.$$

If A is Lipschitz continuous with a coefficient $L \in (0,1)$, then A is called a *contraction*. The metric projection from $\mathcal{X}$ onto E, denoted by P_E, is defined for each $x \in \mathcal{X}$, $P_E x$ is the unique element in E such that $\|x - P_E x\| = \inf_{y \in E} \|x - y\|$. It is known that

$$p^* = P_E x \quad \Longleftrightarrow \quad \bigl\langle x - p^*, y - p^* \bigr\rangle \leq 0, \quad \forall y \in E.$$

The following definition extends the concept of the metric projection.

Definition 2.1 ([2, 20]) Let $h : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ be a proper, lower semi-continuous, and convex function. The *proximity* (or *proximal*) operator of h, denoted by prox_h, is defined for each $x \in \mathcal{X}$, $\operatorname{prox}_h x$ is the unique solution of the minimization problem

$$\underset{y\in\mathcal{X}}{\text{minimize}}\, h(y) + \frac{1}{2}\|x-y\|^2.$$

In particular, if $h := i_E$ is an indicator function on E (defined by $i_E(x) = 0$ if $x \in E$; otherwise $i_E(x) = \infty$), then $\operatorname{prox}_h = P_E$.

Let $h : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ be a proper, lower semi-continuous, and convex function. The subdifferential ∂h of h is defined by

$$\partial h(x) := \bigl\{p \in \mathcal{X} : h(x) + \langle p, y-x\rangle \le h(y), \forall y \in \mathcal{X}\bigr\}, \quad \forall x \in \mathcal{X}.$$

Here, we give some relationships between the proximity operator and the subdifferential operator as follows. For $\alpha > 0$ and $x \in \mathcal{X}$, then

$$\operatorname{prox}_{\alpha h} = (I_d + \alpha\partial h)^{-1} : \mathcal{X} \to \operatorname{dom} h, \tag{2}$$

$$\frac{x - \operatorname{prox}_{\alpha h}(x)}{\alpha} \in \partial h\bigl(\operatorname{prox}_{\alpha h}(x)\bigr). \tag{3}$$

We end this section by giving useful lemmas for proving our main result.

Lemma 2.2 ([25]) *Let $h : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ be a proper, lower semi-continuous, and convex function. Let $\{x_k\}$ and $\{y_k\}$ be two sequences in $\mathcal{X}$ such that $y_k \in \partial h(x_k)$ for all $k \in \mathbb{N}$. If $x_k \rightharpoonup x$ and $y_k \to y$, then $y \in \partial h(x)$.*

Lemma 2.3 ([29]) *Let $x, y \in \mathcal{X}$ and $\xi \in [0,1]$. Then the following properties hold on $\mathcal{X}$:*

(i) $\|\xi x + (1-\xi)y\|^2 = \xi\|x\|^2 + (1-\xi)\|y\|^2 - \xi(1-\xi)\|x-y\|^2$;

(ii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x,y\rangle + \|y\|^2$;

(iii) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$.

Lemma 2.4 ([27]) *Let $\{a_k\} \subset \mathbb{R}_+$, $\{b_k\} \subset \mathbb{R}$, and $\{\xi_k\} \subset (0,1)$ be such that $\sum_{k=1}^{\infty} \xi_k = \infty$ and*

$$a_{k+1} \le (1-\xi_k)a_k + \xi_k b_k, \quad \forall k \in \mathbb{N}.$$

If $\limsup_{i\to\infty} b_{k_i} \le 0$ for every subsequence $\{a_{k_i}\}$ of $\{a_k\}$ satisfying $\liminf_{i\to\infty}(a_{k_i+1} - a_{k_i}) \ge 0$, then $\lim_{k\to\infty} a_k = 0$.

3 Method and convergence result

In this section, by modifying Linesearches A and B, we introduce a new linesearch and present an inertial double forward-backward splitting algorithm based on the viscosity approximation method for solving the convex minimization problem of the sum of two convex functions without any Lipschitz continuity assumption on the gradient. A strong convergence result of our proposed algorithm is analyzed and established.

We now focus on Problem (1) with Assumptions (AI) and (AII). For simplicity, let $\mathbf{h} := h_1 + h_2$ and denote $FB_\alpha := \operatorname{prox}_{\alpha h_2}(I_d - \alpha\nabla h_1)$ for $\alpha > 0$. The set of minimizer of $\mathbf{h}$ is denoted by Γ. Also, assume that $\Gamma \neq \emptyset$. We begin by designing the following linesearch.

Linesearch C: Fix $x \in \mathcal{X}$, $\sigma > 0$, $\delta > 0$, and $\theta \in (0,1)$

Input $\alpha = \sigma$.

While $\frac{\alpha}{2}\{\|\nabla h_1(FB_\alpha^2(x)) - \nabla h_1(FB_\alpha(x))\| + \|\nabla h_1(FB_\alpha(x)) - \nabla h_1(x)\|\}$
$> \delta(\|FB_\alpha^2(x) - FB_\alpha(x)\| + \|FB_\alpha(x) - x\|)$, **do**
$\alpha = \theta\alpha$.

End

Output α.

In other words, if $\alpha :=$ Linesearch C$(x, \sigma, \theta, \delta)$, then $\alpha = \sigma\theta^m$, where m is the smallest nonnegative integer such that

$$\begin{aligned}&\frac{\alpha}{2}\bigl\{\bigl\|\nabla h_1\bigl(FB_\alpha^2(x)\bigr) - \nabla h_1\bigl(FB_\alpha(x)\bigr)\bigr\| + \bigl\|\nabla h_1\bigl(FB_\alpha(x)\bigr) - \nabla h_1(x)\bigr\|\bigr\}\\ &\quad \le \delta\bigl(\bigl\|FB_\alpha^2(x) - FB_\alpha(x)\bigr\| + \bigl\|FB_\alpha(x) - x\bigr\|\bigr).\end{aligned}$$

It can be seen that the terminating condition of the while loop in Linesearch C is somewhat weaker than that in Linesearch B. So, it follows from the well-definedness of Linesearch B that our linesearch also stops after finitely many steps, see [16, Lemma 3.2].

