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On a degenerate parabolic equation with Newtonian fluid~non-Newtonian fluid mixed type

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Abstract

We study the existence of weak solutions to a Newtonian fluid~non-Newtonian fluid mixed-type equation

$$u_t = \operatorname{div}(b(x, t) |\nabla A(u)|^{p(x)-2} \nabla A(u) + \alpha(x, t) \nabla A(u)) + f(u, x, t).$$

We assume that $A'(s) = a(s) \geq 0$, $A(s)$ is a strictly increasing function, $A(0) = 0$, $b(x, t) \geq 0$, and $\alpha(x, t) \geq 0$. If

$$b(x, t) = \alpha(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T],$$

then we prove the stability of weak solutions without the boundary value condition.

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Keywords: Newtonian fluid~non-Newtonian fluid mixed-type equation; The existence; Stability; Boundary value condition

1 Introduction

Consider the nonlinear parabolic equations related to the $p(x)$ -Laplacian

$$u_t = \operatorname{div}(b(x, t) |\nabla A(u)|^{p(x)-2} \nabla A(u) + \alpha(x, t) \nabla A(u)) + f(x, t, u), \quad (x, t) \in Q_T, \quad (1.1)$$

where $Q_T = \Omega \times (0, T)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, the variable exponent $p(x) > 1$ is a $C(\overline{\Omega})$ function, and $\alpha(x, t)$ and $b(x, t) \in C^1(\overline{Q_T})$ are nonnegative,

$$b(x, t) > 0, \quad (x, t) \in \Omega \times [0, T]; \quad b(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \quad (1.2)$$

$A'(s) = a(s) \geq 0$, $a(s) \in C(\mathbb{R})$, and $A(0) = 0$. The evolutionary $p(x)$ -Laplacian equation is a new and interesting topic in this century. Since it is with variable exponent, many mathematical difficulties arise; we refer to [3, 5, 8, 19, 34] for details. If $A(s) = s$, $\alpha(x, t) = 0$,

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and $b(x, t) = 1$, then equation (1.1) becomes the so-called electrorheological fluid equation [1, 19]

$$u_t = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) + f(x, t, u), \quad (x, t) \in Q_T, \quad (1.3)$$

which also has many other important applications, for example, image processing [8] and elasticity [34].

For $p(x)$ satisfying the logarithmic Hölder continuity condition, Antontsev–Shmarev [2] established the existence and uniqueness results of equation (1.3) with usual initial boundary value conditions

$$u(x, t) = u_0(x), \quad x \in \Omega, \quad (1.4)$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \quad (1.5)$$

when $u_0(x) \in L^\infty(\Omega)$. Bendahmane et al. [7] studied the well-posedness (existence and uniqueness) of a renormalized solution to equation (1.3) with $L^1(\Omega)$ -data. Since then, there were many papers on the solvability and regularity of the equation related to equation (1.3); see [4, 6, 14, 16, 21, 22, 33]. By adopting a method of difference in time Liang et al. [16] studied the well-posedness of solutions to equation (1.3) without the logarithmic Hölder continuity condition, provided that f satisfies some other restrictions.

If $\alpha(x) = 0$, $A(s) = s$, and $b(x, t) = b(x)$ satisfies

$$b(x) > 0, \quad x \in \Omega; \quad b(x) = 0, \quad x \in \partial\Omega, \quad (1.6)$$

then equation (1.1) becomes

$$u_t = \operatorname{div}(b(x)|\nabla u|^{p(x)-2} \nabla u) + f(x, t, u), \quad (x, t) \in Q_T. \quad (1.7)$$

This equation was first studied in [23, 27, 32], where some achievements were made; among them, the most important discovery is that the degeneracy of $b(x)$ in (1.6) can be replaced by the Dirichlet boundary value condition (1.5). Recently, Liu and Dong [17] considered the initial boundary value problem of the equation

$$u_t = \operatorname{div}(a|\nabla u^m|^{p(x)-2} \nabla u^m + g(x, t) \nabla u^m) + u^{q(x,t)}, \quad (x, t) \in Q_T, \quad (1.8)$$

where $a > 0$ is a constant, $g(x, t) > 0$ is the convection function satisfying the Carathéodory condition, $m \in (0, 1)$, and $p(x, t)$ and $q(x, t)$ satisfy the logarithmic Hölder continuity condition. Firstly, they proved the existence of weak solutions and obtained suitable energy estimate of solutions in anisotropic Orlicz–Sobolev spaces. Secondly, by applying the energy functional method and the convexity method they showed blowup criteria of solutions. Thirdly, they studied the extinction or nonextinction of solutions by using energy inequalities and comparison principle of ordinary differential equations. Fourthly, they showed some results on global solutions without assumptions on initial data. Moreover, they gave some asymptotic estimates of blowup and extinction solutions.

