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Inertial KM-type extragradient scheme for solving a variational inequality and a hierarchical fixed point problems

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Abstract

We propose an inertial KM-type extragradient scheme to approximate a common solution of a variational inequality problem and a hierarchical fixed point problem for nonexpansive mappings. This scheme generalizes and unifies a number of known iterative schemes. Furthermore, we discuss the weak convergence for the proposed scheme. We also discuss an example to illustrate the main theorem.

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1 Introduction

Let $\mathcal{C}$ be a nonempty convex and closed set in a real Hilbert space $\mathcal{H}$ and $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the inner product and induced norm on $\mathcal{H}$. A mapping $U:\mathcal{C}\to\mathcal{C}$ is said to be nonexpansive if $\|Uu-Uv\|\le\|u-v\|$, $\forall u,v\in\mathcal{C}$. Note that if $\mathrm{F}(U):=\{u\in\mathcal{C}:Uu=u\}\neq\emptyset$ then set $\mathrm{F}(U)$ is convex and closed. Let $\mathrm{F}(U)\neq\emptyset$. The subdifferential of a proper function $g:\mathcal{H}\to(-\infty,+\infty]$ is the set-valued operator $\partial g:\mathcal{H}\to 2^{\mathcal{H}}$ defined by $\partial g(u)=\{w\in\mathcal{H}:\langle y-u,w\rangle+g(u)\le g(y),\forall y\in\mathcal{H}\}$. Let $u\in\mathcal{H}$. Then g is subdifferential at u if $\partial g(u)\neq\emptyset$. The indicator function $\psi_{\mathcal{C}}:\mathcal{H}\to(-\infty,+\infty]$ is given by

$$\partial\psi_{\mathcal{C}}(u)=\begin{cases}0, & u\in\mathcal{C},\\ \infty, & \text{otherwise.}\end{cases}$$

Note that $\psi_{\mathcal{C}}$ is a convex function when $\mathcal{C}$ is a convex set.

In 2006, Moudafi *et al.* [1] discussed the convergence of a scheme for the following hierarchical fixed point problem (in short, H-FPP): Find $\bar{u}\in\mathrm{F}(U)$ such that

$$\langle\bar{u}-V\bar{u},\bar{u}-u\rangle\le 0,\quad\forall u\in\mathrm{F}(U),\tag{1.1}$$

where the mappings $U,V:\mathcal{C}\to\mathcal{C}$ are nonexpansive. Let Φ denote the set of solutions of H-FPP(1.1). If $\bar{u}\in\mathrm{F}(U)$ then (1.1) $\Leftrightarrow\langle-(I-V)\bar{u},u-\bar{u}\rangle+\psi_{\mathrm{F}(U)}(\bar{u})\le\psi_{\mathrm{F}(U)}(u)\Leftrightarrow-(I-V)\bar{u}\in$

$\partial\psi_{\mathrm{F}(U)}(\bar{u})$. Hence H-FPP(1.1) is equivalent to the variational inclusion: Find $\bar{u} \in \mathrm{F}(U)$ such that

$$0 \in (I - V)\bar{u} + N_{\mathrm{F}(U)}(\bar{u}), \tag{1.2}$$

where the mapping I is identity on $\mathcal{C}$ and $N_{\mathrm{F}(U)}(\bar{u})$ denotes the normal cone to $\mathrm{F}(U)$ at $\bar{u}$ given by

$$N_{\mathrm{F}(U)}(\bar{u}) = \partial\psi_{\mathrm{F}(U)}(\bar{u}) = \begin{cases} \{w \in \mathcal{H} : \langle u - \bar{u}, w\rangle \le 0, \forall u \in \mathrm{F}(U)\}, & \text{if } \bar{u} \in \mathrm{F}(U), \\ \emptyset, & \text{otherwise.} \end{cases}$$

If we set $V = I$, then Φ is just $\mathrm{F}(U)$. Furthermore, we mention that H-FPP(1.1) is worth to study because it includes as special cases, the important problems such as the variational inequality on fixed point sets and hierarchical minimization problems; see Moudafi [2].

In 2007, Moudafi [2] proposed the following Krasnoselski–Mann (KM)-type scheme for solving H-FPP(1.1): For given $u_0 \in \mathcal{C}$,

$$u_{k+1} = (1 - \alpha_k)u_k + \alpha_k\bigl(\sigma_k V u_k + (1 - \sigma_n)U u_k\bigr), \quad \forall n \ge 0, \tag{1.3}$$

where $\{\alpha_k\} \subset (0, 1)$ and $\{\sigma_k\} \subset (0, 1)$. For further work related to scheme (1.3), see for example [1, 3–7].

In 2008, Mainge [8] introduced an inertial version of KM-type scheme by unifying the KM-type scheme and the inertial extrapolation, for approximating a fixed point of nonexpansive mappings and discussed the weak convergence. Recently, Bot *et al.* [9] derived some the convergence results of the following inertial KM-type scheme to approximate a fixed point of nonexpansive mapping U on $\mathcal{H}$ which generalize the results of Mainge [8]:

$$\left.\begin{aligned} t_k &= u_k + \eta_k(u_k - u_{k-1}), \\ u_{k+1} &= (1 - \alpha_k)t_k + \alpha_k U t_k, \end{aligned}\right\} \tag{1.4}$$

for each $k \ge 1$, where η_k is a damping-type term and α_k is a relaxation factor. Recently, the interest of studying inertial type algorithms has been increased due to their fast convergence. For further study of scheme (1.4) and its generalizations; see for example [10–13].