Using Linesearch C, we introduce a new viscosity forward-backward splitting algorithm with the inertial technical term as follows.

Method 5: An accelerated viscosity forward-backward algorithm with Linesearch C

Initialization: Pick $x_0 = x_1 \in \mathcal{X}$, $\sigma > 0$, $\delta \in (0, 1/8)$, and $\theta \in (0,1)$.
Take $\{\gamma_k\}, \{\tau_k\} \subset \mathbb{R}_{++}$, and let $\{\mu_k\} \subset \mathbb{R}_+$ be a bounded sequence.
Let $f : \mathcal{X} \to \mathcal{X}$ be a contraction with a coefficient $\eta \in (0,1)$.
Iterative steps: For $k \geq 1$, calculate x_{k+1} as follows:
Step 1. Compute the inertial step:

$$\beta_k = \begin{cases} \min\{\mu_k, \frac{\tau_k}{\|x_k - x_{k-1}\|}\} & \text{if } x_k \neq x_{k-1}, \\ \mu_k & \text{otherwise,} \end{cases} \tag{4}$$

$$w_k = x_k + \beta_k(x_k - x_{k-1}). \tag{5}$$

Step 2. Compute the forward-backward step:

$$z_k = FB_{\alpha_k}(w_k) = \operatorname{prox}_{\alpha_k h_2}\bigl(w_k - \alpha_k \nabla h_1(w_k)\bigr), \tag{6}$$

$$y_k = FB_{\alpha_k}(z_k) = \operatorname{prox}_{\alpha_k h_2}\bigl(z_k - \alpha_k \nabla h_1(z_k)\bigr), \tag{7}$$

where $\alpha_k :=$ Linesearch C$(w_k, \sigma, \theta, \delta)$.
Step 3. Compute the viscosity step:

$$x_{k+1} = \gamma_k f(x_k) + (1 - \gamma_k) y_k. \tag{8}$$

Set $k := k + 1$ and return to Step 1.

To show a strong convergence result of Method 5, the following tool is needed.

Lemma 3.1 *Let $\{x_k\}$ be a sequence generated by Method 5 and $p \in \mathcal{X}$. Then the following inequality holds*:

$$\begin{aligned}\|w_k - p\|^2 - \|y_k - p\|^2 \geq {} & 2\alpha_k\bigl[\boldsymbol{h}(y_k) + \boldsymbol{h}(z_k) - 2\boldsymbol{h}(p)\bigr] \\ & + (1 - 8\delta)\bigl(\|w_k - z_k\|^2 + \|z_k - y_k\|^2\bigr), \quad \forall k \in \mathbb{N}.\end{aligned}$$

Proof From (3), (6), and (7), we get

$$\frac{w_k - z_k}{\alpha_k} - \nabla h_1(w_k) \in \partial h_2(z_k) \quad \text{and} \quad \frac{z_k - y_k}{\alpha_k} - \nabla h_1(z_k) \in \partial h_2(y_k).$$

Let $p \in \mathcal{X}$. By the definition of subdifferential of h_2, the above expressions give

$$\begin{aligned}h_2(p) - h_2(z_k) &\geq \left\langle \frac{w_k - z_k}{\alpha_k} - \nabla h_1(w_k), p - z_k \right\rangle \\ &= \frac{1}{\alpha_k}\langle w_k - z_k, p - z_k\rangle + \bigl\langle \nabla h_1(w_k), z_k - p\bigr\rangle \end{aligned} \tag{9}$$

and

$$\begin{aligned}h_2(p) - h_2(y_k) &\geq \left\langle \frac{z_k - y_k}{\alpha_k} - \nabla h_1(z_k), p - y_k \right\rangle \\ &= \frac{1}{\alpha_k}\langle z_k - y_k, p - y_k\rangle + \bigl\langle \nabla h_1(z_k), y_k - p\bigr\rangle. \end{aligned} \tag{10}$$

By (AI), we obtain the fact

$$h_1(x) - h_1(y) \geq \bigl\langle \nabla h_1(y), x - y\bigr\rangle, \quad \forall x, y \in \mathcal{X}. \tag{11}$$

From (11), we get

$$h_1(p) - h_1(w_k) \geq \bigl\langle \nabla h_1(w_k), p - w_k\bigr\rangle \tag{12}$$

and

$$h_1(p) - h_1(z_k) \geq \bigl\langle \nabla h_1(z_k), p - z_k\bigr\rangle. \tag{13}$$

Combining (9), (10), (12), and (13), we have

$$\begin{aligned}&2\mathbf{h}(p) - \mathbf{h}(z_k) - h_2(y_k) - h_1(w_k) \\ &\quad\geq \bigl\langle \nabla h_1(w_k), z_k - p\bigr\rangle + \bigl\langle \nabla h_1(z_k), y_k - p\bigr\rangle + \bigl\langle \nabla h_1(w_k), p - w_k\bigr\rangle \\ &\qquad + \bigl\langle \nabla h_1(z_k), p - z_k\bigr\rangle + \frac{1}{\alpha_k}\bigl[\langle w_k - z_k, p - z_k\rangle + \langle z_k - y_k, p - y_k\rangle\bigr] \\ &\quad= \bigl\langle \nabla h_1(w_k), z_k - w_k\bigr\rangle + \bigl\langle \nabla h_1(z_k), y_k - z_k\bigr\rangle\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\alpha_k}\big[\langle w_k-z_k,p-z_k\rangle+\langle z_k-y_k,p-y_k\rangle\big]\\
&=\big\langle\nabla h_1(w_k)-\nabla h_1(z_k),z_k-w_k\big\rangle+\big\langle\nabla h_1(z_k),z_k-w_k\big\rangle+\big\langle\nabla h_1(y_k),y_k-z_k\big\rangle\\
&\quad+\big\langle\nabla h_1(z_k)-\nabla h_1(y_k),y_k-z_k\big\rangle+\frac{1}{\alpha_k}\big[\langle w_k-z_k,p-z_k\rangle+\langle z_k-y_k,p-y_k\rangle\big]\\
&\geq\big\langle\nabla h_1(z_k),z_k-w_k\big\rangle+\big\langle\nabla h_1(y_k),y_k-z_k\big\rangle-\big\|\nabla h_1(w_k)-\nabla h_1(z_k)\big\|\|z_k-w_k\|\\
&\quad-\big\|\nabla h_1(z_k)-\nabla h_1(y_k)\big\|\|y_k-z_k\|\\
&\quad+\frac{1}{\alpha_k}\big[\langle w_k-z_k,p-z_k\rangle+\langle z_k-y_k,p-y_k\rangle\big].
\end{aligned}
$$