Certainly, equation (1.1) also can be regarded as a generalization of the following polytropic infiltration equation:

$$u_t = \operatorname{div}(a(x)|\nabla u^m|^{p-2}\nabla u^m) + f(x, t, u, \nabla u), \quad (x, t) \in Q_T, \quad (1.9)$$

where $m > 0$; if $p > 1 + \frac{1}{m}$, then it is called the slow diffusion case, whereas if $p < 1 + \frac{1}{m}$, then it is called the fast diffusion case. There are a great deal of papers devoted to various subjects such as the well-posedness problem, the Harnack inequality, the extinction, positivity, and blowup of solutions, and the large-time behavior of solutions to equation (1.9); we refer to [9, 11, 12, 15, 18, 20, 24–26, 28–30, 35, 36].

In addition, when $A(s) = s$ and $b(x, t) \equiv 0$, equation (1.1) becomes the heat conduction equation (it is also called the Newtonian fluid equation). When $\alpha(x, t) \equiv 0$, it is the electrorheological fluid equation (it is also called the smart non-Newtonian fluid equation when $p(x, t) = p > 1$ is a constant). Thus we can say that equation (1.1) is a Newtonian fluid~non-Newtonian fluid mixed-type equation. Obviously, as we have mentioned before, since $A(s)$ may be a nonlinear function, equation (1.1) has a broader sense. In this paper, we study the existence and uniqueness of weak solutions to equation (1.1).

2 Basic functional spaces and the definition of weak solution

To make the paper sufficiently self-contained and present our discussions in a straightforward manner, let us briefly recall some preliminary results on properties of variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ and variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$ [10, 13].

Set

$$C_+(\bar{\Omega}) = \left\{ h \in C(\bar{\Omega}) : \min_{x \in \bar{\Omega}} h(x) > 1 \right\}.$$

For any $h \in C_+(\bar{\Omega})$, set

$$h^+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h^- = \inf_{x \in \Omega} h(x).$$

For any $p \in C_+(\bar{\Omega})$, let $L^{p(x)}(\Omega)$ be the set of measurable real-valued functions $u(x)$ satisfying

$$\int_{\Omega} |u(x)|^{p(x)} dx < \infty$$

and endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Define

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

endowed with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

Let $W_0^{1,p(x)}(\Omega)$ be the closure space of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Different from the usual Sobolev space $W^{1,p}(\Omega)$, a very important property of the function spaces with variable exponents was found by Zhikov [34], who showed that

$$W_0^{1,p(x)}(\Omega) \neq \{v \in W_0^{1,p(x)}(\Omega) : v|_{\partial\Omega} = 0\} = \mathring{W}^{1,p(x)}(\Omega).$$

However, if the exponent $p(x)$ satisfies the logarithmic Hölder continuity condition

$$|p(x) - p(y)| \leq \omega(|x - y|)$$

for all $x, y \in Q_T$ such that $|x - y| < \frac{1}{2}$ with

$$\overline{\lim}_{s \rightarrow 0^+} \omega(s) \ln\left(\frac{1}{s}\right) = C < \infty,$$

then (see [21])

$$W_0^{1,p(x)}(\Omega) = \mathring{W}^{1,p(x)}(\Omega).$$

From [10, 13] we have the following:

Lemma 2.1 (i) The spaces $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$, $(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)})$, and $W_0^{1,p(x)}(\Omega)$ are reflexive Banach spaces.

(ii) ($p(x)$ -Hölder's inequality) Let $p_1(x)$ and $p_2(x)$ be real functions satisfying $\frac{1}{p_1(x)} + \frac{1}{p_2(x)} = 1$. Then the conjugate space of $L^{p_1(x)}(\Omega)$ is $L^{p_2(x)}(\Omega)$. For any $u \in L^{p_1(x)}(\Omega)$ and $v \in L^{p_2(x)}(\Omega)$, we have

$$\left| \int_{\Omega} uv \, dx \right| \leq 2 \|u\|_{L^{p_1(x)}(\Omega)} \|v\|_{L^{p_2(x)}(\Omega)}.$$

(iii) We have that

$$\text{if } \|u\|_{L^{p(x)}(\Omega)} = 1, \quad \text{then } \int_{\Omega} |u|^{p(x)} \, dx = 1;$$

$$\text{if } \|u\|_{L^{p(x)}(\Omega)} > 1, \quad \text{then } \|u\|_{L^{p(x)}(\Omega)}^{p^-} \leq \int_{\Omega} |u|^{p(x)} \, dx \leq \|u\|_{L^{p(x)}(\Omega)}^{p^+}; \quad \text{and}$$

$$\text{if } \|u\|_{L^{p(x)}(\Omega)} < 1, \quad \text{then } \|u\|_{L^{p(x)}(\Omega)}^{p^+} \leq \int_{\Omega} |u|^{p(x)} \, dx \leq \|u\|_{L^{p(x)}(\Omega)}^{p^-}.$$

(iv) If $p_1(x) \leq p_2(x)$, then

$$L^{p_1(x)}(\Omega) \supset L^{p_2(x)}(\Omega).$$

(v) If $p_1(x) \leq p_2(x)$, then

$$W^{1,p_2(x)}(\Omega) \hookrightarrow W^{1,p_1(x)}(\Omega).$$

(vi) $p(x)$ -Poincaré's inequality. If $p(x) \in C(\Omega)$, then there is a constant $c_0 > 0$ such that

$$\|u\|_{L^{p(x)}(\Omega)} \leq c_0 \|\nabla u\|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega),$$

which implies that $\|\nabla u\|_{L^{p(x)}(\Omega)}$ and $\|u\|_{W_0^{1,p(x)}(\Omega)}$ are equivalent norms of $W_0^{1,p(x)}(\Omega)$.

