RESEARCH   **Open Access**

On semidifferentiable interval-valued programming problems

Kin Keung Lai[1*] ![ORCID], Avanish Shahi[2] and Shashi Kant Mishra[2]

*Correspondence:
mskklai@outlook.com
[1]College of Economics, Shenzhen University, Shenzhen, 518060, China
Full list of author information is available at the end of the article

Abstract

In this paper, we consider the semidifferentiable case of an interval-valued minimization problem and establish sufficient optimality conditions and Wolfe type as well as Mond–Weir type duality theorems under semilocal *E*-preinvex functions. Furthermore, we present saddle-point optimality criteria to relate an optimal solution of the semidifferentiable interval-valued programming problem and a saddle point of the Lagrangian function.

MSC: 90C30; 90C46

Keywords: Interval-valued programming; Semidifferentials; Optimality; Duality

1 Introduction

The technique of solving optimization problems has wide applications in many research areas. Optimization problems having real coefficients are known as the deterministic optimization problems; however, having random variables with known distributions, they are classified as the stochastic optimization problems, see for instance [1, 2]. The specifications of the distributions are more subjective, as many authors invoke the Gaussian distributions for various parameters in the stochastic theory, so it is hard to tackle the large area of real-life problems by these specifications. For resolving such difficulties, interval-valued optimization problems, where coefficients must be chosen as closed intervals, are preferred for studying uncertainty in this optimization problem.

Paper [3] dealt with two types of solutions for an interval-valued optimization problem and established the Karush–Kuhn–Tucker optimality conditions. In addition to that, many solution concepts in the multiobjective view of interval-valued programming problems were proposed in [4]. Further, [5] presented the concept of a nondominated solution for vector optimization problems and established weak and strong duality results for interval-valued programming problems in the presence of an interval-valued Lagrangian function and its dual. For more details on solution concepts of interval-valued programming, one can see [6–9] and the references therein.

The sufficient optimality conditions and duality theorems for Mond–Weir type as well as Wolfe type dual models under generalized invexity assumptions for interval-valued programming problems have been established in [10]. Paper [11] presented the concepts

![Springer]

of invexity and preinvexity for interval-valued functions and studied the KKT optimality conditions for interval-valued programming using Hukuhara differentiability. The Mond–Weir type duality theorems and saddle-point optimality conditions for interval-valued programming problems have been derived in [12]. Recently, [13] extended the invexity assumptions for interval-valued functions with the help of generalized Hukuhara differentiability and presented the Kuhn–Tucker optimality conditions. Further, [14] discussed some properties of the univex mappings for interval-valued functions and established sufficient optimality conditions for the nondominated solution.

On the other hand, [15] introduced the concept of semidifferentiable functions and discussed locally star-shaped functions and generalizations of convex functions using semidifferentiability. Further, [16] extended the concept of semidifferentiability to E-η-semidifferentiability and, using this, introduced (generalized) semilocal E-preinvex functions.

This paper is prepared as follows: in Sect. 2, we give some basic ideas related to interval analysis and semilocal E-preinvex functions. In Sect. 3, we establish sufficient optimality conditions for interval-valued programming using E-η-semidifferentiable and semilocal E-preinvex functions. We give an example to verify our result. In Sect. 4, we propose Wolfe type and Mond–Weir type dual models involving E-η-semidifferentiable functions. Further, we derive weak, strong, and strict converse duality results for the described models. Finally, in the last section, we present relations between an optimal solution of the interval-valued programming problem and a saddle point of the Lagrangian function in case of E-η-semidifferentiability.

2 Preliminaries

Suppose that J is the set of all closed and bounded intervals in $\mathbb{R}$. Then, for $C = [c^L, c^U]$, $D = [d^L, d^U] \in J$, where $c^L(d^L)$ and $c^U(d^U)$ are respectively the lower and upper bounds of $C(D)$ with $c^L \leq c^U$ and $d^L \leq d^U$, we have

$$C + D = \{c + d : c \in C, d \in D\} = \left[c^L + d^L, c^U + d^U\right],$$

$$-C = \{-c : c \in C\} = \left[-c^U, -c^L\right],$$

$$C - D = \left[c^L - d^U, c^U - d^L\right].$$

Now, for any real number μ, we have

$$\mu C = \{\mu c : c \in C\} = \begin{cases} [\mu c^L, \mu c^U] & \text{if } \mu \geq 0, \\ [\mu c^U, \mu c^L] & \text{if } \mu < 0. \end{cases}$$

For further details on interval analysis, one can see [17].

For $C \preceq D$ if and only if $c^L \leq d^L$ and $c^U \leq d^U$. Clearly, $\preceq$ is a partial ordering on J.

Again, $C \prec D$ if and only if $C \preceq D$ and $C \neq D$. This means $C \prec D$ if and only if

$$\begin{cases} c^L < d^L, \\ c^U < d^U \end{cases} \quad \text{or} \quad \begin{cases} c^L \leq d^L, \\ c^U < d^U \end{cases} \quad \text{or} \quad \begin{cases} c^L < d^L, \\ c^U \leq d^U. \end{cases}$$

Suppose that $\mathbb{R}^n$ denotes an n-dimensional Euclidean space, $E : \mathbb{R}^n \to \mathbb{R}^n$ and $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are two fixed mappings.

Definition 2.1 ([18]) A set $X \subset \mathbb{R}^n$ is called *E-invex* with respect to η if

$$E\big(x^*\big) + \lambda\eta\big(E(x), E\big(x^*\big)\big) \in X, \quad \forall x, x^* \in X, \lambda \in [0, 1].$$

Definition 2.2 ([18]) A set $X \subset \mathbb{R}^n$ is called *local E-invex* with respect to η if $\forall x, x^* \in X$ there exists $u(x, x^*) \in (0, 1]$ such that

$$E\big(x^*\big) + \lambda\eta\big(E(x), E\big(x^*\big)\big) \in X, \quad \forall \lambda \in \big[0, u\big(x, x^*\big)\big]. \tag{2.1}$$

Definition 2.3 ([16]) A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *semilocal E-preinvex* on $X \subset \mathbb{R}^n$ with respect to η if, for all $x, x^* \in X$ (with a maximal positive number $u(x, x^*) \leq 1$ satisfying (2.1)), there exists $0 < v(x, x^*) \leq u(x, x^*)$ such that X is a local *E*-invex set and

$$f\big(E\big(x^*\big) + \lambda\eta\big(E(x), E\big(x^*\big)\big)\big) \leq \lambda f(x) + (1 - \lambda)f\big(x^*\big), \quad \forall \lambda \in \big[0, v\big(x, x^*\big)\big].$$

Definition 2.4 ([16]) Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$, where X is a local *E*-invex set. Then f is said to be *pseudo-semilocal E-preinvex* with respect to η if, for all $x, x^* \in X$ (with a maximal positive number $u(x, x^*) \leq 1$ satisfying (2.1)), there are positive numbers $v(x, x^*) \leq u(x, x^*)$ and $w(x, x^*)$ such that

$$f(x) < f\big(x^*\big) \quad \Rightarrow$$
$$f\big(E\big(x^*\big) + \lambda\eta\big(E(x), E\big(x^*\big)\big)\big) \leq f\big(x^*\big) - \lambda w\big(x, x^*\big), \quad \forall \lambda \in \big[0, v\big(x, x^*\big)\big].$$