On the other hand, we consider the classical variational inequality (in short, VI): Find $\bar{u} \in \mathcal{C}$ such that

$$\bigl\langle h(\bar{u}), v - \bar{u}\bigr\rangle \ge 0, \quad \forall v \in \mathcal{C}, \tag{1.5}$$

introduced in [14] where $h : \mathcal{H} \to \mathcal{H}$. The set of solutions of VI(1.5) is denoted by Sol(VI(1.5)). Note that the projected gradient scheme for solving VI(1.5) is

$$u_{k+1} = \mathcal{P}_{\mathcal{C}}\bigl(u_k - \mu h(u_k)\bigr), \tag{1.6}$$

where $\mu > 0$ and $\mathcal{P}_{\mathcal{C}}$ is the metric projection onto $\mathcal{C}$. In order to converge, this scheme requires the restrictive condition that h is inverse strongly (or strongly) monotone. To overcome this difficulty, Korpelevich [15] proposed an extragradient iterative scheme by

$$\left.\begin{aligned} v_k &= \mathcal{P}_{\mathcal{C}}(u_k - \mu h(u_k)),\\ u_{k+1} &= \mathcal{P}_{\mathcal{C}}(u_k - \mu h(v_k)), \end{aligned}\right\} \tag{1.7}$$

where $\mu \in (0, \frac{1}{L})$, where $L > 0$ is Lipschitz constant of h. Since then many researchers improved scheme (1.7) in various directions; see, e.g. [16–24] and the references therein. Note that the calculation of two projections onto $\mathcal{C}$ might affect the efficiency of such scheme. Therefore, Dong et al. [25] proposed the following inertial KM-type extragradient scheme for VI(1.5):

$$\left.\begin{aligned} t_k &= u_k + \eta_k(u_k - u_{k-1}),\\ v_k &= \mathcal{P}_{\mathcal{C}}(t_k - \mu h(t_k)),\\ u_{k+1} &= (1-\alpha_k)t_k + \alpha_k \mathcal{P}_{\mathcal{C}}(t_k - \mu h(v_k)), \end{aligned}\right\} \tag{1.8}$$

where $\{\eta_k\} \subset [0, \eta]$, $\forall k$ is nondecreasing with $\eta_1 = 0$ and $0 \le \eta_k \le \eta < 1$, for every $k \ge 1$ such that

$$\delta > \frac{\eta[(1+\mu L)^2\eta(1+\eta) + (1-\mu^2L^2)\eta\sigma + \sigma(1+\mu L)^2]}{1-\mu^2L^2}$$

and

$$0 < \alpha \le \alpha_k \le \frac{\delta(1-\mu^2L^2) - \eta[(1+\mu L)^2\eta(1+\eta) + (1-\mu^2L^2)\eta\sigma + \sigma(1+\mu L)^2]}{\delta[(1+\mu L)^2\eta(1+\eta) + (1-\mu^2L^2)\eta\sigma + \sigma(1+\mu L)^2]},$$

where $\alpha, \sigma, \delta > 0$.

They proved the weak convergence theorem for scheme (1.8).

In this paper, we propose an inertial version of KM-type extragradient scheme by combining iterative schemes (1.3) and (1.8) to approximate a common solution of H-FPP(1.1) and VI(1.5). We prove a weak convergence theorem for the proposed scheme. Furthermore, we discuss an example to illustrate the main theorem. The theorems of the paper unify and generalize previously known corresponding theorems; see for example [2, 8, 9, 25–27].

2 Preliminaries

We give some definitions and results of convex and nonlinear analysis, which will be used in the proof of the weak convergence theorem.

A mapping $\mathcal{P}_{\mathcal{C}}$ is called the *metric projection* of $\mathcal{H}$ onto $\mathcal{C}$ if for every point $u \in \mathcal{H}$, there exists a unique point in $\mathcal{C}$ denoted by $\mathcal{P}_{\mathcal{C}}u$ such that

$$\|u - \mathcal{P}_{\mathcal{C}}u\| \le \|u - v\|, \quad \forall v \in \mathcal{C}.$$

Note that $\mathcal{P}_{\mathcal{C}}$ is nonexpansive and satisfies

$$\langle u - v, \mathcal{P}_{\mathcal{C}}u - \mathcal{P}_{\mathcal{C}}v\rangle \ge \|\mathcal{P}_{\mathcal{C}}u - \mathcal{P}_{\mathcal{C}}v\|^2, \quad \forall u \in \mathcal{H}.$$

Moreover, $\mathcal{P}_{\mathcal{C}}u$ is characterized by the fact $\mathcal{P}_{\mathcal{C}}u \in \mathcal{C}$ and

$$\langle u - \mathcal{P}_{\mathcal{C}}u, v - \mathcal{P}_{\mathcal{C}}u\rangle \leq 0, \quad \forall v \in \mathcal{C},$$

which implies that

$$\|u - v\|^2 \geq \|u - \mathcal{P}_{\mathcal{C}}u\|^2 + \|v - \mathcal{P}_{\mathcal{C}}u\|^2, \quad \forall u \in \mathcal{H}, v \in \mathcal{C}.$$