Definition 2.2 A function $u(x, t)$ is said to be a weak solution of equation (1.1) with initial condition (1.5) if $u \in L^\infty(Q_T)$,

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^u \sqrt{a(s)} ds &\in L^2(Q_T), \\ b(x, t) |\nabla A(u)|^{p(x)} &\in L^1(Q_T), \\ \alpha(x, t) |\nabla A(u)|^2 &\in L^1(Q_T), \end{aligned} \tag{2.1}$$

and for any function $\varphi \in C_0^1(Q_T)$, we have the following integral equivalence:

$$\begin{aligned} &\iint_{Q_T} \left[\frac{\partial u}{\partial t} \varphi(x, t) + b(x, t) |\nabla A(u)|^{p(x)-2} \nabla A(u) \cdot \nabla \varphi + \alpha(x, t) \nabla A(u) \nabla \varphi \right] dx dt \\ &= \iint_{Q_T} f(x, t, u) \varphi(x, t) dx dt. \end{aligned} \tag{2.2}$$

The initial condition (1.5) is satisfied in the sense of

$$\lim_{t \rightarrow 0} \int_{\Omega} \left| \int_0^{u(x,t)} \sqrt{a(s)} - \int_0^{u_0(x)} \sqrt{a(s)} ds \right| dx = 0. \tag{2.3}$$

In our paper, we first study the existence of a weak solution.

Theorem 2.3 If $p^- \geq 2$, $A(s)$ is a strictly increasing continuous function, $A(0) = 0$, $b(x, t)$ satisfies (1.2) and

$$\left| \frac{\partial b(x, t)}{\partial t} \right| \leq cb(x, t), \tag{2.4}$$

$$\frac{\partial \alpha(x, t)}{\partial t} \geq 0, \tag{2.5}$$

$\alpha(x, t)$ satisfies $\alpha(x, t)^{p(x)} \leq cb(x, t)$, $u_0(x) \geq 0$, and

$$u_0 \in L^\infty(\Omega), \quad b(x, 0)u_0(x) \in W^{1,p(x)}(\Omega), \tag{2.6}$$

then equation (1.1) with initial value (1.5) has a solution.

Theorem 2.4 Suppose $b(x, t)$ satisfies (1.2), $A(s)$ is a strictly increasing function, $A(0) = 0$,

$$\nabla \alpha(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \quad (2.7)$$

and for large enough n ,

$$n^{1-\frac{1}{p^+}} \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} |\nabla b|^{p(x)} dx \right)^{\frac{1}{p^+}} \leq c(T); \quad (2.8)$$

Let $u(x, t)$ and $v(x, t)$ be two solutions of equation (1.1) with the initial values $u_0(x)$ and $v_0(x)$, respectively. Then

$$\int_{\Omega} |u(x, t) - v(x, t)| dx \leq c \int_{\Omega} |u(x, 0) - v(x, 0)| dx. \quad (2.9)$$

In this paper, ∇b represents the gradient of the spatial variable x , and for any $t \in [0, T]$,

$$\Omega_{\frac{1}{n}t} = \left\{ x \in \Omega : b(x, t) > \frac{1}{n} \right\}.$$

3 Proof of Theorem 2.3

Without loss the generality, we assume that $A(s)$ is a strictly increasing C^1 function and $A'(s) = a(s) \geq 0$. Consider the parabolically regularized system

$$\begin{aligned} u_t &= \operatorname{div}((b(x, t) + \varepsilon) |\nabla A(u)|^{p(x)-2} \nabla A(u) \\ &\quad + (\alpha(x, t) + \varepsilon) \nabla A(u)) + f(x, t, u), \quad (x, t) \in Q_T, \end{aligned} \quad (3.1)$$

$$u(x, 0) = u_0(x) + \varepsilon, \quad x \in \Omega, \quad (3.2)$$

$$u(x, t) = \varepsilon, \quad (x, t) \in \partial\Omega \times (0, T). \quad (3.3)$$

Proof of Theorem 2.3 Similarly to [31], by the monotone convergence method we are able to prove that the solution u_ε of the initial-boundary value problem (3.1)–(3.3), $u_{\varepsilon t} \in L^2(Q_T)$, $A(u_\varepsilon) \in L^\infty(0, T; W^{1,p(x)}(\Omega)) \cap L^\infty(0, T; W^{1,2}(\Omega))$, and

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leq c. \quad (3.4)$$

Also, we can obtain the existence of weak solutions in another sense, for example, $u_{\varepsilon t} \in W'(Q_T)$ in [17, 23, 27, 32], where $W(Q_T)$ is a specified reflexive Banach space, and $W'(Q_T)$ is its dual space.