Definition 2.5 ([16]) Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$, where X is a local *E*-invex set. Then f is said to be *quasi-semilocal E-preinvex* with respect to η if, for all $x, x^* \in X$ (with a maximal positive number $u(x, x^*) \leq 1$ satisfying (2.1)), there is a positive number $v(x, x^*) \leq u(x, x^*)$ such that

$$f(x) \leq f\big(x^*\big) \quad \Rightarrow \quad f\big(E\big(x^*\big) + \lambda\eta\big(E(x), E\big(x^*\big)\big)\big) \leq f\big(x^*\big), \quad \forall \lambda \in \big[0, v\big(x, x^*\big)\big].$$

Definition 2.6 ([16]) A function $f : X \to \mathbb{R}$ is said to be *E-η-semidifferentiable* at $\bar{x} \in X$, where $X \subset \mathbb{R}^n$ is a local *E*-invex set with respect to η, if $E(\bar{x}) = \bar{x}$ and

$$(df)^+\big(\bar{x}; \eta\big(E(x), \bar{x}\big)\big) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}\big[f\big(\bar{x} + \lambda\eta\big(E(x), \bar{x}\big)\big) - f(\bar{x})\big] \quad \text{exists} \quad \forall x \in X.$$

Remark 2.1 When E becomes an identity map, then the notion of *E*-η-semidifferentiability is η-semidifferentiability, and for E as an identity map with $\eta(x, \bar{x}) = x - \bar{x}$, the same is converted into a semidifferentiable function (see [15]).

Lemma 2.1 (see [16]) (*i*) *Suppose that f is (strictly) semilocal E-preinvex and E-η-semidifferentiable at $\bar{x} \in X \subset \mathbb{R}^n$, where X is a local E-invex set with respect to η. Then*

$$f(x) - f(\bar{x})(>) \geq (df)^+\big(\bar{x}; \eta\big(E(x), \bar{x}\big)\big), \quad \forall x \in X.$$

(*ii*) *Suppose that f is pseudo(quasi)-semilocal E-preinvex and E-η-semidifferentiable at $\bar{x} \in X \subset \mathbb{R}^n$, where X is a local E-invex set with respect to η. Then*

$$f(x) < (\leq)f(\bar{x}) \quad \Rightarrow \quad (df)^+\big(\bar{x}; \eta\big(E(x), \bar{x}\big)\big) < (\leq)0, \quad \forall x \in X.$$

3 Optimality conditions for interval-valued programming problem

Consider the following interval-valued minimization problem:

$$(\text{IVP}) \quad \min \quad F(x) = \left[F^L(x), F^U(x) \right]$$

$$\text{subject to} \quad g_j(x) \leq 0, \quad j = 1, 2, \ldots, l,$$

where $F : X \to I$ is an interval-valued function and $F^L(x)$, $F^U(x)$ ($F^L(x) \leq F^U(x)$), and $g_j : X \to \mathbb{R}$, $j = 1, 2, \ldots, l$, are real-valued functions on a local E-invex set $X \subset \mathbb{R}^n$.

Let $P = \{x \in X : g_j(x) \leq 0, j = 1, 2, \ldots, l\}$ be a feasible set of (IVP).

Definition 3.1 ([19]) Let $\bar{x}$ be a feasible solution of problem (IVP). We say that $\bar{x}$ is an *LU optimal solution* of the problem if there exists no $x' \in P$ such that

$$F(x') \prec F(\bar{x}).$$

Now, we define semilocal E-preinvexity for interval-valued functions as follows.

Lemma 3.1 *Suppose that F is an E-η-semidifferentiable interval-valued function. Then F is semilocal E-preinvex with respect to η at $\bar{x}$ if both real-valued functions F^L and F^U are semilocal E-preinvex with respect to the same η at $\bar{x}$.*

Motivated by [19] and [20], we state the Karush–Kuhn–Tucker type necessary conditions for interval-valued programming problems in terms of E-η-semidifferentiable functions.

Theorem 3.1 *Let $E(\bar{x}) = \bar{x}$. Suppose that $\bar{x}$ is an LU optimal solution to (IVP) and the suitable constraint qualification is satisfied, and all functions F^L, F^U, and g_j are E-η-semidifferentiable at $\bar{x}$. Then there exist scalars $w^L, w^U (> 0) \in \mathbb{R}$, and $\tau_j (\geq 0) \in \mathbb{R}$, $j = 1, 2, \ldots, l$, such that*

$$w^L \left(dF^L \right)^+ \left(\bar{x}; \eta \left(E(x'), \bar{x} \right) \right) + w^U \left(dF^U \right)^+ \left(\bar{x}; \eta \left(E(x'), \bar{x} \right) \right)$$

$$+ \sum_{j=1}^{l} \tau_j (dg_j)^+ \left(\bar{x}; \eta \left(E(x'), \bar{x} \right) \right) \geq 0, \quad \forall x' \in P, \tag{3.1}$$

$$\sum_{j=1}^{l} \tau_j g_j(\bar{x}) = 0. \tag{3.2}$$

Now, we present some sufficient optimality conditions for (IVP).

Theorem 3.2 *Let $E(\bar{x}) = \bar{x}$ and $\bar{x} \in P$. Suppose that functions F^L, F^U, and g_j are E-η-semidifferentiable at $\bar{x}$ and there exist scalars $w^L, w^U (> 0) \in \mathbb{R}$, and $\tau_j (\geq 0) \in \mathbb{R}$, $j = 1, 2, \ldots, l$, such that*

$$(i) \quad w^L \left(dF^L \right)^+ \left(\bar{x}; \eta \left(E(x'), \bar{x} \right) \right) + w^U \left(dF^U \right)^+ \left(\bar{x}; \eta \left(E(x'), \bar{x} \right) \right)$$

$$+ \sum_{j=1}^{l} \tau_j (dg_j)^+ \left(\bar{x}; \eta \left(E(x'), \bar{x} \right) \right) \geq 0, \quad \forall x' \in P,$$

(ii) $\displaystyle\sum_{j=1}^{l} \tau_j g_j(\bar{x}) = 0,$

(iii)   *F and* $\displaystyle\sum_{j=1}^{l} \tau_j g_j$ *are semilocal E-preinvex at* $\bar{x}$.

Then $\bar{x}$ *is an LU optimal solution to* (IVP).

Proof Suppose that $\bar{x}$ is not an LU optimal solution to (IVP), then there exists a point $x' \in P$ such that $F(x') \prec F(\bar{x})$.