Definition 2.1 A mapping $h : \mathcal{H} \to \mathcal{H}$ is called:

(i) *monotone*, if for all $u, v \in \mathcal{H}$, we have

$$\langle hu - hv, u - v\rangle \geq 0;$$

(ii) *L-Lipschitz continuous*, if there exists a constant $L > 0$ such that, for all $u, v \in \mathcal{H}$, we have

$$\|hu - hv\| \leq L\|u - v\|.$$

Lemma 2.1 *If a mapping U is nonexpansive on $\mathcal{H}$ then $I - U$ is maximal monotone* [28] *and demiclosed* [29] *on $\mathcal{H}$.*

Lemma 2.2 ([30]) *Let $\{\psi_k\}$, $\{\delta_k\}$ and $\{\eta_k\}$ be the sequences in $[0,\infty)$ such that $\psi_{k+1} \leq \psi_k + \eta_k(\psi_k - \psi_{k-1}) + \gamma_k$, $\forall k \geq 1$, $\sum_{k=1}^{\infty} \gamma_k < +\infty$ and there is a number η with $0 \leq \eta_k \leq \eta < 1$, $\forall k \geq 1$. Then the following hold*:

(a) $\sum_{k=1}^{\infty} [\psi_k - \psi_{k-1}]_+ < +\infty$, where $[r]_+ := \max\{r, 0\}$;

(b) *there is a* $\psi^* \in [0,\infty)$ such that $\lim_{k\to\infty} \psi_k = \psi^*$.

Lemma 2.3 ([31]) *Let $\mathcal{C}$ be a nonempty subset of $\mathcal{H}$ and the sequence $\{u_k\}$ in $\mathcal{H}$ satisfy the conditions*:

(a) $\lim_{k\to\infty} \|u_k - u\|$ *exists for every* $u \in \mathcal{C}$;

(b) *any weak cluster point of* $\{u_k\}$ *is in* $\mathcal{C}$.

Then $\{u_k\}$ is weak convergent to a point in $\mathcal{C}$.

3 Weak convergence theorem

We propose the following inertial KM-type extragradient scheme for solving H-FPP(1.1) and VI(1.5).

Scheme Choose initial values $u_0, u_1 \in \mathcal{H}$ arbitrarily. The sequence $\{u_k\}$ be generated by the scheme:

$$\left.\begin{aligned} t_k &= u_k + \eta_k(u_k - u_{k-1}),\\ v_k &= \mathcal{P}_{\mathcal{C}}(t_k - \mu h(t_k)),\\ w_k &= \mathcal{P}_{\mathcal{C}}(t_k - \mu h(v_k)),\\ u_{k+1} &= (1 - \alpha_k)t_k + \alpha_n(\sigma_k V w_k + (1 - \sigma_n)U w_k), \end{aligned}\right\} \tag{3.1}$$

where $\{\eta_k\} \subset [0,\eta]$, $\forall k$, is nondecreasing with $\eta_1 = 0$ and $0 \le \eta_k \le \eta < 1$, $\{\sigma_k\} \subseteq [c,d]$, $c,d \in (0,1)$, $\mu \in (0,\frac{1}{L})$, $L > 0$ and $\{\alpha_k\}$ is a real sequence with conditions:

$$\delta > \frac{\eta^2(1+\eta)+\eta\sigma}{1-\eta^2} \quad \text{and} \quad 0 < \alpha \le \alpha_k \le \frac{\delta - \eta[\eta(1+\eta)+\eta\delta+\sigma]}{\delta[1+\eta(1+\eta)+\eta\delta+\sigma]}, \quad \text{where } \alpha, \sigma, \delta > 0.$$

Now, we discuss the weak convergence for scheme (3.1).

Theorem 3.1 *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{C} \subset \mathcal{H}$ be a nonempty, convex and closed set; let the mappings $U,V:\mathcal{C}\to\mathcal{C}$ be nonexpansive and $h:\mathcal{H}\to\mathcal{H}$ be L-Lipschitz continuous and monotone. Assume that $\Gamma = \mathrm{Sol}(\mathrm{VI}(1.5)) \cap \Phi \cap \mathrm{F}(V) \neq \emptyset$. Let the sequence $\{u_k\}$ be defined by scheme (3.1). Then the following results hold:*

(a) $\sum_{k=1}^{\infty} \|u_{k+1}-u_k\|^2 < +\infty$;

(b) *the sequence $\{u_k\}$ converges weakly to $\bar{u} \in \Gamma$.*

Proof (a). Let for any $q \in \Gamma$. Since h is L-Lipschitz continuous and monotone then we can easily get

$$\|w_k - q\|^2 \le \|t_k - q\|^2 - \left(1-\mu^2L^2\right)\|t_k - v_k\|^2; \tag{3.2}$$

see [3]. From the nonexpansivity of $\mathcal{P}_\mathcal{C}$ and Lipschitz continuity of h, it follows that

$$\begin{aligned}\|v_k - w_k\| = \left\|\mathcal{P}_\mathcal{C}\left(t_k - \mu h(t_k)\right) - \mathcal{P}_\mathcal{C}\left(t_k - \mu h(v_k)\right)\right\| &\le \mu\left\|h(t_k) - h(v_k)\right\| \\ &\le \mu L\|t_k - v_k\|, \end{aligned} \tag{3.3}$$

which yields

$$\|t_k - w_k\| \le \|t_k - v_k\| + \|v_k - w_k\| \le (1+\mu L)\|t_k - v_k\|. \tag{3.4}$$