Multiplying (3.1) by $A(u_\varepsilon) - A(\varepsilon)$, integrating over $Q_t = \Omega \times (0, t)$ for any $t \in [0, T]$, and denoting

$$\int_0^r A(s) ds = \mathbb{A}(r),$$

we have

$$\begin{aligned}
& \int_{\Omega} \mathbb{A}(u_{\varepsilon}(x, t)) dx + \iint_{Q_t} (b(x, t) + \varepsilon) |\nabla A(u_{\varepsilon})|^{p(x)} dx dt \\
& + \iint_{Q_t} (\alpha(x, t) + \varepsilon) |\nabla A(u_{\varepsilon})|^2 dx dt \\
& = \int_{\Omega} \mathbb{A}(u_0(x)) dx + A(\varepsilon) \int_{\Omega} [u(x, t) - u_0(x)] dx \\
& + \iint_{Q_t} f(x, t, u_{\varepsilon}) [A(u_{\varepsilon}) - A(\varepsilon)] dx dt,
\end{aligned} \tag{3.5}$$

which implies

$$\begin{aligned}
& \iint_{Q_t} b(x, t) |\nabla A(u_{\varepsilon})|^{p(x)} dx dt \\
& \leq c \iint_{Q_t} (b(x, t) + \varepsilon) |\nabla A(u_{\varepsilon})|^{p(x)} dx dt \\
& \leq c,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
& \iint_{Q_t} \alpha(x, t) |\nabla A(u_{\varepsilon})|^2 dx dt \\
& \leq c \iint_{Q_t} (b(x, t) + \varepsilon) |\nabla A(u_{\varepsilon})|^2 dx dt \\
& \leq c.
\end{aligned} \tag{3.7}$$

If $p^- \geq 2$, then since $\alpha(x, t)^{p(x)} \leq cb(x, t)$, by the Young inequality we have

$$\begin{aligned}
& \iint_{Q_t} \alpha(x, t) |\nabla A(u_{\varepsilon})|^2 dx dt \\
& \leq c \iint_{Q_t} [1 + \alpha(x, t)^{p(x)}] |\nabla A(u_{\varepsilon})|^{p(x)} dx dt \\
& \leq c \iint_{Q_t} [1 + b(x, t)] |\nabla A(u_{\varepsilon})|^{p(x)} dx dt \\
& \leq c.
\end{aligned} \tag{3.8}$$

Multiplying (3.1) by $[A(u_{\varepsilon}) - A(\varepsilon)]_t$ and integrating over $Q_t = \Omega \times (0, t)$, we have

$$\begin{aligned}
& \iint_{Q_t} (A(u_{\varepsilon}))_t u_{\varepsilon t} dx dt + \iint_{Q_t} (b(x, t) + \varepsilon) |\nabla A(u_{\varepsilon})|^{p(x)-2} \nabla A(u_{\varepsilon}) \nabla (A(u_{\varepsilon}))_t dx dt \\
& + \iint_{Q_t} (\alpha(x, t) + \varepsilon) \nabla A(u_{\varepsilon}) \nabla (A(u_{\varepsilon}))_t dx dt \\
& = \iint_{Q_t} f(x, t, u_{\varepsilon}) [A(u_{\varepsilon}) - A(\varepsilon)]_t dx dt.
\end{aligned} \tag{3.9}$$

Hence

$$|\nabla A(u_{\varepsilon})|^{p(x)-2} \nabla (A(u_{\varepsilon}))_t = \frac{1}{2} \int_0^{|\nabla A(u_{\varepsilon})|^2} s^{\frac{p(x)-2}{2}} ds,$$

and $|\frac{\partial b(x,t)}{\partial t}| \leq cb(x,t)$

$$\begin{aligned}
& \iint_{Q_t} (b(x,t) + \varepsilon) |\nabla A(u_\varepsilon)|^{p(x)-2} \nabla(A(u_\varepsilon))_t dx dt \\
&= \frac{1}{2} \iint_{Q_t} \frac{\partial}{\partial t} \left[(b(x,t) + \varepsilon) \int_0^{|\nabla A(u_\varepsilon)|^2} s^{\frac{p(x)-2}{2}} ds \right] dx dt \\
&\quad - \frac{1}{2} \iint_{Q_t} \int_0^{|\nabla A(u_\varepsilon)|^2} s^{\frac{p(x)-2}{2}} ds \frac{\partial b(x,t)}{\partial t} dx dt \\
&= \frac{1}{2} \int_{\Omega} \frac{2}{p(x)} [(b(x,t) + \varepsilon) |\nabla A(u_\varepsilon)|^{p(x)} - (b(x,0) + \varepsilon) |\nabla A(u_0)|^{p(x)}] dx \\
&\quad + \frac{c}{2} \iint_{Q_t} \frac{2}{p(x)} b(x,t) |\nabla A(u_\varepsilon)|^{p(x)} dx dt \\
&\leq c.
\end{aligned} \tag{3.10}$$