   This means

$$\begin{cases} F^L(x') < F^L(\bar{x}), \\ F^U(x') < F^U(\bar{x}) \end{cases} \quad \text{or} \quad \begin{cases} F^L(x') \le F^L(\bar{x}), \\ F^U(x') < F^U(\bar{x}) \end{cases} \quad \text{or} \quad \begin{cases} F^L(x') < F^L(\bar{x}), \\ F^U(x') \le F^U(\bar{x}). \end{cases}$$

For $w^L, w^U > 0$, we can write

$$w^L F^L(x') + w^U F^U(x') < w^L F^L(\bar{x}) + w^U F^U(\bar{x}). \tag{3.3}$$

Since F is semilocal E-preinvex with respect to η at $\bar{x}$, then

$$F^L(x') - F^L(\bar{x}) \ge (dF^L)^+(\bar{x}; \eta(E(x'), \bar{x})) \tag{3.4}$$

and

$$F^U(x') - F^U(\bar{x}) \ge (dF^U)^+(\bar{x}; \eta(E(x'), \bar{x})). \tag{3.5}$$

Multiplying (3.4) by w^L and (3.5) by w^U and adding them, we get

$$\begin{aligned} \big[\{w^L F^L(x') + w^U F^U(x')\} &- \{w^L F^L(\bar{x}) + w^U F^U(\bar{x})\}\big] \\ &\ge w^L (dF^L)^+(\bar{x}; \eta(E(x'), \bar{x})) + w^U (dF^U)^+(\bar{x}; \eta(E(x'), \bar{x})). \end{aligned}$$

Using (3.3), the above inequality becomes

$$w^L (dF^L)^+(\bar{x}; \eta(E(x'), \bar{x})) + w^U (dF^U)^+(\bar{x}; \eta(E(x'), \bar{x})) < 0. \tag{3.6}$$

From *(ii)* with the feasibility of x' to (IVP), we find that

$$\sum_{j=1}^{l} \tau_j g_j(x') \le \sum_{j=1}^{l} \tau_j g_j(\bar{x}). \tag{3.7}$$

Since $\sum_{j=1}^{l} \tau_j g_j$ is semilocal E-preinvex with respect to η at $\bar{x}$, then

$$\sum_{j=1}^{l} \tau_j g_j(x') - \sum_{j=1}^{l} \tau_j g_j(\bar{x}) \ge \sum_{j=1}^{l} \tau_j (dg_j)^+(\bar{x}; \eta(E(x'), \bar{x})).$$

Using (3.7), the above inequality becomes

$$\sum_{j=1}^{l} \tau_j (dg_j)^+ \big(\bar{x}; \eta\big(E(x'),\bar{x}\big)\big) \le 0. \tag{3.8}$$

Adding (3.6) and (3.8), we obtain

$$w^L \big(dF^L\big)^+ \big(\bar{x}; \eta\big(E(x'),\bar{x}\big)\big) + w^U \big(dF^U\big)^+ \big(\bar{x}; \eta\big(E(x'),\bar{x}\big)\big)$$

$$+ \sum_{j=1}^{l} \tau_j (dg_j)^+ \big(\bar{x}; \eta\big(E(x'),\bar{x}\big)\big) < 0, \quad \forall x' \in P,$$

a contradiction to assumption *(i)*. Hence, $\bar{x}$ is an LU optimal solution to (IVP). □

Example 3.1 Consider the interval-valued programming problem:

$$\min \quad F(x) = \big[x^3 + 2, 2x^3 + 3x + 4\big], \quad x \ge 0,$$
$$\text{subject to} \quad g_1(x) = -3x + 2.$$

It is easy to see that F^L, F^U, and g_1 are E-η-semidifferentiable functions, where $E : \mathbb{R} \to \mathbb{R}$, defined by

$$E(x) = \begin{cases} -2, & x < 0, \\ 4, & 1 < x \le 2, \\ x, & 0 \le x \le 1 \text{ or } x > 2; \end{cases}$$

and the map $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$\eta(x, x^*) = \begin{cases} 0, & x = x^*, \\ x - 1, & x \ne x^*. \end{cases}$$

The feasible set of the problem is $P = \{x : -3x + 2 \le 0\}$. Clearly, $\bar{x} = 1$ is feasible. Choose another point $x' = 3 \in P$.

Now,

$$\big(dF^L\big)^+ \big(\bar{x}; \eta\big(E(x'),\bar{x}\big)\big) = \lim_{\lambda \to 0^+} \frac{1}{\lambda} \big[F^L\big(\bar{x} + \lambda \eta\big(E(x'),\bar{x}\big)\big) - F^L(\bar{x})\big]$$

$$= \lim_{\lambda \to 0^+} \frac{1}{\lambda} \big[F^L\big(1 + \lambda \eta\big(E(3), 1\big)\big) - F^L(1)\big]$$

$$= \lim_{\lambda \to 0^+} \frac{1}{\lambda} \big[F^L\big(1 + \lambda \eta(3, 1)\big) - F^L(1)\big]$$

$$= \lim_{\lambda \to 0^+} \frac{1}{\lambda} \big[F^L(1 + 2\lambda) - F^L(1)\big]$$

$$= \lim_{\lambda \to 0^+} \frac{1}{\lambda} \big[(1 + 2\lambda)^3 + 2 - 3\big]$$

$$= 6,$$

$$\left(dF^U\right)^+\left(\bar{x}; \eta\left(E(x'), \bar{x}\right)\right) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}\left[F^U\left(\bar{x} + \lambda\eta\left(E(x'), \bar{x}\right)\right) - F^U(\bar{x})\right]$$

$$= \lim_{\lambda \to 0^+} \frac{1}{\lambda}\left[F^U(1 + 2\lambda) - F^U(1)\right]$$

$$= \lim_{\lambda \to 0^+} \frac{1}{\lambda}\left[2(1 + 2\lambda)^3 + 3(1 + 2\lambda) + 4 - 9\right]$$

$$= 18,$$

$$\left(dg_1\right)^+\left(\bar{x}; \eta\left(E(x'), \bar{x}\right)\right) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}\left[g_1\left(\bar{x} + \lambda\eta\left(E(x'), \bar{x}\right)\right) - g_1(\bar{x})\right]$$

$$= \lim_{\lambda \to 0^+} \frac{1}{\lambda}\left[g_1(1 + 2\lambda) - g_1(1)\right]$$

$$= \lim_{\lambda \to 0^+} \frac{1}{\lambda}\left[-3(1 + 2\lambda) + 2 + 1\right]$$

$$= -6.$$

If we choose $w^L = 1$, $w^U = 1$, and $\tau_1 = 0$, then

$$w^L\left(dF^L\right)^+\left(\bar{x}; \eta\left(E(x'), \bar{x}\right)\right) + w^U\left(dF^U\right)^+\left(\bar{x}; \eta\left(E(x'), \bar{x}\right)\right)$$

$$+ \tau_1(dg_1)^+\left(\bar{x}; \eta\left(E(x'), \bar{x}\right)\right) = 24 > 0,$$

and $\tau_1 g_1(\bar{x}) = 0$.

Moreover, functions F and $\tau_1 g_1$ are semilocal E-preinvex at $\bar{x} = 1$. Therefore, $\bar{x} = 1$ is an LU optimal solution to the given problem. Thus, Theorem 3.2 is verified.