As follows from (3.2), (3.4) and $\mu L \in (0,1)$, we have

$$\|w_k - q\|^2 \le \|t_k - q\|^2 - \frac{1-\mu^2L^2}{(1+\mu L)^2}\|t_k - w_k\|^2. \tag{3.5}$$

Let for any $q \in \Gamma$ and $T_{\sigma_k} := \sigma_k V + (1-\sigma_k)U$. Now, by using (3.5), we estimate

$$\begin{aligned}\|u_{k+1} - q\|^2 &= \left\|(1-\alpha_k)t_k + \alpha_k T_{\sigma_k} w_k - q\right\|^2 \\ &\le (1-\alpha_k)\|t_k - q\|^2 + \alpha_k\|T_{\sigma_n} w_k - q\|^2 - \alpha_k(1-\alpha_k)\|T_{\sigma_k} w_k - t_k\|^2 \\ &\le (1-\alpha_k)\|t_k - q\|^2 + \alpha_k\left(\sigma_k\|Vw_k - q\|^2 + (1-\sigma_k)\|Uw_k - q\|^2\right. \\ &\quad \left. - \sigma_k(1-\sigma_k)\|Vw_k - Uw_k\|^2\right) - \alpha_k(1-\alpha_k)\|T_{\sigma_k} w_k - t_k\|^2 \\ &\le \|t_k - q\|^2 - \alpha_k\sigma_k(1-\sigma_k)\|Vw_k - Uw_k\|^2 - \frac{1-\mu^2L^2}{(1+\mu L)^2}\|t_k - v_k\|^2 \\ &\quad - \alpha_k(1-\alpha_k)\|T_{\sigma_k} w_k - t_k\|^2 \qquad (3.6) \\ &\le \|t_k - q\|^2 - \alpha_k(1-\alpha_k)\|T_{\sigma_k} w_k - t_k\|^2. \qquad (3.7)\end{aligned}$$

Next, we estimate

$$\begin{aligned}\|t_k - q\|^2 &= \left\|u_k + \eta_k(u_k - u_{k-1}) - q\right\|^2\\ &= (1+\eta_k)\|u_k - q\|^2 - \eta_k\|u_{k-1} - q\|^2\\ &\quad + \eta_k(1+\eta_k)\|u_k - u_{k-1}\|^2.\end{aligned}\tag{3.8}$$

From (3.7) and (3.8), we have

$$\begin{aligned}\|u_{k+1} - q\|^2 - (1+\eta_k)\|u_k - q\|^2 + \eta_k\|u_{k-1} - q\|^2 &\le -\alpha_k(1-\alpha_k)\|T_{\sigma_k}u_k - t_k\|^2\\ &\quad + \eta_k(1+\eta_k)\|u_k - u_{k-1}\|^2.\end{aligned}\tag{3.9}$$

Furthermore, from scheme (3.1), we have

$$\begin{aligned}\|T_{\sigma_k}w_k - t_k\|^2 &= \left\|\frac{1}{\alpha_k}(u_{k+1} - u_k) + \frac{\eta_k}{\alpha_k}(u_{k-1} - u_k)\right\|^2\\ &\ge \frac{1}{\alpha_k^2}\|u_{k+1} - u_k\|^2 + \frac{\eta_k^2}{\alpha_k^2}\|u_k - u_{k-1}\|^2\\ &\quad + \frac{\eta_k}{\alpha_k^2}\left(-\rho_k\|u_{k+1} - u_k\|^2 - \frac{1}{\rho_k}\|u_k - u_{k-1}\|^2\right),\end{aligned}\tag{3.10}$$

where $\rho_k := \frac{1}{\eta_k + \delta\alpha_k}$. Thus, it follows from (3.9) and (3.10) that

$$\begin{aligned}\|u_{k+1} - q\|^2 - (1+\eta_k)\|u_k - q\|^2 + \eta_k\|u_{k-1} - q\|^2 &\le \frac{(1-\alpha_k)(\eta_k\rho_k - 1)}{\alpha_k}\|u_{k+1} - u_k\|^2\\ &\quad + \gamma_k\|u_k - u_{k-1}\|^2,\end{aligned}\tag{3.11}$$

where

$$\gamma_k := \eta_k(1+\eta_k) + \eta_k(1-\alpha_k)\frac{(1-\eta_k\rho_k)}{\alpha_k\rho_k} > 0,\tag{3.12}$$

since $\eta_k\rho_k < 1$ and $\alpha_k \in (0,1)$. It follows from $\delta = \frac{(1-\eta_k\rho_k)}{\alpha_k\rho_k}$ and (3.12) that

$$\gamma_k := \eta_k(1+\eta_k) + \eta_k(1-\alpha_k)\delta \le \eta(1+\eta) + \eta\delta, \quad \forall k \ge 1.\tag{3.13}$$

Next, we define the sequences $\{\phi_k\}$ and $\{\psi_k\}$ by

$$\phi_k := \|x_k - q\|^2, \qquad \psi_k := \phi_k - \eta_k\phi_{k-1} + \gamma_k\|u_k - u_{k-1}\|^2, \quad \forall k \ge 1.\tag{3.14}$$