Again, since $\frac{\partial \alpha(x,t)}{\partial t} \geq 0$,

$$\begin{aligned}
& \iint_{Q_t} (\alpha(x,t) + \varepsilon) \nabla A(u_\varepsilon) (\nabla(A(u_\varepsilon))_t) dx dt \\
&= \frac{1}{2} \iint_{Q_t} \frac{\partial}{\partial t} [(\alpha(x,t) + \varepsilon) |\nabla A(u_\varepsilon)|^2] dx dt \\
&\quad - \frac{1}{2} \iint_{Q_t} \frac{\partial}{\partial t} (\alpha(x,t) + \varepsilon) |\nabla A(u_\varepsilon)|^2 dx dt \\
&= \frac{1}{2} \int_{\Omega} |\nabla A(u_\varepsilon)|^2 (\alpha(x,t) + \varepsilon) dx - \frac{1}{2} \int_{\Omega} |\nabla A(u_\varepsilon)|^2 (\alpha(x,0) + \varepsilon) dx \\
&\quad - \frac{1}{2} \iint_{Q_t} \frac{\partial}{\partial t} (\alpha(x,t) + \varepsilon) |\nabla A(u_\varepsilon)|^2 dx dt \\
&\leq c - \frac{1}{2} \iint_{Q_t} \frac{\partial \alpha(x,t)}{\partial t} |\nabla A(u_\varepsilon)|^2 dx dt \\
&\leq c.
\end{aligned} \tag{3.11}$$

Once more,

$$\begin{aligned}
& \left| \iint_{Q_t} f(x,t,u_\varepsilon) [A(u_\varepsilon) - A(\varepsilon)]_t dx dt \right| \\
&\leq \iint_{Q_t} |f(x,t,u_\varepsilon)(A(u_\varepsilon))_t| dx dt \\
&\leq \iint_{Q_t} \left[\frac{1}{2} |f(x,t,u_\varepsilon)\sqrt{a(u_\varepsilon)}|^2 + \frac{1}{2} a(u_\varepsilon) |u_{et}|^2 \right] dx dt \\
&\leq c + \frac{1}{2} \iint_{Q_t} a(u_\varepsilon) |u_{et}|^2 dx dt.
\end{aligned} \tag{3.12}$$

Thus from (3.9)–(3.12) we deduce that

$$\iint_{Q_t} (A(u_\varepsilon))_t u_{et} dx dt = \iint_{Q_t} a(u_\varepsilon) |u_{et}|^2 dx dt \leq c, \tag{3.13}$$

which implies that

$$\iint_{Q_t} |(A(u_\varepsilon))_t|^2 dx dt = \iint_{Q_t} a(u_\varepsilon)^2 |u_{\varepsilon t}|^2 dx dt \leq c \iint_{Q_t} a(u_\varepsilon) |u_{\varepsilon t}|^2 dx dt \leq c. \quad (3.14)$$

By (3.6), $u_\varepsilon \rightharpoonup u$ weakly star in $L^\infty(Q_T)$. For any $\varphi(x, t) \in C_0^1(Q_T)$, we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \iint_{Q_T} \frac{\partial}{\partial t} \left(\int_0^{u_\varepsilon} \sqrt{a(s)} ds - \int_0^u \sqrt{a(s)} ds \right) dx dt \\ &= - \lim_{\varepsilon \rightarrow 0} \iint_{Q_T} \int_u^{u_\varepsilon} \sqrt{a(s)} ds \varphi_t(x, t) dx dt \\ &= - \lim_{\varepsilon \rightarrow 0} \iint_{Q_T} \sqrt{a(\xi)} (u - u_\varepsilon) \varphi_t(x, t) dx dt \\ &= 0, \end{aligned} \quad (3.15)$$

from which we can extrapolate that

$$\frac{\partial}{\partial t} \int_0^{u_\varepsilon} \sqrt{a(s)} ds \rightharpoonup \frac{\partial}{\partial t} \int_0^u \sqrt{a(s)} ds, \quad \text{in } L^2(Q_T). \quad (3.16)$$

At the same time, for any $\varphi(x, t) \in C_0^1(Q_T)$, we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \iint_{Q_T} \alpha(x, t) [\nabla A(u_\varepsilon) - \nabla A(u)] \varphi(x, t) dx dt \\ &= - \lim_{\varepsilon \rightarrow 0} \iint_{Q_T} [A(u_\varepsilon) - A(u)] [\nabla \alpha(x, t) \varphi(x, t) + \alpha(x, t) \nabla \varphi(x, t)] dx dt \\ &= - \lim_{\varepsilon \rightarrow 0} \iint_{Q_T} \int_u^{u_\varepsilon} a(s) ds [\nabla \alpha(x, t) \varphi(x, t) + \alpha(x, t) \nabla \varphi(x, t)] dx dt \\ &= - \lim_{\varepsilon \rightarrow 0} \iint_{Q_T} a(\xi) (u_\varepsilon - u) [\nabla \alpha(x, t) \varphi(x, t) + \alpha(x, t) \nabla \varphi(x, t)] dx dt \\ &= 0. \end{aligned} \quad (3.17)$$