Again, applying (11), the above inequality becomes

$$
\begin{aligned}
&2\mathbf{h}(p)-\mathbf{h}(z_k)-h_2(y_k)-h_1(w_k)\\
&\quad\geq h_1(y_k)-h_1(w_k)-\big\|\nabla h_1(w_k)-\nabla h_1(z_k)\big\|\|z_k-w_k\|\\
&\qquad-\big\|\nabla h_1(z_k)-\nabla h_1(y_k)\big\|\|y_k-z_k\|\\
&\qquad+\frac{1}{\alpha_k}\big[\langle w_k-z_k,p-z_k\rangle+\langle z_k-y_k,p-y_k\rangle\big]\\
&\quad\geq h_1(y_k)-h_1(w_k)-\big\|\nabla h_1(w_k)-\nabla h_1(z_k)\big\|\big(\|y_k-z_k\|+\|z_k-w_k\|\big)\\
&\qquad-\big\|\nabla h_1(z_k)-\nabla h_1(y_k)\big\|\big(\|y_k-z_k\|+\|z_k-w_k\|\big)\\
&\qquad+\frac{1}{\alpha_k}\big[\langle w_k-z_k,p-z_k\rangle+\langle z_k-y_k,p-y_k\rangle\big]\\
&\quad=h_1(y_k)-h_1(w_k)+\frac{1}{\alpha_k}\big[\langle w_k-z_k,p-z_k\rangle+\langle z_k-y_k,p-y_k\rangle\big]\\
&\qquad-\big(\big\|\nabla h_1(w_k)-\nabla h_1(z_k)\big\|+\big\|\nabla h_1(z_k)-\nabla h_1(y_k)\big\|\big)\big(\|y_k-z_k\|+\|z_k-w_k\|\big).
\end{aligned}
\tag{14}
$$

Since $\alpha_k :=$ Linesearch C$(w_k,\sigma,\theta,\delta)$, then

$$
\begin{aligned}
&\frac{\alpha_k}{2}\big\{\big\|\nabla h_1(y_k)-\nabla h_1(z_k)\big\|+\big\|\nabla h_1(z_k)-\nabla h_1(w_k)\big\|\big\}\\
&\quad\leq\delta\big(\|y_k-z_k\|+\|z_k-w_k\|\big).
\end{aligned}
\tag{15}
$$

From (14) and (15), we have

$$
\begin{aligned}
&\frac{1}{\alpha_k}\big[\langle w_k-z_k,z_k-p\rangle+\langle z_k-y_k,y_k-p\rangle\big]\\
&\quad\geq\mathbf{h}(y_k)+\mathbf{h}(z_k)-2\mathbf{h}(p)\\
&\qquad-\big(\big\|\nabla h_1(w_k)-\nabla h_1(z_k)\big\|+\big\|\nabla h_1(z_k)-\nabla h_1(y_k)\big\|\big)\big(\|y_k-z_k\|+\|z_k-w_k\|\big)\\
&\quad\geq\mathbf{h}(y_k)+\mathbf{h}(z_k)-2\mathbf{h}(p)-\frac{2\delta}{\alpha_k}\big(\|y_k-z_k\|+\|z_k-w_k\|\big)^2\\
&\quad\geq\mathbf{h}(y_k)+\mathbf{h}(z_k)-2\mathbf{h}(p)-\frac{4\delta}{\alpha_k}\big(\|y_k-z_k\|^2+\|z_k-w_k\|^2\big).
\end{aligned}
\tag{16}
$$

By Lemma 2.3(ii), we get

$$\langle w_k - z_k, z_k - p\rangle = \frac{1}{2}\bigl(\|w_k - p\|^2 - \|w_k - z_k\|^2 - \|z_k - p\|^2\bigr), \tag{17}$$

and

$$\langle z_k - y_k, y_k - p\rangle = \frac{1}{2}\bigl(\|z_k - p\|^2 - \|z_k - y_k\|^2 - \|y_k - p\|^2\bigr). \tag{18}$$

Hence, we can conclude from (16)–(18) that

$$\begin{aligned}\|w_k - p\|^2 - \|y_k - p\|^2 \geq{}& 2\alpha_k\bigl[\mathbf{h}(y_k) + \mathbf{h}(z_k) - 2\mathbf{h}(p)\bigr]\\ &+ (1 - 8\delta)\bigl(\|w_k - z_k\|^2 + \|z_k - y_k\|^2\bigr), \quad \forall k \in \mathbb{N}.\end{aligned}$$ □

Now we are in a position to prove our main theorem.

Theorem 3.2 *Let* $\{x_k\} \subset \mathcal{X}$ *be a sequence generated by Method 5. Then*:

(i) *For* $p \in \Gamma$, *we have*

$$\|x_{k+1} - p\| \leq \max\left\{\|x_k - p\|, \frac{\frac{\beta_k}{\gamma_k}\|x_k - x_{k-1}\| + \|f(p) - p\|}{1 - \eta}\right\}, \quad \forall k \in \mathbb{N}.$$

(ii) *If the sequences* $\{\alpha_k\}$, $\{\gamma_k\}$, *and* $\{\tau_k\}$ *satisfy the following conditions*:

(Ci) $\alpha_k \geq \alpha$ *for some* $a \in \mathbb{R}_{++}$;

(Cii) $\gamma_k \in (0,1)$ *such that* $\lim_{k\to\infty}\gamma_k = 0$ *and* $\sum_{k=1}^{\infty}\gamma_k = \infty$;

(Ciii) $\lim_{k\to\infty}\tau_k/\gamma_k = 0$,

then $\{x_k\}$ *converges strongly to a point* $p^* \in \Gamma$, *where* $p^* = P_\Gamma f(p^*)$.