Theorem 3.3 *Let $E(\bar{x}) = \bar{x}$ and $\bar{x} \in P$. Suppose that functions F^L, F^U, and g_j are E-η-semidifferentiable at $\bar{x}$ and there exist scalars $w^L, w^U(> 0) \in \mathbb{R}$, and $\tau_j(\geq 0) \in \mathbb{R}$, $j = 1, 2, \ldots, l$, such that*

(i) $w^L\left(dF^L\right)^+\left(\bar{x}; \eta\left(E(x'), \bar{x}\right)\right) + w^U\left(dF^U\right)^+\left(\bar{x}; \eta\left(E(x'), \bar{x}\right)\right)$

$$+ \sum_{j=1}^{l} \tau_j(dg_j)^+\left(\bar{x}; \eta\left(E(x'), \bar{x}\right)\right) \geq 0, \quad \forall x' \in P,$$

(ii) $\displaystyle\sum_{j=1}^{l} \tau_j g_j(\bar{x}) = 0,$

(iii) $w^L F^L + w^U F^U$ *is pseudo-semilocal E-preinvex and* $\displaystyle\sum_{j=1}^{l} \tau_j g_j$

is quasi-semilocal E-preinvex at $\bar{x}$.

Then $\bar{x}$ is an LU optimal solution to (IVP).

Proof On the contrary, suppose that $\bar{x}$ is not an LU optimal solution to (IVP), then there exists a point $x' \in P$ such that $F(x') \prec F(\bar{x})$.

That is,

$$
\begin{cases} F^L(x') < F^L(\bar{x}), \\ F^U(x') < F^U(\bar{x}) \end{cases} \quad \text{or} \quad \begin{cases} F^L(x') \le F^L(\bar{x}), \\ F^U(x') < F^U(\bar{x}) \end{cases} \quad \text{or} \quad \begin{cases} F^L(x') < F^L(\bar{x}), \\ F^U(x') \le F^U(\bar{x}). \end{cases}
$$

Since $w^L, w^U > 0$, we can write

$$
w^L F^L(x') + w^U F^U(x') < w^L F^L(\bar{x}) + w^U F^U(\bar{x}).
$$

The above inequality together with the pseudo-semilocal E-preinvexity of $w^L F^L + w^U F^U$ with respect to η at $\bar{x}$ gives

$$
w^L (dF^L)^+ (\bar{x}; \eta(E(x'), \bar{x})) + w^U (dF^U)^+ (\bar{x}; \eta(E(x'), \bar{x})) < 0. \tag{3.9}
$$

Again, from the feasibility of x' to (IVP) and by *(ii)*, we have

$$
\sum_{j=1}^{l} \tau_j g_j(x') \le \sum_{j=1}^{l} \tau_j g_j(\bar{x}).
$$

With above inequality, use the fact that $\sum_{j=1}^{l} \tau_j g_j$ is quasi-semilocal E-preinvex with respect to η at $\bar{x}$, then

$$
\sum_{j=1}^{l} \tau_j (dg_j)^+ (\bar{x}; \eta(E(x'), \bar{x})) \le 0. \tag{3.10}
$$

Adding (3.9) and (3.10), we get

$$
w^L (dF^L)^+ (\bar{x}; \eta(E(x'), \bar{x})) + w^U (dF^U)^+ (\bar{x}; \eta(E(x'), \bar{x}))
$$

$$
+ \sum_{j=1}^{l} \tau_j (dg_j)^+ (\bar{x}; \eta(E(x'), \bar{x})) < 0, \quad \forall x' \in P,
$$

a contradiction to assumption *(i)*. Thus, $\bar{x}$ is an LU optimal solution to (IVP). □

4 Duality
4.1 Wolfe type duality
Consider the following Wolfe type dual model:

$$
\text{(IVWD)} \quad \max \quad F(z) + \sum_{j=1}^{l} \tau_j g_j(z) = \left[F^L(z) + \sum_{j=1}^{l} \tau_j g_j(z), F^U(z) + \sum_{j=1}^{l} \tau_j g_j(z) \right]
$$

subject to

$$
w^L (dF^L)^+ (z; \eta(E(\bar{x}), z)) + w^U (dF^U)^+ (z; \eta(E(\bar{x}), z))
$$

$$
+ \sum_{j=1}^{l} \tau_j (dg_j)^+ (z; \eta(E(\bar{x}), z)) \ge 0, \quad \forall \bar{x} \in X, \tag{4.1}
$$

$$
w^L, w^U > 0, \qquad \tau_j \ge 0, \quad j = 1, 2, \dots, l. \tag{4.2}
$$

Definition 4.1 Let $(\bar{z}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ be a feasible solution of the dual problem. Then $(\bar{z}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is said to be an *LU optimal solution* of dual problem (IVWD) if there exists no $(z, \bar{w}^L, \bar{w}^U, \bar{\tau})$ such that

$$F(\bar{z}) + \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{z}) \prec F(z) + \sum_{j=1}^{l} \bar{\tau}_j g_j(z).$$

Theorem 4.1 (Weak duality) *Let $\bar{x}$ and (z, w^L, w^U, τ) be the feasible solutions to (IVP) and (IVWD), respectively, with $E(z) = z$. Assume that F and $\sum_{j=1}^{l} \tau_j g_j$ are semilocal E-preinvex, and all functions are E-η-semidifferentiable at z such that $w^L + w^U = 1$. Then*

$$F(\bar{x}) \succeq F(z) + \sum_{j=1}^{l} \tau_j g_j(z).$$

Proof On the contrary, suppose that $F(\bar{x}) \prec F(z) + \sum_{j=1}^{l} \tau_j g_j(z)$.
  That is,

$$\begin{cases} F^L(\bar{x}) < F^L(z) + \sum_{j=1}^{l} \tau_j g_j(z), \\ F^U(\bar{x}) < F^U(z) + \sum_{j=1}^{l} \tau_j g_j(z) \end{cases} \text{or} \quad \begin{cases} F^L(\bar{x}) \leq F^L(z) + \sum_{j=1}^{l} \tau_j g_j(z), \\ F^U(\bar{x}) < F^U(z) + \sum_{j=1}^{l} \tau_j g_j(z) \end{cases} \text{or}$$

$$\begin{cases} F^L(\bar{x}) < F^L(z) + \sum_{j=1}^{l} \tau_j g_j(z), \\ F^U(\bar{x}) \leq F^U(z) + \sum_{j=1}^{l} \tau_j g_j(z). \end{cases}$$

Using the fact that $w^L, w^U > 0$ and $w^L + w^U = 1$ with the feasibility of $\bar{x}$ to (IVP), the above inequalities become

$$w^L F^L(\bar{x}) + w^U F^U(\bar{x}) + \sum_{j=1}^{l} \tau_j g_j(\bar{x}) < w^L F^L(z) + w^U F^U(z) + \sum_{j=1}^{l} \tau_j g_j(z). \tag{4.3}$$

Since F is semilocal E-preinvex with respect to η at z, then

$$F^L(\bar{x}) - F^L(z) \geq \left(dF^L\right)^+ \left(z; \eta\left(E(\bar{x}), z\right)\right)$$

and

$$F^U(\bar{x}) - F^U(z) \geq \left(dF^U\right)^+ \left(z; \eta\left(E(\bar{x}), z\right)\right).$$