Thus

$$\alpha(x, t) \nabla A(u_\varepsilon) \rightharpoonup \alpha(x, t) \nabla A(u), \quad \text{in } L^1(Q_T). \quad (3.18)$$

Moreover, by (3.6), since $b(x, t) > 0$ for $x \in \Omega$, for any compact $\Omega_1 \subset \Omega$, we have

$$\int_0^T \int_{\Omega_1} |\nabla A(u_\varepsilon)|^{p(x)} dx dt \leq c. \quad (3.19)$$

Combining this with (3.14), since $A(s)$ is a strictly increasing function, we get that

$$A(u_\varepsilon) = \int_0^{u_\varepsilon} a(s) ds \rightarrow \int_0^u a(s) ds = A(u), \quad \text{in } L^1(0, T : L^{r(x)}(\Omega_1)),$$

which implies that $A(u_\varepsilon) \rightarrow A(u)$ a.e. in $\Omega_1 \times [0, T]$. By the arbitrariness of Ω_1 we extrapolate that $A(u_\varepsilon) \rightarrow A(u)$ a.e. in $Q_T = \Omega \times (0, T)$. So $u_\varepsilon \rightarrow u$ a.e. in $Q_T = \Omega \times (0, T)$.

Hence by (3.6) we easily get that there exists an n -dimensional vector $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$ such that

$$|\vec{\zeta}| \in L^{p^-}(0, T; W^{1,p(x)}(\Omega))$$

and

$$b(x, t)|A(\nabla u_\varepsilon)|^{p(x)-2}\nabla u_\varepsilon \rightharpoonup \vec{\zeta} \quad \text{in } L^{p^-}(0, T; W^{1,p(x)}(\Omega)).$$

To prove that u is the solution of equation (1.1), we notice that for any function $\varphi \in C_0^1(Q_T)$,

$$\begin{aligned} & \iint_{Q_T} [u_{\varepsilon t}\varphi + (b(x, t) + \varepsilon)|\nabla A(u_\varepsilon)|^{p(x)-2}\nabla A(u_\varepsilon) \cdot \nabla \varphi] dx dt \\ & + \iint_{Q_T} \alpha(x, t)\nabla A(u_\varepsilon)\nabla \varphi dx dt \\ & = \iint_{Q_T} f(x, t, u_\varepsilon)\varphi(x, t) dx dt. \end{aligned} \tag{3.20}$$

As $\varepsilon \rightarrow 0$, since $b(x, t)$ is a $C^1(\overline{Q_T})$ function with $b(x, t)|_{\partial\Omega \times [0, T]} = 0$ and $b(x, t) > 0$, $(x, t) \in \Omega \times [0, T]$, we have $c > \max_{\text{supp } \varphi} \frac{|\nabla \varphi|}{b(x, t)} > 0$ by $\varphi \in C_0^\infty(Q_T)$, and, accordingly,

$$\begin{aligned} & \varepsilon \left| \iint_{Q_T} |\nabla A(u_\varepsilon)|^{p(x)-2} A(\nabla u_\varepsilon) \cdot \nabla \varphi dx dt \right| \\ & \leq \varepsilon \sup_{\text{supp } \varphi} \frac{|\nabla \varphi|}{b(x, t)} \iint_{Q_T} b(x, t) (|\nabla A(u_\varepsilon)|^{p(x)} + c) dx dt \\ & \rightarrow 0, \end{aligned}$$

Since $p(x) > 1$, by the Young inequality,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon \left| \iint_{Q_T} A(\nabla u_\varepsilon) \cdot \nabla \varphi dx dt \right| \\ & \leq \varepsilon \sup_{\text{supp } \varphi} \frac{|\nabla \varphi|}{b(x, t)} \iint_{Q_T} b(x, t) (|\nabla A(u_\varepsilon)|^{p(x)} + c) dx dt \\ & = 0, \end{aligned}$$

and thus

$$\begin{aligned} & \iint_{Q_T} \vec{\zeta} \cdot \nabla \varphi dx dt = \lim_{\varepsilon \rightarrow 0} \iint_{Q_T} b(x, t) |\nabla A(u_\varepsilon)|^{p(x)-2} \nabla A(u_\varepsilon) \cdot \nabla \varphi dx dt \\ & = \lim_{\varepsilon \rightarrow 0} \iint_{Q_T} (b(x, t) + \varepsilon) |\nabla A(u_\varepsilon)|^{p(x)-2} \nabla A(u_\varepsilon) \cdot \nabla \varphi dx dt \\ & - \lim_{\varepsilon \rightarrow 0} \varepsilon \iint_{Q_T} |\nabla A(u_\varepsilon)|^{p(x)-2} \nabla A(u_\varepsilon) \cdot \nabla \varphi dx dt \\ & = \lim_{\varepsilon \rightarrow 0} \iint_{Q_T} (b(x, t) + \varepsilon) |\nabla A(u_\varepsilon)|^{p(x)-2} \nabla A(u_\varepsilon) \cdot \nabla \varphi dx dt. \end{aligned}$$