Proof Let $p \in \Gamma$. Applying Lemma 3.1, we have

$$\begin{aligned}\|w_k - p\|^2 - \|y_k - p\|^2 &\geq 2\alpha_k\bigl[\mathbf{h}(y_k) - \mathbf{h}(p) + \mathbf{h}(z_k) - \mathbf{h}(p)\bigr]\\ &\quad + (1 - 8\delta)\bigl(\|w_k - z_k\|^2 + \|z_k - y_k\|^2\bigr)\\ &\geq (1 - 8\delta)\bigl(\|w_k - z_k\|^2 + \|z_k - y_k\|^2\bigr) \qquad (19)\\ &\geq 0. \qquad (20)\end{aligned}$$

From (19) and (5) and by Lemma 2.3(ii), we get

$$\begin{aligned}\|y_k - p\|^2 &\leq \|w_k - p\|^2 - (1 - 8\delta)\bigl(\|w_k - z_k\|^2 + \|z_k - y_k\|^2\bigr)\\ &= \|x_k - p\|^2 + \beta_k^2\|x_k - x_{k-1}\|^2 + 2\beta_k\langle x_k - p, x_k - x_{k-1}\rangle\\ &\quad - (1 - 8\delta)\bigl(\|w_k - z_k\|^2 + \|z_k - y_k\|^2\bigr).\\ &\leq \|x_k - p\|^2 + \beta_k^2\|x_k - x_{k-1}\|^2 + 2\beta_k\|x_k - p\|\|x_k - x_{k-1}\|\\ &\quad - (1 - 8\delta)\bigl(\|w_k - z_k\|^2 + \|z_k - y_k\|^2\bigr). \qquad (21)\end{aligned}$$

From (20) and (5), we get

$$\|y_k - p\| \leq \|w_k - p\| \leq \|x_k - p\| + \beta_k\|x_k - x_{k-1}\|. \tag{22}$$

By (8) and (22), we have

$$\begin{aligned}\|x_{k+1}-p\| &\le \gamma_k\big\|f(x_k)-f(p)\big\| + \gamma_k\big\|f(p)-p\big\| + (1-\gamma_k)\|y_k-p\| \\ &\le \gamma_k\eta\|x_k-p\| + \gamma_k\big\|f(p)-p\big\| + (1-\gamma_k)\|y_k-p\| \\ &\le \big(1-\gamma_k(1-\eta)\big)\|x_k-p\| + \gamma_k\big\|f(p)-p\big\| \\ &\quad + (1-\gamma_k)\beta_k\|x_k-x_{k-1}\| \\ &\le \big(1-\gamma_k(1-\eta)\big)\|x_k-p\| + \gamma_k\left(\frac{\beta_k}{\gamma_k}\|x_k-x_{k-1}\| + \big\|f(p)-p\big\|\right) \\ &\le \max\left\{\|x_k-p\|, \frac{\frac{\beta_k}{\gamma_k}\|x_k-x_{k-1}\| + \|f(p)-p\|}{1-\eta}\right\}.\end{aligned}$$

Therefore, we obtain (i). By (4) and using (Ciii), we have $\frac{\beta_k}{\gamma_k}\|x_k-x_{k-1}\| \to 0$ as $k\to\infty$, and so there exists $M>0$ such that $\frac{\beta_k}{\gamma_k}\|x_k-x_{k-1}\| \le M$ for all $k\in\mathbb{N}$. Thus,

$$\|x_{k+1}-p\| \le \max\left\{\|x_k-p\|, \frac{M+\|f(p)-p\|}{1-\eta}\right\}.$$

By mathematical induction, we deduce that

$$\|x_k-p\| \le \max\left\{\|x_1-p\|, \frac{M+\|f(p^*)-p\|}{1-\eta}\right\}, \quad \forall k\in\mathbb{N}.$$

Hence, $\{x_k\}$ is bounded. One can see that the operator $P_\Gamma f$ is a contraction. By the Banach contraction principle, there is a unique point $p^*\in\Gamma$ such that $p^* = P_\Gamma f(p^*)$. It follows from the characterization of P_Γ that

$$\big\langle f\big(p^*\big)-p^*, p-p^*\big\rangle \le 0, \quad \forall p\in\Gamma. \tag{23}$$

Using Lemma 2.3(i), (iii) and (21), we have

$$\begin{aligned}\big\|x_{k+1}-p^*\big\|^2 &\le \big\|(1-\gamma_k)\big(y_k-p^*\big) + \gamma_k\big(f(x_k)-f\big(p^*\big)\big)\big\|^2 \\ &\quad + 2\gamma_k\big\langle f\big(p^*\big)-p^*, x_{k+1}-p^*\big\rangle \\ &\le (1-\gamma_k)\big\|y_k-p^*\big\|^2 + \gamma_k\big\|f(x_k)-f\big(p^*\big)\big\|^2 \\ &\quad + 2\gamma_k\big\langle f\big(p^*\big)-p^*, x_{k+1}-p^*\big\rangle \\ &\le (1-\gamma_k)\big\|x_k-p^*\big\|^2 + \beta_k^2\|x_k-x_{k-1}\|^2 \\ &\quad + 2\beta_k\big\|x_k-p^*\big\|\|x_k-x_{k-1}\| \\ &\quad + \gamma_k\eta\big\|x_k-p^*\big\|^2 + 2\gamma_k\big\langle f\big(p^*\big)-p^*, x_{k+1}-p^*\big\rangle \\ &\quad - (1-\gamma_k)(1-8\delta)\big(\|w_k-z_k\|^2 + \|z_k-y_k\|^2\big) \\ &= \big(1-\gamma_k(1-\eta)\big)\big\|x_k-p^*\big\|^2 + \gamma_k(1-\eta)b_k \\ &\quad - (1-\gamma_k)(1-8\delta)\big(\|w_k-z_k\|^2 + \|z_k-y_k\|^2\big),\end{aligned} \tag{24}$$

where

$$b_k := \frac{1}{1-\eta}\left(2\langle f(p^*) - p^*, x_{k+1} - p^*\rangle + \frac{\beta_k^2}{\gamma_k}\|x_k - x_{k-1}\|^2 + 2\frac{\beta_k}{\gamma_k}\|x_k - p^*\|\|x_k - x_{k-1}\|\right).$$

It follows that

$$\begin{aligned}(1-\gamma_k)(1-8\delta)\left(\|w_k - z_k\|^2 + \|z_k - y_k\|^2\right) \le \left\|x_k - p^*\right\|^2 - \left\|x_{k+1} - p^*\right\|^2 \\ + \gamma_k(1-\eta)M',\end{aligned} \tag{25}$$

where $M' = \sup\{b_k : k \in \mathbb{N}\}$.