Now, using the monotonicity of $\{\eta_k\}$ and the fact that $\phi_k \ge 0$ for all $k \in \mathbb{N}$, we have

$$\psi_{k+1} - \psi_k \le \phi_{k+1} - (1+\eta_k)\phi_k + \eta_k\phi_{k-1} + \gamma_{k+1}\|u_{k+1} - u_k\|^2 - \gamma_k\|u_k - u_{k-1}\|^2.\tag{3.15}$$

Hence, it follows from (3.11) and (3.15) that

$$\begin{aligned}\psi_{k+1}-\psi_k &\leq \frac{(1-\alpha_k)(\eta_k\rho_k-1)}{\alpha_k}\|u_{k+1}-u_k\|^2+\gamma_{k+1}\|u_{k+1}-u_k\|^2\\ &=\left(\frac{(1-\alpha_k)(\eta_k\rho_k-1)}{\alpha_k}+\gamma_{k+1}\right)\|u_{k+1}-u_k\|^2. \end{aligned}\tag{3.16}$$

Now, we note that

$$\frac{(1-\alpha_k)(\eta_k\rho_k-1)}{\alpha_k}+\gamma_{k+1}\leq -\sigma,\quad \forall k\geq 1;\tag{3.17}$$

see [9].

Therefore, it follows from (3.16) and (3.17) that

$$\psi_{k+1}-\psi_k\leq -\sigma\|u_{k+1}-u_k\|^2.\tag{3.18}$$

Since $\eta_1=0$, it follows from (3.14) that $\psi_1=\phi_1\geq 0$ and hence (3.18) shows that $\{\psi_k\}$ is bounded. Furthermore, (3.14) and the boundedness of $\{\eta_k\}$ yield

$$-\eta\phi_{k-1}\leq\phi_k-\eta\phi_{k-1}\leq\psi_k\leq\psi_1.\tag{3.19}$$

Thus, we obtain

$$\phi_k\leq\eta^k\phi_0+\psi_1\sum_{j=1}^{k-1}\eta^j\leq\eta^k\phi_0+\frac{1}{1-\eta}\psi_1.\tag{3.20}$$

Now, it follows from (3.18), (3.19), (3.20) and the boundedness of $\{\psi_k\}$ that

$$\sigma\sum_{j=1}^{k}\|u_{j+1}-u_j\|^2\leq\psi_1-\psi_{k+1}\leq\psi_1+\eta\phi_k\leq\psi_1+\eta^k\phi_0+\frac{1}{1-\eta}\psi_1,\tag{3.21}$$

which implies that $\sum_{k=1}^{\infty}\|u_{k+1}-u_k\|^2<+\infty$.

Proof of (b). Since $\eta_k\rho_k<1$, it follows from (3.11), (3.13), $\sum_{k=1}^{\infty}\|u_{k+1}-u_k\|^2<+\infty$, and Lemma 2.2 that

$$\lim_{k\to\infty}\|u_k-q\|\quad\text{exists and finite,}\tag{3.22}$$

and hence $\{u_k\}$ is bounded. It follows furthermore from $\sum_{k=1}^{\infty}\|u_{k+1}-u_k\|^2<+\infty$ that

$$\lim_{k\to\infty}\|u_{k+1}-u_k\|=0.\tag{3.23}$$

Next, by the definition of t_k in (3.1) and $\eta_k\leq\eta$, $\forall k$, we have

$$\|t_k-u_k\|=\eta_k\|u_k-u_{k-1}\|\leq\eta\|u_k-u_{k-1}\|,$$

which implies that

$$\lim_{k\to\infty} \|t_k - u_k\| = 0, \tag{3.24}$$

and hence $\{t_k\}$ is bounded. Since

$$\|t_k - u_{k+1}\| \le \|t_k - u_k\| + \|u_k - u_{k+1}\|, \tag{3.25}$$

it follows from (3.23), (3.24) and (3.25) that

$$\lim_{k\to\infty} \|t_k - u_{k+1}\| = 0. \tag{3.26}$$

From (3.6) and (3.26), and $\{\alpha_k\} \subseteq (0,1)$, $\{\sigma_k\} \subseteq [c,d]$, $c,d \in (0,1)$, we have

$$\begin{aligned}
\alpha_k\sigma_k(1-\sigma_k)\|Vw_k - Uw_k\|^2 &= \|t_k - q\|^2 - \|u_{k+1} - q\|^2 \\
&\le \|t_k - u_{k+1}\|\bigl(\|t_k - q\| + \|u_{k+1} - q\|\bigr) \\
&= \|t_k - u_{k+1}\|M_1,
\end{aligned}$$

where $M_1 := \sup_k\{\|t_k - q\| + \|u_{k+1} - q\|\}$. Hence, it follows

$$\lim_{k\to\infty} \|Vw_k - Uw_k\| = 0. \tag{3.27}$$

From (3.6) and (3.26), and $\mu L \in (0,1)$, we have

$$\begin{aligned}
\frac{1-\mu^2L^2}{(1+\mu L)^2}\|t_k - w_k\|^2 &\le \|t_k - q\|^2 - \|u_{k+1} - q\|^2 \\
&= \|t_k - u_{k+1}\|M_1,
\end{aligned}$$

it follows that

$$\lim_{k\to\infty} \|t_k - w_k\| = 0. \tag{3.28}$$

It follows from (3.26) and (3.28) that

$$\lim_{k\to\infty} \bigl\|t_k - u_{k+1} - \alpha_k(t_k - w_k)\bigr\| = 0. \tag{3.29}$$