Now, for any function $\varphi \in C_0^1(Q_T)$,

$$\iint_{Q_T} (u\varphi_t + \vec{\zeta} \cdot \nabla \varphi + \alpha(x, t)\nabla A(u) \cdot \nabla \varphi) dx dt = \iint_{Q_T} f(x, t, u)\varphi dx dt. \quad (3.21)$$

We will prove that

$$\iint_{Q_T} b(x, t)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla \varphi dx dt = \iint_{Q_T} \vec{\zeta} \cdot \nabla \varphi dx dt. \quad (3.22)$$

We choose $0 \leq \psi \in C_0^\infty(Q_T)$ and $\psi = 1$ in $\text{supp } \varphi$. Let $v \in L^\infty(Q_T)$, $b(x, t)|\nabla A(v)|^{p(x)} \in L^1(Q_T)$. Then

$$\begin{aligned} & \iint_{Q_T} \psi(b(x, t) + \varepsilon)(|A(\nabla u_\varepsilon)|^{p(x)-2}\nabla A(u_\varepsilon) - |\nabla A(v)|^{p(x)-2}\nabla A(v)) \\ & \quad \cdot (\nabla A(u_\varepsilon) - \nabla A(v)) dx dt \\ & \geq 0, \end{aligned} \quad (3.23)$$

$$\iint_{Q_T} \psi(\alpha(x, t) + \varepsilon)|\nabla A(u_\varepsilon) - \nabla A(v)|^2 dx dt \geq 0. \quad (3.24)$$

Let $\varphi = \psi A(u_\varepsilon)$ in (3.20). Then

$$\begin{aligned} & \iint_{Q_T} \psi(b(x, t) + \varepsilon)|\nabla A(u_\varepsilon)|^{p(x)} dx dt + \iint_{Q_T} \psi(\alpha(x, t) + \varepsilon)|\nabla A(u_\varepsilon)|^2 dx dt \\ & = \iint_{Q_T} \psi_t \mathbb{A}(u_\varepsilon) dx dt \\ & \quad - \iint_{Q_T} (b(x, t) + \varepsilon)A(u_\varepsilon)|\nabla A(u_\varepsilon)|^{p(x)-2}\nabla A(u_\varepsilon)\nabla \psi dx dt \\ & \quad - \iint_{Q_T} (\alpha(x, t) + \varepsilon)A(u_\varepsilon)\nabla A(u_\varepsilon)\nabla \psi dx dt \\ & \quad + \iint_{Q_T} f(x, t, u_\varepsilon)\psi A(u_\varepsilon) dx dt. \end{aligned} \quad (3.25)$$

Accordingly,

$$\begin{aligned} & \iint_{Q_T} \psi_t \mathbb{A}(u_\varepsilon) dx dt - \iint_{Q_T} (b(x, t) + \varepsilon)A(u_\varepsilon)|\nabla A(u_\varepsilon)|^{p(x)-2}\nabla A(u_\varepsilon) \cdot \nabla \psi dx dt \\ & \quad - \iint_{Q_T} (\alpha(x, t) + \varepsilon)A(u_\varepsilon)\nabla A(u_\varepsilon)\nabla \psi dx dt \\ & \quad - \iint_{Q_T} (b(x, t) + \varepsilon)\psi|\nabla A(u_\varepsilon)|^{p(x)-2}\nabla A(u_\varepsilon)\nabla A(v) dx dt \\ & \quad - \iint_{Q_T} (\alpha(x, t) + \varepsilon)\psi\nabla A(u_\varepsilon)\nabla A(v) dx dt \\ & \quad - \iint_{Q_T} (b(x, t) + \varepsilon)\psi|\nabla A(v)|^{p(x)-2}\nabla A(v) \cdot \nabla(A(u_\varepsilon) - A(v)) dx dt \end{aligned} \quad (3.26)$$

$$\begin{aligned}
& - \iint_{Q_T} (\alpha(x, t) + \varepsilon) \psi \nabla A(\nu) \cdot \nabla (A(u_\varepsilon) - A(\nu)) dx dt \\
& + \iint_{Q_T} f(x, t, u_\varepsilon) \psi A(u_\varepsilon) dx dt \\
& \geq 0.
\end{aligned}$$