Let us show that $\{x_k\}$ converges to p^*. Set $a_k := \|x_k - p^*\|^2$ and $\xi_k := \gamma_k(1-\eta)$. From (24), we have the following inequality:

$$a_{k+1} \le (1-\xi_k)a_k + \xi_k b_k.$$

To apply Lemma 2.4, we have to show that $\limsup_{i\to\infty} b_{k_i} \le 0$ whenever a subsequence $\{a_{k_i}\}$ of $\{a_k\}$ satisfies

$$\liminf_{i\to\infty}(a_{k_i+1} - a_{k_i}) \ge 0. \tag{26}$$

To do this, suppose that $\{a_{k_i}\} \subseteq \{a_k\}$ is a subsequence satisfying (26). Then, by (25) and (Cii), we have

$$\begin{aligned}&\limsup_{i\to\infty}(1-\gamma_{k_i})(1-8\delta)\left(\|w_{k_i} - z_{k_i}\|^2 + \|z_{k_i} - y_{k_i}\|^2\right) \\ &\quad\le \limsup_{i\to\infty}(a_{k_i} - a_{k_i+1}) + (1-\eta)M'\lim_{i\to\infty}\gamma_{k_i} \\ &\quad= -\liminf_{i\to\infty}(a_{k_i+1} - a_{k_i}) \\ &\quad\le 0,\end{aligned}$$

which implies

$$\lim_{i\to\infty}\|w_{k_i} - z_{k_i}\| = \lim_{i\to\infty}\|z_{k_i} - y_{k_i}\| = 0. \tag{27}$$

Using (Cii), (Ciii), and (27), we have

$$\begin{aligned}\|x_{k_i+1} - x_{k_i}\| &\le \gamma_{k_i}\left\|f(x_{k_i}) - y_{k_i}\right\| + \|y_{k_i} - x_{k_i}\| \\ &\le \gamma_{k_i}\left\|f(x_{k_i}) - y_{k_i}\right\| + \|y_{k_i} - w_{k_i}\| + \|w_{k_i} - x_{k_i}\| \\ &\le \gamma_{k_i}\left\|f(x_{k_i}) - y_{k_i}\right\| + \|y_{k_i} - z_{k_i}\| + \|z_{k_i} - w_{k_i}\| \\ &\quad + \frac{\beta_{k_i}}{\gamma_{k_i}}\|x_{k_i} - x_{k_i-1}\| \\ &\to 0\end{aligned} \tag{28}$$

as $i \to \infty$. We next show that $\limsup_{i\to\infty} b_{k_i} \le 0$. Clearly, it suffices to show that

$$\limsup_{i\to\infty} \langle f(p^*) - p^*, x_{k_i+1} - p^* \rangle \le 0.$$

Let $\{x_{k_{i_j}}\}$ be a subsequence of $\{x_{k_i}\}$ such that

$$\lim_{j\to\infty} \langle f(p^*) - p^*, x_{k_{i_j}} - p^* \rangle = \limsup_{i\to\infty} \langle f(p^*) - p^*, x_{k_i} - p^* \rangle.$$

Since $\{x_{k_{i_j}}\}$ is bounded, there exists a subsequence $\{x_{k_{i_{j_p}}}\}$ of $\{x_{k_{i_j}}\}$ such that $x_{k_{i_{j_p}}} \rightharpoonup \bar{p} \in \mathcal{X}$. Without loss of generality, we may assume that $x_{k_{i_j}} \rightharpoonup \bar{p}$. Thus, we also have $z_{k_{i_j}} \rightharpoonup \bar{p}$. From (AI), we have $\|\nabla h_1(w_{k_{i_j}}) - \nabla h_1(z_{k_{i_j}})\| \to 0$ as $j \to \infty$. This together with (27) and (Ci) yields

$$\lim_{j\to\infty} \left\| \frac{w_{k_{i_j}} - z_{k_{i_j}}}{\alpha_{k_{i_j}}} + \nabla h_1(z_{k_{i_j}}) - \nabla h_1(w_{k_{i_j}}) \right\| = 0. \tag{29}$$

By (3), we get

$$\frac{w_{k_{i_j}} - z_{k_{i_j}}}{\alpha_{k_{i_j}}} + \nabla h_1(z_{k_{i_j}}) - \nabla h_1(w_{k_{i_j}}) \in \partial h_2(z_{k_{i_j}}) + \nabla h_1(z_{k_{i_j}}) = \partial \mathbf{h}(z_{k_{i_j}}). \tag{30}$$

Now, by (29), (30), and $z_{k_{i_j}} \rightharpoonup \bar{p}$, it follows from Lemma 2.2 that $0 \in \partial \mathbf{h}(\bar{p})$. Hence, $\bar{p} \in \Gamma$. From (28) and (23), we have

$$\begin{aligned}
\limsup_{i\to\infty} \langle f(p^*) - p^*, x_{k_i+1} - p^* \rangle &\le \limsup_{i\to\infty} \langle f(p^*) - p^*, x_{k_i+1} - x_{k_i} \rangle \\
&\quad + \limsup_{i\to\infty} \langle f(p^*) - p^*, x_{k_i} - p^* \rangle \\
&= \lim_{j\to\infty} \langle f(p^*) - p^*, x_{k_{i_j}} - p^* \rangle \\
&= \langle f(p^*) - p^*, \bar{p} - p^* \rangle \\
&\le 0.
\end{aligned}$$

By Lemma 2.4, we can conclude that $\{x_k\}$ converges to p^*. The proof is complete. □

Note that the stepsize condition on $\{\alpha_k\}$ in Theorem 3.2 needs the boundedness from below by a positive real number. Next, we show that this condition can be ensured by the Lipschitz continuity assumption on ∇h_1.