Since $w^L, w^U > 0$, then the above inequalities become

$$w^L F^L(\bar{x}) - w^L F^L(z) \geq w^L \left(dF^L\right)^+ \left(z; \eta\left(E(\bar{x}), z\right)\right) \tag{4.4}$$

and

$$w^U F^U(\bar{x}) - w^U F^U(z) \geq w^U \left(dF^U\right)^+ \left(z; \eta\left(E(\bar{x}), z\right)\right). \tag{4.5}$$

From the semilocal E-preinvexity of $\sum_{j=1}^{l} \tau_j g_j$ with respect to η at z, we have

$$\sum_{j=1}^{l} \tau_j g_j(\bar{x}) - \sum_{j=1}^{l} \tau_j g_j(z) \geq \sum_{j=1}^{l} \tau_j (dg_j)^+ \big(z; \eta\big(E(\bar{x}), z\big)\big). \tag{4.6}$$

Adding (4.4), (4.5), and (4.6), we get

$$\left[\left\{ w^L F^L(\bar{x}) + w^U F^U(\bar{x}) + \sum_{j=1}^{l} \tau_j g_j(\bar{x}) \right\} - \left\{ w^L F^L(z) + w^U F^U(z) + \sum_{j=1}^{l} \tau_j g_j(z) \right\} \right]$$

$$\geq w^L \big(dF^L\big)^+ \big(z; \eta\big(E(\bar{x}), z\big)\big) + w^U \big(dF^U\big)^+ \big(z; \eta\big(E(\bar{x}), z\big)\big) + \sum_{j=1}^{l} \tau_j (dg_j)^+ \big(z; \eta\big(E(\bar{x}), z\big)\big).$$

On using inequality (4.3), the above inequality becomes

$$w^L \big(dF^L\big)^+ \big(z; \eta\big(E(\bar{x}), z\big)\big) + w^U \big(dF^U\big)^+ \big(z; \eta\big(E(\bar{x}), z\big)\big)$$

$$+ \sum_{j=1}^{l} \tau_j (dg_j)^+ \big(z; \eta\big(E(\bar{x}), z\big)\big) < 0,$$

a contradiction to dual constraint (4.1) with (z, w^L, w^U, τ) feasible to (IVWD). This completes the proof. □

Theorem 4.2 (Strong duality) *Let $E(\bar{x}) = \bar{x}$. Suppose that $\bar{x}$ is an LU optimal solution to (IVP) and suitable constraint qualification is satisfied, and all functions are E-η-semidifferentiable at $\bar{x}$. Then there exist $\bar{w}^L > 0$, $\bar{w}^U > 0$, and $\bar{\tau} \geq 0$ such that $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is a feasible solution to (IVWD) and the two objective values are the same. Further, if the assumptions of weak duality Theorem 4.1 hold for all feasible solutions $(\bar{z}, \bar{w}^L, \bar{w}^U, \bar{\tau})$, then $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is an LU optimal solution to (IVWD).*

Proof As $\bar{x}$ is an LU optimal solution to (IVP) and suitable constraint qualification holds at $\bar{x}$, so by Theorem 3.1 there exist scalars $\bar{w}^L > 0$, $\bar{w}^U > 0$, $\bar{\tau}_j \geq 0$, $j = 1, 2, \ldots, l$, such that

$$\bar{w}^L \big(dF^L\big)^+ \big(\bar{x}; \eta\big(E(x'), \bar{x}\big)\big) + \bar{w}^U \big(dF^U\big)^+ \big(\bar{x}; \eta\big(E(x'), \bar{x}\big)\big)$$

$$+ \sum_{j=1}^{l} \bar{\tau}_j (dg_j)^+ \big(\bar{x}; \eta\big(E(x'), \bar{x}\big)\big) \geq 0, \quad \forall x' \in P,$$

$$\text{and} \quad \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{x}) = 0$$

show that $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is a feasible solution to (IVWD) and the analogous objective values are the same. Assume that $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is not an LU optimal solution to (IVWD), this means there is a feasible solution $(\bar{z}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ to (IVWD) such that

$$F(\bar{x}) \prec F(\bar{z}) + \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{z}),$$

which is a contradiction to weak duality Theorem 4.1. Thus, $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is an LU optimal solution to (IVWD). □

Theorem 4.3 (Strict converse duality) *Let $\bar{x}$ and $(\bar{z}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ be the feasible solutions to* (IVP) *and* (IVWD), *respectively, with $E(\bar{z}) = \bar{z}$. Assume that F is strictly semilocal E-preinvex and $\sum_{j=1}^{l} \bar{\tau}_j g_j$ is semilocal E-preinvex, and all functions are E-η-semidifferentiable at $\bar{z}$ with*

$$\bar{w}^L F^L(\bar{x}) + \bar{w}^U F^U(\bar{x}) + \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{x}) \le \bar{w}^L F^L(\bar{z}) + \bar{w}^U F^U(\bar{z}) + \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{z}). \qquad (4.7)$$

Then $\bar{x} = \bar{z}$.

Proof Suppose, on the contrary, that $\bar{x} \ne \bar{z}$. Since F is strictly semilocal E-preinvex with respect to η at $\bar{z}$, i.e.,

$$F^L(\bar{x}) - F^L(\bar{z}) > \left(dF^L\right)^+ \left(\bar{z}; \eta\left(E(\bar{x}), \bar{z}\right)\right)$$

and

$$F^U(\bar{x}) - F^U(\bar{z}) > \left(dF^U\right)^+ \left(\bar{z}; \eta\left(E(\bar{x}), \bar{z}\right)\right).$$

Multiplying the above inequalities by $\bar{w}^L$ and $\bar{w}^U$, respectively, and adding them, we get

$$\begin{aligned}
&\left[\left\{\bar{w}^L F^L(\bar{x}) + \bar{w}^U F^U(\bar{x})\right\} - \left\{\bar{w}^L F^L(\bar{z}) + \bar{w}^U F^U(\bar{z})\right\}\right] \\
&\quad > \bar{w}^L \left(dF^L\right)^+ \left(\bar{z}; \eta\left(E(\bar{x}), \bar{z}\right)\right) + \bar{w}^U \left(dF^U\right)^+ \left(\bar{z}; \eta\left(E(\bar{x}), \bar{z}\right)\right).
\end{aligned} \qquad (4.8)$$

By the semilocal E-preinvexity of $\sum_{j=1}^{l} \bar{\tau}_j g_j$ with respect to η at $\bar{z}$, we have

$$\sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{x}) - \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{z}) \ge \sum_{j=1}^{l} \bar{\tau}_j (dg_j)^+ \left(\bar{z}; \eta\left(E(\bar{x}), \bar{z}\right)\right). \qquad (4.9)$$