Furthermore, we have

$$\begin{aligned}
&\alpha_k\|Uw_k - w_k\| \le \|u_{k+1} - t_k\| + \alpha_k\|t_k - w_k\| + \alpha_k\sigma_k\|Uw_k - Vw_k\|, \\
&\|Uw_k - w_k\| \le \frac{1}{\alpha_k}\|u_{k+1} - t_k\| + \|t_k - w_k\| + \sigma_k\|Uw_k - Vw_k\|.
\end{aligned} \tag{3.30}$$

Since $\alpha_k > \alpha > 0$, $\forall k$, it follows from (3.26), (3.27), (3.28) and (3.30) that

$$\lim_{k\to\infty} \|Uw_k - w_k\| = 0. \tag{3.31}$$

From (3.27) and (3.31), we have

$$\lim_{k\to\infty} \|Vw_k - w_k\| = 0. \tag{3.32}$$

Now, let $\bar{u}$ be a sequential weak cluster point of $\{u_k\}$, that is, there exists a subsequence $\{u_{k_i}\}$ of $\{u_k\}$ such that $\{u_{k_i}\}$ converges weakly to $\bar{u}$, say, in $\mathcal{H}$. Furthermore, (3.24) and (3.28) imply that $\{u_k\}$, $\{t_k\}$ and $\{w_k\}$ all have the same asymptotic behavior and hence there exist subsequences $\{t_{k_i}\}$ of $\{t_k\}$ and $\{w_{k_i}\}$ of $\{w_k\}$ and such that t_{k_i} and w_{k_i} both converge weakly to $\bar{u}$. Now, Lemma 2.1, (3.31) and (3.32) imply that $\bar{u} \in \mathrm{F}(U)$ and $\bar{u} \in \mathrm{F}(V)$.

Next, we prove that $\bar{u} \in \Phi$. Since

$$u_{k+1} - t_k = \alpha_k(w_k - t_k) + \alpha_k\bigl(\sigma_k(Vw_k - w_k) + (1-\sigma_k)(Uw_k - w_k)\bigr), \tag{3.33}$$

and hence

$$\frac{1}{\alpha_k\sigma_k}\bigl(t_k - u_{k+1} - \alpha_k(t_k - w_k)\bigr) = (I-V)w_k + \left(\frac{1-\sigma_k}{\sigma_k}\right)(I-U)w_k, \tag{3.34}$$

and therefore for all $z \in \mathrm{F}(U)$ and by making use of the monotonicity of $I - V$, we have

$$\begin{aligned}\left\langle \frac{1}{\alpha_k\sigma_k}\bigl(t_k - u_{k+1} - \alpha_k(t_k - w_k)\bigr), w_k - z\right\rangle &= \bigl\langle (I-V)w_k - (I-V)z, w_k - z\bigr\rangle \\ &\quad + \bigl\langle (I-V)z, w_k - z\bigr\rangle \\ &\quad + \frac{1-\sigma_k}{\sigma_k}\langle w_k - Uw_k, w_k - z\rangle \\ &\geq \bigl\langle (I-V)z, w_k - z\bigr\rangle \\ &\quad + \frac{1-\sigma_k}{\sigma_k}\langle w_k - Uw_k, w_k - z\rangle. \end{aligned} \tag{3.35}$$

Hence,

$$\begin{aligned}&\left\langle \frac{1}{\alpha_{k_i}\sigma_{k_i}}\bigl(t_{k_i} - u_{k_i+1} - \alpha_{k_i}(t_{k_i} - w_{k_i})\bigr), w_{k_i} - z\right\rangle \\ &\quad \geq \bigl\langle (I-V)z, w_{k_i} - z\bigr\rangle \\ &\qquad + \frac{1-\sigma_{k_i}}{\sigma_{k_i}}\langle w_{k_i} - Uw_{k_i}, w_{k_i} - z\rangle. \end{aligned} \tag{3.36}$$

Using (3.29), (3.31), and the conditions on the parameters α_k and σ_k in (3.36), we have

$$\limsup_{i\to\infty}\langle z - Vz, w_{k_i} - z\rangle \leq 0 \quad \forall z \in \mathrm{F}(U). \tag{3.37}$$

Since w_{k_i} converges weakly to $\bar{u}$, we get

$$\bigl\langle (I-V)z, \bar{u} - z\bigr\rangle \leq 0, \quad \forall z \in \mathrm{F}(U). \tag{3.38}$$

Since $\mathrm{F}(U)$ is convex, $\beta z + (1-\beta)\hat{u} \in \mathrm{F}(U)$ for $\beta \in (0,1)$ and hence

$$\bigl\langle (I-V)\bigl(\beta z + (1-\beta)\bar{u}\bigr), \bar{u} - \bigl(\beta z + (1-\beta)\bar{u}\bigr)\bigr\rangle \tag{3.39}$$

$$= \beta\big\langle (I-V)\big(\beta z + (1-\beta)\bar{u}\big), \bar{u} - z\big\rangle \tag{3.40}$$
$$\leq 0, \quad \forall z \in \mathrm{F}(\mathcal{U}), \tag{3.41}$$

which implies

$$\big\langle (I-V)\big(\beta z + (1-\beta)\bar{u}\big), \bar{u} - z\big\rangle \leq 0, \quad \forall z \in \mathrm{F}(\mathcal{U}).$$

On taking the limit $\beta \to 0_+$, we have

$$\big\langle (I-V)\bar{u}, \bar{u} - z\big\rangle \leq 0, \quad \forall z \in \mathrm{F}(\mathcal{U}), \tag{3.42}$$

which implies $\bar{u} \in \Phi$.