Proposition 3.3 *Let $\{\alpha_k\}$ be the sequence generated by Linesearch C of Method 5. If $\nabla h_1 : \mathcal{X} \to \mathcal{X}$ is Lipschitz continuous with a constant $L > 0$, then $\alpha_k \ge \min\{\sigma, 2\delta\theta/L\}$ for all $k \in \mathbb{N}$.*

Proof Let ∇h_1 be L-Lipschitz continuous on $\mathcal{X}$. Since $\alpha_k :=$ Linesearch $\mathrm{C}(w_k, \sigma, \theta, \delta)$, then $\alpha_k \le \sigma$ for all $k \in \mathbb{N}$. If $\alpha_k < \sigma$, then $\alpha_k = \sigma\theta^{m_k}$ where m_k is the smallest positive integer such that

$$\begin{aligned}
&\frac{\alpha_k}{2} \{ \| \nabla h_1(FB^2_{\alpha_k}(w_k)) - \nabla h_1(FB_{\alpha_k}(w_k)) \| + \| \nabla h_1(FB_{\alpha_k}(w_k)) - \nabla h_1(w_k) \| \} \\
&\quad \le \delta \big(\| FB^2_{\alpha_k}(w_k) - FB_{\alpha_k}(w_k) \| + \| FB_{\alpha_k}(w_k) - w_k \| \big).
\end{aligned}$$

Set $\hat{\alpha_k} := \alpha_k/\theta$. By the Lipschitz continuity of ∇h_1 and the above expression, we have

$$\begin{aligned}&\frac{\hat{\alpha_k}L}{2}\big(\big\|FB^2_{\hat{\alpha_k}}(w_k)-FB_{\hat{\alpha_k}}(w_k)\big\|+\big\|FB_{\hat{\alpha_k}}(w_k)-w_k\big\|\big)\\&\quad\geq\frac{\hat{\alpha_k}}{2}\big(\big\|\nabla h_1\big(FB^2_{\hat{\alpha_k}}(w_k)\big)-\nabla h_1\big(FB_{\hat{\alpha_k}}(w_k)\big)\big\|+\big\|\nabla h_1\big(FB_{\hat{\alpha_k}}(w_k)\big)-\nabla h_1(w_k)\big\|\big)\\&\quad>\delta\big(\big\|FB^2_{\hat{\alpha_k}}(w_k)-FB_{\hat{\alpha_k}}(w_k)\big\|+\big\|FB_{\hat{\alpha_k}}(w_k)-w_k\big\|\big),\end{aligned}$$

it follows that $\alpha_k > 2\delta\theta/L$. Therefore, $\alpha_k \geq \min\{\sigma, 2\delta\theta/L\}$ for all $k \in \mathbb{N}$. □

Remark 3.4 It is worth mentioning that the Lipschitz continuity assumption on the gradient of h_1 is sufficient for Assumption (AI). However, if we assume this assumption further, the computation of the stepsize α_k generated by Linesearch C is still independent of the Lipschitz constant.

4 Numerical experiments in image and signal recovery

In this section, we apply the convex minimization problem, Problem (1), to image and signal recovery problems. We analyze and illustrate the convergence behavior of Method 5 for recovering images and signals, and also compare its efficiency with Methods 1–4. All experiments and visualizations are performed on a laptop computer (Intel Core-i5/4.00 GB RAM/Windows 8/64-bit) with MATLAB.

Many problems in image and signal processing, especially the image/signal recovery, are the problems of inferring an image/signal $x \in \mathbb{R}^N$ from the observation of an image/signal $y \in \mathbb{R}^M$ via the linear equation

$$y = Tx + \varepsilon, \tag{31}$$

where $T : \mathbb{R}^N \to \mathbb{R}^M$ is a bounded linear operator and ε is an additive noise. To approximate the original image/signal in (31), we need to minimize the value of ε by using the LASSO problem [31]

$$\min_{x\in\mathbb{R}^N}\left\{\frac{1}{2}\|y-Tx\|_2^2+\lambda\|x\|_1\right\}, \tag{32}$$

where λ is a positive parameter, $\|\cdot\|_1$ is the l_1-norm, and $\|\cdot\|_2$ is the Euclidean norm. It is worth noting that Problem (1) can be applied to the LASSO problem (32) by setting

$$h_1(x) = \frac{1}{2}\|y-Tx\|_2^2 \quad \text{and} \quad h_2(x) = \lambda\|x\|_1.$$

4.1 Image recovery

In the following two examples, we set a regularization parameter in the LASSO problem (32) by $\lambda := 10^{-5}$. Signal-to-noise ratio (PSNR) in decibel (dB) [30] and structural similarity index metric (SSIM) [33] are used as image quality metrics. The maximum iteration number for all deblurring methods is fixed at 500.

Example 4.1 Consider a prototype image (Lenna) with size of 256×256, which is contaminated by Gaussian blur of filter size 7×7 with standard deviation $\hat{\sigma} = 6$ and noise

10^{-5}, see the original image (a) and the blurred image (b) in Fig. 1. The values of PSNR and SSIM of the blurred image are 24.6547 dB and 0.4770, respectively. The parameters of our method (Method 5) are chosen as follows:

$$\sigma = 2, \qquad \theta = 0.9, \qquad \delta = 0.1, \qquad \tau_k = \frac{10^{50}}{k^2}, \qquad \gamma_k = \frac{1}{50k}, \qquad \mu_k = \frac{t_k - 1}{t_{k+1}},$$

$$t_{k+1} = \frac{1 + \sqrt{1 + 4t_k^2}}{2}, \qquad t_1 = 1.$$

Consider a contraction f in the form of $f(x) = \eta x$, where $0 < \eta < 1$. We take the parameter η as the following five cases:

Case 1: $\eta = 0.1$, *Case 2:* $\eta = 0.3$, *Case 3:* $\eta = 0.5$, *Case 4:* $\eta = 0.8$,

Case 5: $\eta = 0.99$.

Now, the experiments for recovering the Lenna image of Method 5 with Cases 1–5 are shown in Figs. 1 and 2. It is observed from Fig. 2 that Case 5 gives the higher values of PSNR and SSIM than other cases.