Adding (4.8) and (4.9), we get

$$\begin{aligned}
&\left[\left\{\bar{w}^L F^L(\bar{x}) + \bar{w}^U F^U(\bar{x}) + \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{x})\right\} - \left\{\bar{w}^L F^L(\bar{z}) + \bar{w}^U F^U(\bar{z}) + \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{z})\right\}\right] \\
&\quad > \bar{w}^L \left(dF^L\right)^+ \left(\bar{z}; \eta\left(E(\bar{x}), \bar{z}\right)\right) + \bar{w}^U \left(dF^U\right)^+ \left(\bar{z}; \eta\left(E(\bar{x}), \bar{z}\right)\right) + \sum_{j=1}^{l} \bar{\tau}_j (dg_j)^+ \left(\bar{z}; \eta\left(E(\bar{x}), \bar{z}\right)\right).
\end{aligned}$$

The above inequality with the feasibility of $(\bar{z}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ to (IVWD) (i.e., with dual constraint (4.1)) becomes

$$\bar{w}^L F^L(\bar{x}) + \bar{w}^U F^U(\bar{x}) + \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{x}) > \bar{w}^L F^L(\bar{z}) + \bar{w}^U F^U(\bar{z}) + \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{z}),$$

a contradiction to inequality (4.7). Hence, $\bar{x} = \bar{z}$. □

4.2 Mond–Weir type duality

Consider the following Mond–Weir type dual model:

$$(\text{IVMWD}) \quad \max \quad F(z) = \left[F^L(z), F^U(z) \right]$$

subject to

$$w^L \left(dF^L \right)^+ \left(z; \eta \left(E(\bar{x}), z \right) \right) + w^U \left(dF^U \right)^+ \left(z; \eta \left(E(\bar{x}), z \right) \right)$$

$$+ \sum_{j=1}^{l} \tau_j (dg_j)^+ \left(z; \eta \left(E(\bar{x}), z \right) \right) \geq 0, \quad \forall \bar{x} \in X, \tag{4.10}$$

$$\sum_{j=1}^{l} \tau_j g_j(z) \geq 0, \tag{4.11}$$

$$w^L, w^U > 0, \qquad \tau_j \geq 0, \quad j = 1, 2, \ldots, l. \tag{4.12}$$

Definition 4.2 Let $(\bar{z}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ be a feasible solution of the dual problem. Then $(\bar{z}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is said to be an *LU optimal solution* of dual problem (IVMWD) if there exists no $(z, \bar{w}^L, \bar{w}^U, \bar{\tau})$ such that

$$F(\bar{z}) \prec F(z).$$

Theorem 4.4 (Weak duality) *Let $\bar{x}$ and (z, w^L, w^U, τ) be the feasible solutions to (IVP) and (IVMWD), respectively, with $E(z) = z$. Suppose that $w^L F^L + w^U F^U$ is pseudo-semilocal E-preinvex and $\sum_{j=1}^{l} \tau_j g_j$ is quasi-semilocal E-preinvex, and all functions are E-η-semidifferentiable at z. Then*

$$F(\bar{x}) \succeq F(z).$$

Proof On the contrary, suppose that $F(\bar{x}) \prec F(z)$.
  This means

$$\begin{cases} F^L(\bar{x}) < F^L(z), \\ F^U(\bar{x}) < F^U(z) \end{cases} \quad \text{or} \quad \begin{cases} F^L(\bar{x}) \leq F^L(z), \\ F^U(\bar{x}) < F^U(z) \end{cases} \quad \text{or} \quad \begin{cases} F^L(\bar{x}) < F^L(z), \\ F^U(\bar{x}) \leq F^U(z). \end{cases}$$

For $w^L > 0$ and $w^U > 0$, we can write

$$w^L F^L(\bar{x}) + w^U F^U(\bar{x}) < w^L F^L(z) + w^U F^U(z),$$

which together with the pseudo-semilocal *E*-preinvexity of $w^L F^L + w^U F^U$ with respect to η at z gives

$$w^L \left(dF^L \right)^+ \left(z; \eta \left(E(\bar{x}), z \right) \right) + w^U \left(dF^U \right)^+ \left(z; \eta \left(E(\bar{x}), z \right) \right) < 0. \tag{4.13}$$

Again, from the feasibility of $\bar{x}$ and (z, w^L, w^U, τ) to (IVP) and (IVMWD), respectively, we have

$$\sum_{j=1}^{l} \tau_j g_j(\bar{x}) \leq \sum_{j=1}^{l} \tau_j g_j(z).$$

With the above inequality, use the fact that $\sum_{j=1}^{l} \tau_j g_j$ is quasi-semilocal E-preinvex with respect to η at z, then

$$\sum_{j=1}^{l} \tau_j (dg_j)^+ \big(z; \eta\big(E(\bar{x}), z\big)\big) \leq 0. \tag{4.14}$$

Adding (4.13) and (4.14), we get

$$w^L \big(dF^L\big)^+ \big(z; \eta\big(E(\bar{x}), z\big)\big) + w^U \big(dF^U\big)^+ \big(z; \eta\big(E(\bar{x}), z\big)\big)$$
$$+ \sum_{j=1}^{l} \tau_j (dg_j)^+ \big(z; \eta\big(E(\bar{x}), z\big)\big) < 0,$$

a contradiction to dual constraint (4.10) for (z, w^L, w^U, τ) feasible to (IVMWD). Hence, $F(\bar{x}) \succeq F(z)$. $\qquad\square$

Theorem 4.5 (Strong duality) *Let $E(\bar{x}) = \bar{x}$. Suppose that $\bar{x}$ is an LU optimal solution to (IVP) and suitable constraint qualification is satisfied, and all functions are E-η-semidifferentiable at $\bar{x}$. Then there exist $\bar{w}^L > 0$, $\bar{w}^U > 0$, and $\bar{\tau} \geq 0$ such that $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is a feasible solution to (IVMWD) and the analogous objective values are the same. Again, if the assumptions of weak duality Theorem 4.4 are satisfied for all feasible solutions $(\bar{z}, \bar{w}^L, \bar{w}^U, \bar{\tau})$, then $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is an LU optimal solution to (IVMWD).*

Proof As $\bar{x}$ is an LU optimal solution to (IVP) and suitable constraint qualification holds at $\bar{x}$, so by Theorem 3.1 there exist scalars $\bar{w}^L > 0$, $\bar{w}^U > 0$, and $\bar{\tau}_j \geq 0$, $j = 1, 2, \ldots, l$, such that

$$\bar{w}^L \big(dF^L\big)^+ \big(\bar{x}; \eta\big(E(x'), \bar{x}\big)\big) + \bar{w}^U \big(dF^U\big)^+ \big(\bar{x}; \eta\big(E(x'), \bar{x}\big)\big)$$
$$+ \sum_{j=1}^{l} \bar{\tau}_j (dg_j)^+ \big(\bar{x}; \eta\big(E(x'), \bar{x}\big)\big) \geq 0, \quad \forall x' \in P,$$

$$\text{and} \quad \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{x}) = 0$$

show that $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is a feasible solution to (IVMWD) and the analogous objective values are the same. Assume that $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is not an LU optimal solution to (IVMWD), this means there is a feasible solution $(\bar{z}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ to (IVMWD) such that

$$F(\bar{x}) \prec F(\bar{z}),$$

which is a contradiction to weak duality Theorem 4.4. Thus, $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is an LU optimal solution to (IVMWD).   □

Theorem 4.6 (Strict converse duality) *Let $\bar{x}$ and $(\bar{z}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ be the feasible solutions to (IVP) and (IVMWD), respectively, with $E(\bar{z}) = \bar{z}$. Assume that $\bar{w}^L F^L + \bar{w}^U F^U$ is strictly pseudo-semilocal E-preinvex and $\sum_{j=1}^{l} \bar{\tau}_j g_j$ is quasi-semilocal E-preinvex, and all functions are E-η-semidifferentiable at $\bar{z}$ with*

$$\bar{w}^L F^L(\bar{x}) + \bar{w}^U F^U(\bar{x}) \leq \bar{w}^L F^L(\bar{z}) + \bar{w}^U F^U(\bar{z}). \tag{4.15}$$

Then $\bar{x} = \bar{z}$.