Now, we show that $\bar{u} \in \mathrm{Sol}(\mathrm{VI}(1.5))$. Since $\lim_{k\to\infty} \|v_k - t_k\| = 0$ and $\lim_{k\to\infty} \|t_k - u_k\| = 0$, there exist subsequences $\{t_{k_i}\}$ of $\{t_k\}$ and $\{v_{k_i}\}$ of $\{v_k\}$, respectively, such that $\{t_{k_i}\}$, $\{v_{k_i}\}$ both converge weakly to $\bar{u}$. Let

$$Gv = \begin{cases} hv + N_{\mathcal{C}}(v), & \text{if } v \in \mathcal{C}; \\ \emptyset, & \text{if } v \notin \mathcal{C}, \end{cases}$$

then the monotone mapping G is maximal [32] and hence $0 \in Gv$ if and only if $v \in \mathrm{Sol}(\mathrm{VI}(1.5))$ [33]. Let $(v, w) \in \mathrm{graph}(G)$, then $w \in Gv = hv + N_{\mathcal{C}}(v)$ and hence $w - hv \in N_{\mathcal{C}}(v)$, i.e., $\langle v - u, w - hv\rangle \geq 0$, for all $u \in \mathcal{C}$.

On the other hand, from $v_k = \mathcal{P}_{\mathcal{C}}(I - \mu h)t_k$ and $v \in \mathcal{C}$, we get

$$\big\langle (I - \mu h)t_k - v_k, v_k - v\big\rangle \geq 0.$$

This implies that

$$\left\langle v^* - v_k, \frac{v_k - t_k}{\mu} + ht_k \right\rangle \geq 0.$$

Since $\langle v - u, w - hv\rangle \geq 0$, for all $u \in \mathcal{C}$ and $v_{k_i} \in \mathcal{C}$, using the monotonicity of h, we have

$$\begin{aligned} \langle v - v_{k_i}, w\rangle &\geq \langle v - v_{k_i}, hv\rangle \\ &\geq \langle v - v_{k_i}, hv\rangle - \left\langle v - v_{k_i}, \frac{v_{k_i} - t_{k_i}}{\mu} + ht_{k_i}\right\rangle \\ &= \langle v - v_{k_i}, hv - hv_{k_i}\rangle + \langle v - v_{k_i}, hv_{k_i} - ht_{k_i}\rangle - \left\langle v - y_{k_i}, \frac{v_{k_i} - t_{k_i}}{\mu}\right\rangle \\ &\geq \langle v - v_{k_i}, hv_{k_i} - ht_{k_i}\rangle - \left\langle v - v_{k_i}, \frac{v_{k_i} - t_{k_i}}{\mu}\right\rangle. \end{aligned}$$

Since h is continuous, on taking the limit $i \to \infty$ we have $\langle v - \bar{u}, w\rangle \geq 0$. Since G is maximal monotone, we have $\bar{u} \in G^{-1}0$ and hence $\bar{u} \in \mathrm{Sol}(\mathrm{VI}(1.5))$ and thus $\bar{u} \in \Gamma$.

Finally, it follows from (3.22) and Lemma 2.3 that the sequence $\{u_k\}$ converges weakly to $\bar{u} \in \Gamma$. □

Remark 3.2 One can derive a number of schemes from scheme (3.1); some special cases are as follows:

(i) Setting $\eta_k = 0$, $\forall k$ then scheme (3.1) reduces to extragradient scheme for solving VI(1.5) and H-FPP(1.1).

(ii) Setting $\sigma_k = 0$, $\forall k$, and $V = I$, $U = I$ then scheme (3.1) reduces to scheme (1.8) for solving VI(1.5) and hence we recover Theorem 3.1 [25].

(iii) Setting $V = I$, $\sigma_k = 0$, $U = J^B_{\lambda_k} := (I + \lambda_k B)^{-1}$ (where $B : \mathcal{H} \to 2^{\mathcal{H}}$ is maximal monotone and $\lambda_k \in (0,\infty)$), and $\alpha_k = \alpha$ $\forall k$, scheme (3.1) takes the following form:

$$\left.\begin{aligned} t_k &= u_k + \eta_k(u_k - u_{k-1}),\\ v_k &= \mathcal{P}_{\mathcal{C}}(t_k - \mu h(t_k)),\\ w_k &= \mathcal{P}_{\mathcal{C}}(t_k - \mu h(v_k)),\\ u_{k+1} &= (1-\alpha)t_k + \alpha J^B_{\lambda_k} w_k, \end{aligned}\right\} \tag{3.43}$$

which was considered with an additional error tolerance strategy in [34].

4 Numerical example

We discuss an example to illustrate Theorem 3.1.