Figure 1 Restoration for the Lenna image at the 500th iteration. (**a**) Original image; (**b**) Blurry image contaminated by Gaussian blur; (**c**)–(**g**) Restored images by Method 5 with different parameters η

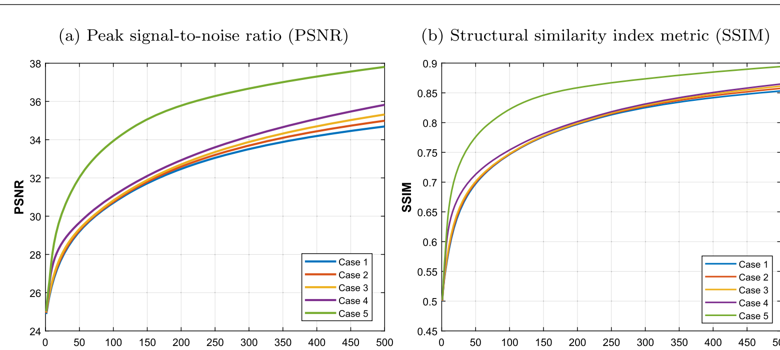

Figure 2 Plot of PSNR and SSIM of restored images by Method 5

Table 1 The parameters for the deblurring methods

Parameters	Methods				
	1	2	3	4	5
$\alpha_k = \frac{k}{(k+1)L}, L = \lambda_{\max}(T^\top T)$	–	–	–	√	–
$\sigma = 1, \theta = 0.9, \delta = 0.1$	√	√	√	–	√
$\mu_k = \frac{k}{k+1}, \tau_k = \frac{10^{50}}{k^2}$	–	–	–	–	√
$\gamma_k = \frac{1}{50k}$	–	–	–	√	√

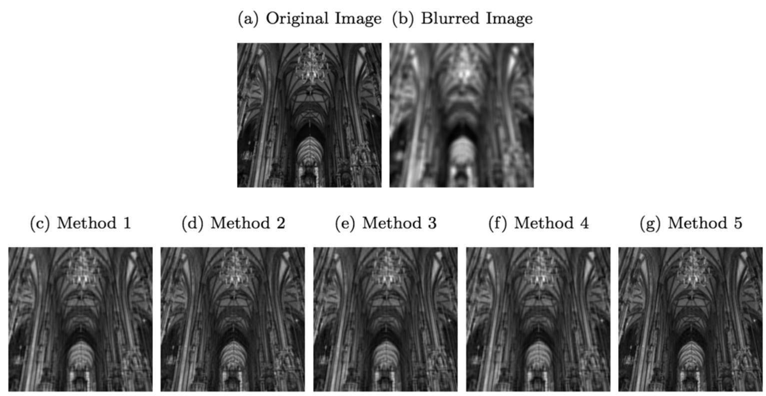

Figure 3 Restoration for the hall image at the 500th iteration. (**a**) Original image; (**b**) Blurry image contaminated by Gaussian blur; (**c**)–(**g**) Restored images by Methods 1–5

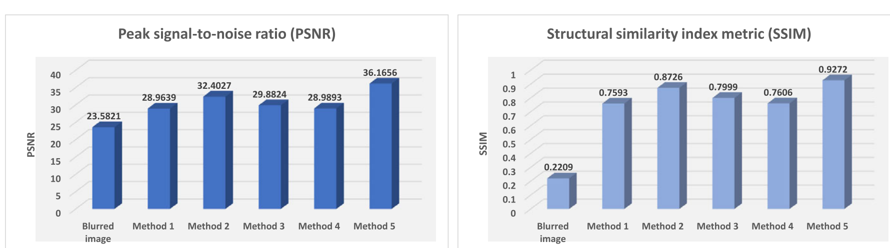

Figure 4 The comparison of PSNR and SSIM values for the blurred image and restored images by Methods 1–5 at the 500th iteration

Example 4.2 Consider a prototype image (hall) with size of 256×256, which is contaminated by Gaussian blur of filter size 9×9 with standard deviation $\hat{\sigma} = 4$ and noise 10^{-5}, see the original image (a) and the blurred image (b) in Fig. 3. The parameters for each deblurring method are set as in Table 1.

Also, we define a contraction f by $f(x) = 0.99x$ for Methods 4 and 5.

Let us see the comparative experiments for recovering the hall images of Methods 1–5 as shown in Figs. 3–5. It can be seen that Method 5 gives the higher values of PSNR and SSIM than the other tested methods. So, our method has the highest image recovery efficiency compared with other methods.

4.2 Signal recovery

Example 4.3 In the LASSO problem (32), the matrix $T \in \mathbb{R}^{M \times N}$ is generated by the normal distribution with mean zero and variance one. The vector $x \in \mathbb{R}^N$ is generated by a uniform