Proof Suppose, contrary to the result, that $\bar{x} \neq \bar{z}$. Using $\bar{\tau}_j \geq 0$, $j = 1, 2, \ldots, l$, with the feasibility of $\bar{x}$ and $(\bar{z}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ to (IVP) and (IVMWD), respectively, we have

$$\sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{x}) \leq \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{z}).$$

The above inequality, with the fact that $\sum_{j=1}^{l} \bar{\tau}_j g_j$ is quasi-semilocal E-preinvex with respect to η at $\bar{z}$, gives

$$\sum_{j=1}^{l} \bar{\tau}_j (dg_j)^+ \big(\bar{z}; \eta\big(E(\bar{x}), \bar{z}\big)\big) \leq 0.$$

Since $(\bar{z}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is feasible to (IVMWD), then from dual constraint (4.10) and the above inequality, we obtain

$$\bar{w}^L \big(dF^L\big)^+ \big(\bar{z}; \eta\big(E(\bar{x}), \bar{z}\big)\big) + \bar{w}^U \big(dF^U\big)^+ \big(\bar{z}; \eta\big(E(\bar{x}), \bar{z}\big)\big) \geq 0.$$

With the above inequality using the strict pseudo-semilocal E-preinvexity of $\bar{w}^L F^L + \bar{w}^U F^U$ with respect to η at $\bar{z}$, we get

$$\bar{w}^L F^L(\bar{x}) + \bar{w}^U F^U(\bar{x}) > \bar{w}^L F^L(\bar{z}) + \bar{w}^U F^U(\bar{z}),$$

a contradiction to inequality (4.15). Hence, $\bar{x} = \bar{z}$.   □

5 Lagrangian function and saddle-point criteria

Consider the following Lagrangian function for interval-valued optimization problem (IVP):

$$L\big(x, w^L, w^U, \tau\big) = w^L F^L(x) + w^U F^U(x) + \sum_{j=1}^{l} \tau_j g_j(x), \tag{5.1}$$

where $x \in X$, $w^L \geq 0$, $w^U \geq 0$, $\tau \in \mathbb{R}_+^l$.

Definition 5.1 ([20]) Suppose that $\bar{w}^L \geq 0$ and $\bar{w}^U \geq 0$ are fixed. A point $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau}) \in X \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^l$ is called a *saddle point* of the real-valued function $L(x, w^L, w^U, \tau)$ if the following condition holds:

$$L(\bar{x}, \bar{w}^L, \bar{w}^U, \tau) \leq L(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau}) \leq L(x, \bar{w}^L, \bar{w}^U, \bar{\tau}), \quad \forall x \in X, \forall \tau \in \mathbb{R}_+^l. \tag{5.2}$$

Theorem 5.1 ([20]) *Suppose that $\bar{w}^L > 0$ and $\bar{w}^U > 0$ are fixed and $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is a saddle point of $L(x, w^L, w^U, \tau)$. Then $\bar{x}$ is an LU optimal solution to (IVP).*

Theorem 5.2 *Let $E(\bar{x}) = \bar{x}$ and $\bar{x}$ be an LU optimal solution to (IVP), with all functions F^L, F^U, and g_j being E-η-semidifferentiable at $\bar{x}$. Suppose that there exist scalars $\bar{w}^L, \bar{w}^U (> 0) \in \mathbb{R}$, and $\bar{\tau}_j (\geq 0) \in \mathbb{R}$, $j = 1, 2, \ldots, l$, such that*

(i) $\bar{w}^L (dF^L)^+ (\bar{x}; \eta(E(x), \bar{x})) + \bar{w}^U (dF^U)^+ (\bar{x}; \eta(E(x), \bar{x}))$

$$+ \sum_{j=1}^{l} \bar{\tau}_j (dg_j)^+ (\bar{x}; \eta(E(x), \bar{x})) \geq 0, \quad \forall x \in X,$$

(ii) $\displaystyle\sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{x}) = 0,$

(iii)   *F and $\displaystyle\sum_{j=1}^{l} \bar{\tau}_j g_j$ are semilocal E-preinvex at $\bar{x}$.*

Then $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is a saddle point of $L(x, w^L, w^U, \tau)$.

Proof As F is semilocal E-preinvex with respect to η at $\bar{x}$, then

$$F^L(x) - F^L(\bar{x}) \geq (dF^L)^+ (\bar{x}; \eta(E(x), \bar{x}))$$

and

$$F^U(x) - F^U(\bar{x}) \geq (dF^U)^+ (\bar{x}; \eta(E(x), \bar{x})).$$

Multiplying the above inequalities by $\bar{w}^L$ and $\bar{w}^U$, respectively, and adding them, we get

$$\left[\{ \bar{w}^L F^L(x) + \bar{w}^U F^U(x) \} - \{ \bar{w}^L F^L(\bar{x}) + \bar{w}^U F^U(\bar{x}) \} \right]$$
$$\geq \bar{w}^L (dF^L)^+ (\bar{x}; \eta(E(x), \bar{x})) + \bar{w}^U (dF^U)^+ (\bar{x}; \eta(E(x), \bar{x})).$$

By the semilocal E-preinvexity of $\sum_{j=1}^{l} \bar{\tau}_j g_j$ with respect to η at $\bar{x}$, we get

$$\sum_{j=1}^{l} \bar{\tau}_j g_j(x) - \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{x}) \geq \sum_{j=1}^{l} \bar{\tau}_j (dg_j)^+ (\bar{x}; \eta(E(x), \bar{x})).$$

On adding the above two inequalities

$$\left[\left\{\bar{w}^L F^L(x) + \bar{w}^U F^U(x) + \sum_{j=1}^{l} \bar{\tau}_j g_j(x)\right\} - \left\{\bar{w}^L F^L(\bar{x}) + \bar{w}^U F^U(\bar{x}) + \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{z})\right\}\right]$$

$$\geq \bar{w}^L \left(dF^L\right)^+ \left(\bar{x}; \eta\left(E(x),\bar{x}\right)\right) + \bar{w}^U \left(dF^U\right)^+ \left(\bar{x}; \eta\left(E(x),\bar{x}\right)\right) + \sum_{j=1}^{l} \bar{\tau}_j (dg_j)^+ \left(\bar{x}; \eta\left(E(x),\bar{x}\right)\right).$$