Example 4.1 Let $\mathcal{H} = \mathbb{R}$. Let $\mathcal{C} = (-\infty, +\infty)$, the mappings $h : \mathcal{H} \to \mathcal{H}$ be defined by $h(u) = 3u - 2$, $\forall u \in \mathcal{C}$; and $U, V : \mathcal{C} \to \mathcal{C}$ be defined by $Uu = \frac{u+4}{7}$, $Vu = \frac{u+6}{10}$, $\forall u \in \mathcal{C}$, respectively. Setting $\{\alpha_k\} = 0.8$, $\{\eta_k\} = 0.4$ and $\{\sigma_k\} = \{\frac{1}{1000} + \frac{0.9}{k^2}\}$, $\forall k \geq 1$. Then there are unique sequences $\{u_k\}$, $\{v_k\}$ and $\{w_k\}$ obtained by scheme (3.1) converging to $\bar{u} = \frac{2}{3} \in \Gamma$.

Proof Since h is Lipschitz continuous with $L = 3$ and monotone and hence $\mu \in (0, \frac{1}{3})$, we take $\mu = \frac{1}{4}$. Observe that the mappings U, V are nonexpansive with $\mathrm{F}(U) = \{\frac{2}{3}\}$, $\mathrm{F}(V) = \{\frac{2}{3}\}$, and hence $\Phi = \mathrm{Sol(H\text{-}FPP(1.1))} = \{\frac{2}{3}\}$. One can also obtain $\mathrm{Sol(VI(1.5))} = \{\frac{2}{3}\}$. Hence, $\Gamma = \mathrm{Sol(VI(1.5))} \cap \Phi \cap \mathrm{F}(S) = \{\frac{2}{3}\} \neq \emptyset$. Furthermore, scheme (3.1) reduces to the following scheme: Given initial values u_0, u_1,

$$\left.\begin{aligned} t_k &= u_k + \eta_k(u_k - u_{k-1}),\\ v_k &= \mathcal{P}_{\mathcal{C}}(t_k - \mu h(t_k)) = \begin{cases} 0, & \text{if } u < 0,\\ 1, & \text{if } u > 1,\\ \frac{1}{4}t_k + \frac{1}{2}, & \text{otherwise,} \end{cases}\\ w_k &= \mathcal{P}_{\mathcal{C}}(t_k - \mu h(v_k)) = \begin{cases} t_k + \frac{1}{2}, & \text{if } u < 0,\\ t_k + \frac{1}{4}, & \text{if } u > 1,\\ t_k - \frac{1}{4}(3y_k - 2), & \text{otherwise,} \end{cases}\\ u_{k+1} &= (1-\alpha_k)t_k + \alpha_k\Big(\sigma_k \frac{w_k+6}{10} + (1-\sigma_k)\frac{w_k+7}{4}\Big). \end{aligned}\right\} \tag{4.1}$$

Finally, using MATLAB, we have Fig. 1 and Table 1, which show that $\{u_k\}$, $\{v_k\}$ and $\{w_k\}$ converge to $\bar{u} = \frac{2}{3}$ as $k \to +\infty$. □

Figure 1 Convergence of $\{u_k\}$, $\{v_k\}$ and $\{w_k\}$ when $u_0 = 1, u_1 = 2$

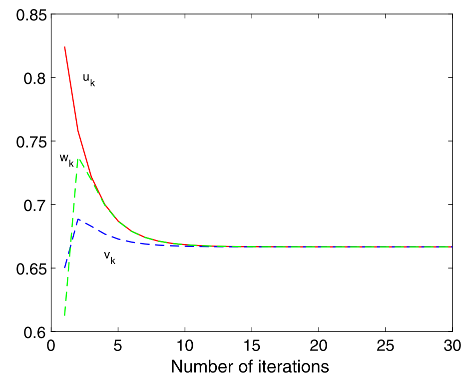

Table 1 Values of u_k, v_k and w_k

No. of iterations	u_k ($u_0 = 1, u_1 = 2$)	v_k	w_k
1	0.824408	0.650000	0.612500
2	0.758075	0.688543	0.737764
3	0.721779	0.682885	0.719378
4	0.700135	0.676815	0.699649
5	0.687021	0.672869	0.686825
6	0.679051	0.670444	0.678942
7	0.674203	0.668966	0.674138
8	0.671253	0.668066	0.671214
9	0.669458	0.667518	0.669434
10	0.668366	0.667185	0.668351
11	0.667701	0.666982	0.667692
12	0.667296	0.666859	0.667291
13	0.667050	0.666784	0.667047
14	0.666900	0.666738	0.666898
15	0.666809	0.666710	0.666807
20	0.666679	0.666670	0.666678
25	0.666668	0.666667	0.666668
29	0.666667	0.666667	0.666667
30	0.666667	0.666667	0.666667

Concluding remark 4.1 In this paper, we considered a variational inequality problem (VI) and a hierarchical fixed point problem (H-FPP) in Hilbert space. We proposed an inertial version of Krasnoselski–Mann (KM)-type extragradient scheme (3.1) by combining the KM-type scheme (1.3) and an inertial version of the extragradient scheme (1.8) to approximate a common solution of H-FPP(1.1) and VI(1.5). Furthermore, we proved a weak convergence theorem for the proposed scheme (3.1). Finally, we discussed an example to illustrate Theorem 3.1.

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