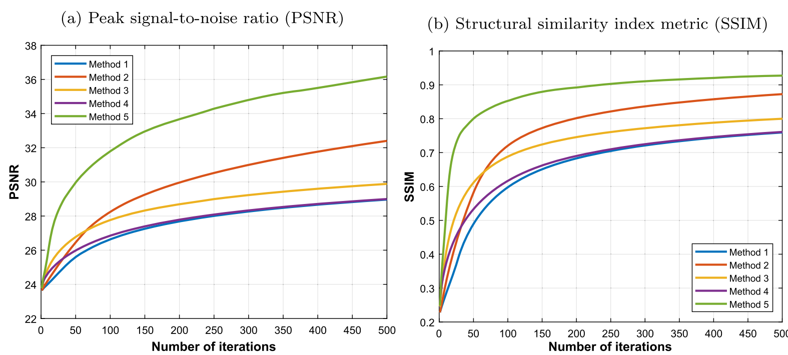

Figure 5 Plot of PSNR and SSIM of restored images by Methods 1–5

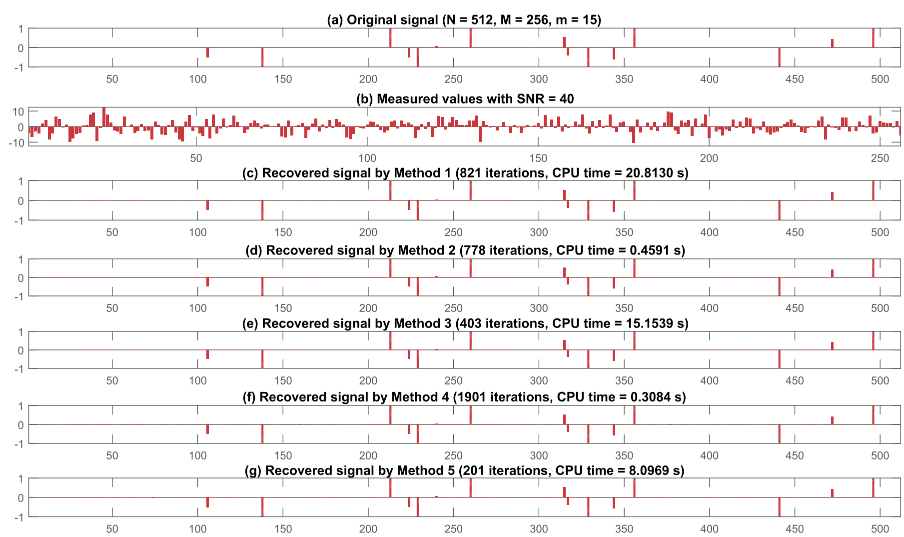

Figure 6 Signal recovery in case of $N = 512$, $M = 256$, $m = 15$. (**a**) Original signal; (**b**) Observed data; (**c**)–(**g**) Recovered signals by Methods 1–5

distribution in $[-2, 2]$ with m nonzero elements. The vector y is generated by the Gaussian noise with the signal-to-noise ratio (SNR) as 40 dB. The regularization parameter is taken by $\lambda = 1$. The parameters of Methods 1–5 are set as in Table 1 in Example 4.2. We use the *mean squared error* (MSE) as the stopping criterion defined by

$$\mathrm{MSE}(k) := \frac{1}{N}\left\|x_k - p^*\right\|_2^2 \leq 10^{-5},$$

where p^* is an original signal.

Now, the experiments for recovering two signals by Methods 1–5 are shown in Figs. 6–7, and the graphs of the MSE for two cases are shown in Fig. 8. It is observed from Figs. 6–8 that the convergence speed of Method 5 is better than that of Methods 1–4 and hence our method has a better convergence behavior than the other tested methods in terms of the number of iterations.

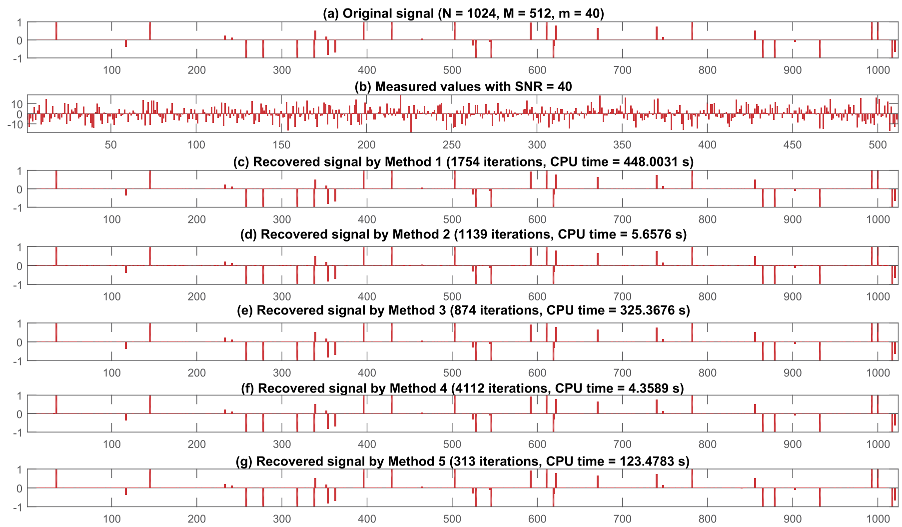

Figure 7 Signal recovery in case of $N = 1024$, $M = 512$, $m = 40$. (**a**) Original signal; (**b**) Observed data; (**c**)–(**g**) Recovered signals by Methods 1–5

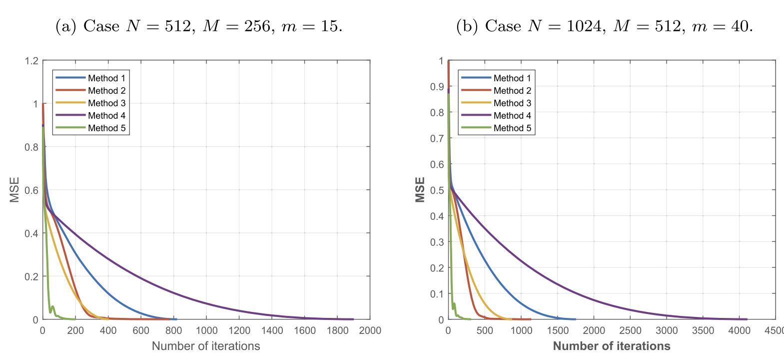

Figure 8 The MSE versus the number of iterations for recovering the signals by Methods 1–5

5 Conclusion

In this work, we discuss the convex minimization problem of the sum of two convex functions in a Hilbert space. The challenge of removing the Lipschitz continuity assumption on the gradient of the function attracts us to study the concept of the linesearch method. We introduce a new linesearch and propose an inertial viscosity forward-backward algorithm whose stepsize does not depend on any Lipschitz constant for solving the considered problem without any Lipschitz continuity condition on the gradient. We prove that the sequence generated by our proposed method converges strongly to a minimizer of the sum of those two convex functions under some mild control conditions. As applications, we apply our method to solving image and signal recovery problems. The comparative experiments show that our method has a higher efficiency than the well-known methods in [9, 16, 18].

Acknowledgements
This work was supported by Chiang Mai University and Thailand Science Research and Innovation under the project IRN62W0007.

Funding
Chiang Mai University and Thailand Science Research and Innovation under the project IRN62W0007.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Publisher's Note

Received: 26 October 2020 Accepted: 16 February 2021 Published online: 27 February 2021

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