The above inequality together with given assumption *(i)* gives

$$\bar{w}^L F^L(\bar{x}) + \bar{w}^U F^U(\bar{x}) + \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{x}) \leq \bar{w}^L F^L(x) + \bar{w}^U F^U(x) + \sum_{j=1}^{l} \bar{\tau}_j g_j(x),$$

i.e., $L\left(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau}\right) \leq L\left(x, \bar{w}^L, \bar{w}^U, \bar{\tau}\right).$ $\qquad$ (5.3)

As $\bar{x}$ is feasible to (IVP) and $\tau \in \mathbb{R}_+^l$, then

$$\sum_{j=1}^{l} \tau_j g_j(\bar{x}) \leq 0. \qquad (5.4)$$

By inequality (5.4) with assumption *(ii)*, we find that

$$L\left(\bar{x}, \bar{w}^L, \bar{w}^U, \tau\right) \leq L\left(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau}\right). \qquad (5.5)$$

From (5.3) and (5.5), it is clear that $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is a saddle point of $L(x, w^L, w^U, \tau)$. $\qquad \square$

Theorem 5.3 *Let $E(\bar{x}) = \bar{x}$ and $\bar{x}$ be an LU optimal solution to* (IVP), *with all functions F^L, F^U, and g_j being E-η-semidifferentiable at $\bar{x}$. Suppose that there exist scalars $\bar{w}^L, \bar{w}^U(> 0) \in \mathbb{R}$, and $\bar{\tau}_j(\geq 0) \in \mathbb{R}, j = 1, 2, \ldots, l$, such that*

(i) $\quad \bar{w}^L \left(dF^L\right)^+ \left(\bar{x}; \eta\left(E(x),\bar{x}\right)\right) + \bar{w}^U \left(dF^U\right)^+ \left(\bar{x}; \eta\left(E(x),\bar{x}\right)\right)$

$$+ \sum_{j=1}^{l} \bar{\tau}_j (dg_j)^+ \left(\bar{x}; \eta\left(E(x),\bar{x}\right)\right) \geq 0, \quad \forall x \in X,$$

(ii) $\quad \displaystyle\sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{x}) = 0,$

(iii) $\quad \bar{w}^L F^L + \bar{w}^U F^U + \displaystyle\sum_{j=1}^{l} \bar{\tau}_j g_j$ *is semilocal E-preinvex at $\bar{x}$.*

Then $(\bar{x}, \bar{w}^L, \bar{w}^U, \bar{\tau})$ is a saddle-point of $L(x, w^L, w^U, \tau)$.

Proof Assumption *(i)*, together with the semilocal E-preinvexity of $\bar{w}^L F^L + \bar{w}^U F^U + \sum_{j=1}^{l} \bar{\tau}_j g_j$ with respect to η at $\bar{x}$, yields

$$\bar{w}^L F^L(\bar{x}) + \bar{w}^U F^U(\bar{x}) + \sum_{j=1}^{l} \bar{\tau}_j g_j(\bar{x}) \leq \bar{w}^L F^L(x) + \bar{w}^U F^U(x) + \sum_{j=1}^{l} \bar{\tau}_j g_j(x).$$

Now, the rest of the proof is the same as the proof of Theorem 5.2. $\qquad \square$

6 Conclusions

We have considered interval-valued programming problems (IVP) for E-η-semidifferentiable functions. We established sufficient optimality conditions for (IVP) and illustrated the result with the help of an example. We formulated the Wolfe type and Mond–Weir type dual models for (IVP) and established the usual duality results for the described models. Further, we presented the saddle-point optimality criteria to establish the relation between an optimal solution of semidifferentiable (IVP) and a saddle point of the Lagrangian function. In the future, the results obtained in this paper can be extended to multiobjective case and generalized type-I functions.

Author details
[1]College of Economics, Shenzhen University, Shenzhen, 518060, China. [2]Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, 221005, India.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

1. Birge, J.R., Louveaux, F.: Introduction to Stochastic Programming. Springer, New York (1997)
2. Stancu-Minasian, I.M.: Stochastic Programming with Multiple Objective Functions. Reidel, Dordrecht (1984)
3. Wu, H.-C.: The Karush–Kuhn–Tucker optimality conditions in an optimization problem with interval-valued objective function. Eur. J. Oper. Res. **176**, 46–59 (2007)
4. Wu, H.-C.: The Karush–Kuhn–Tucker optimality conditions in multiobjective programming problems with interval-valued objective functions. Eur. J. Oper. Res. **196**, 49–60 (2009)
5. Wu, H.-C.: Duality theory for optimization problems with interval-valued objective functions. J. Optim. Theory Appl. **144**, 615–628 (2010)
6. Bhurjee, A.K., Panda, G.: Sufficient optimality conditions and duality theory for interval optimization problem. Ann. Oper. Res. **243**, 335–348 (2016)
7. Ishibuchi, H., Tanaka, H.: Multiobjective programming in optimization of the interval objective function. Eur. J. Oper. Res. **48**, 219–225 (1990)
8. Wu, H.-C.: On interval-valued nonlinear programming problems. J. Math. Anal. Appl. **338**, 299–316 (2008)
9. Wu, H.-C.: Wolfe duality for interval-valued optimization. J. Optim. Theory Appl. **138**, 497–509 (2008)
10. Jayswal, A., Stancu-Minasian, I.M., Ahmad, I.: On sufficiency and duality for a class of interval-valued programming problems. Appl. Math. Comput. **218**, 4119–4127 (2011)
11. Zhang, J., Liu, S., Li, L., Feng, Q.: The KKT optimality conditions in a class of generalized convex optimization problems with an interval-valued objective function. Optim. Lett. **8**, 607–631 (2014)
12. Sun, Y., Xu, X., Wang, L.: Duality and saddle-point type optimality for interval-valued programming. Optim. Lett. **8**, 1077–1091 (2014)
13. Li, L., Liu, S., Zhang, J.: On interval-valued invex mappings and optimality conditions for interval-valued optimization problems. J. Inequal. Appl. **2015**, 179 (2015)
14. Li, L., Zhang, J., Zhou, C.: Optimality conditions for interval-valued univex programming. J. Inequal. Appl. **2019**, 49 (2019)
15. Kaul, R.N., Kaur, S.: Generalizations of convex and related functions. Eur. J. Oper. Res. **9**, 369–377 (1982)
16. Jiao, H., Liu, S.: Semilocal *E*-preinvexity and its applications in nonlinear multiple objective fractional programming. J. Inequal. Appl. **2011**, 116 (2011)
17. Moore, R.E.: Methods and Applications of Interval Analysis. SIAM, Philadelphia (1979)

18.　Fulga, C., Preda, V.: Nonlinear programming with *E*-preinvex and local *E*-preinvex functions. Eur. J. Oper. Res. **192**, 737–743 (2009)

19.　Sun, Y., Wang, L.: Optimality conditions and duality in nondifferentiable interval-valued programming. J. Ind. Manag. Optim. **9**, 131–142 (2013)

20.　Jayswal, A., Ahmad, I., Banerjee, J.: Nonsmooth interval-valued optimization and saddle-point optimality criteria. Bull. Malays. Math. Sci. Soc. **39**, 1391–1411 (2016)