Thus

$$\begin{aligned}
& \iint_{Q_T} \psi_t \mathbb{A}(u_\varepsilon) dx dt - \iint_{Q_T} (b(x, t) + \varepsilon) A(u_\varepsilon) |\nabla A(u_\varepsilon)|^{p(x)-2} \nabla A(u_\varepsilon) \cdot \nabla \psi dx dt \\
& - \iint_{Q_T} (\alpha(x, t) + \varepsilon) A(u_\varepsilon) \nabla A(u_\varepsilon) \nabla \psi dx dt \\
& - \iint_{Q_T} (b(x, t) + \varepsilon) \psi |\nabla A(u_\varepsilon)|^{p(x)-2} \nabla A(u_\varepsilon) \nabla \nu dx dt \\
& - \iint_{Q_T} (b(x, t) + \varepsilon) \psi \nabla A(u_\varepsilon) \nabla \nu dx dt \\
& - \iint_{Q_T} \psi b(x, t) |\nabla A(\nu)|^{p(x)-2} \nabla A(\nu) \cdot (\nabla A(u_\varepsilon) - \nabla A(\nu)) dx dt \\
& - \varepsilon \iint_{Q_T} \psi |\nabla A(\nu)|^{p(x)-2} \nabla A(\nu) \cdot (\nabla A(u_\varepsilon) - \nabla A(\nu)) dx dt \\
& - \varepsilon \iint_{Q_T} \psi \nabla A(\nu) \cdot (\nabla A(u_\varepsilon) - \nabla A(\nu)) dx dt \\
& + \iint_{Q_T} f(x, t, u_\varepsilon) \psi A(u_\varepsilon) dx dt \\
& \geq 0.
\end{aligned} \tag{3.27}$$

Now we have

$$\begin{aligned}
& \varepsilon \left| \iint_{Q_T} \psi |\nabla A(\nu)|^{p(x)-2} \nabla A(\nu) \cdot (\nabla A(u_\varepsilon) - \nabla A(\nu)) dx dt \right| \\
& \leq \varepsilon \sup_{(x,t) \in Q_T} \frac{|\psi|}{b(x, t)} \iint_{Q_T} b(x, t) |\nabla A(\nu)|^{p(x)-1} |\nabla A(u_\varepsilon) - \nabla A(\nu)| dx dt \\
& \leq \varepsilon \sup_{(x,t) \in Q_T} \frac{|\psi|}{b(x, t)} \left(\iint_{Q_T} b(x, t) |\nabla A(\nu)|^{p(x)} dx dt \right. \\
& \quad \left. + \iint_{Q_T} b(x, t) |\nabla A(\nu)|^{p(x)-1} |\nabla A(u_\varepsilon)| dx dt \right),
\end{aligned} \tag{3.28}$$

which converges to 0 as $\varepsilon \rightarrow 0$. Since $p^- \geq 2$, by the Young inequality we have

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \varepsilon \left| \iint_{Q_T} \psi \nabla A(\nu) \cdot (\nabla A(u_\varepsilon) - \nabla A(\nu)) dx dt \right| \\
& \leq \lim_{\varepsilon \rightarrow 0} \varepsilon \iint_{Q_T} \frac{|\psi|}{b(x, t)} b(x, t) |\nabla A(\nu) \cdot (\nabla A(u_\varepsilon) - \nabla A(\nu))| dx dt \\
& \leq \sup_{(x,t) \in Q_T} \frac{|\psi|}{b(x, t)} \lim_{\varepsilon \rightarrow 0} \varepsilon \iint_{Q_T} b(x, t) [|\nabla A(\nu) \cdot \nabla A(u_\varepsilon)| + |\nabla A(\nu)|^2] dx dt
\end{aligned} \tag{3.29}$$

$$\begin{aligned} &\leq c \sup_{(x,t) \in Q_T} \frac{|\psi|}{b(x,t)} \lim_{\varepsilon \rightarrow 0} \varepsilon \iint_{Q_T} b(x,t) [|\nabla A(u_\varepsilon)|^{p(x)} + |\nabla A(v)|^{p(x)}] dx dt \\ &= 0. \end{aligned}$$

Thus

$$\begin{aligned} &\iint_{Q_T} \psi_t \mathbb{A}(u) dx dt - \iint_{Q_T} A(u) \vec{\zeta} \cdot \nabla \psi dx dt \\ &- \iint_{Q_T} A(u) \nabla A(u) \cdot \nabla \psi dx dt - \iint_{Q_T} \psi \vec{\zeta} \cdot \nabla A(v) dx dt \\ &- \iint_{Q_T} \psi b(x,t) |\nabla A(v)|^{p(x)-2} \nabla A(v) \cdot (\nabla A(u) - \nabla A(v)) dx dt \\ &+ \iint_{Q_T} f(x,t,u) \psi A(u) dx dt \\ &\geq 0. \end{aligned}$$

Let $\varphi = \psi A(u)$ in (3.21). We obtain

$$\begin{aligned} &\iint_{Q_T} \psi \vec{\zeta} \cdot \nabla A(u) dx dt + \iint_{Q_T} \psi |\nabla A(u)|^2 dx dt - \iint_{Q_T} \mathbb{A}(u) \psi_t dx dt \\ &+ \iint_{Q_T} A(u) \vec{\zeta} \cdot \nabla \psi dx dt + \iint_{Q_T} A(u) \nabla A(u) \cdot \nabla \psi dx dt \\ &= \iint_{Q_T} f(x,t,u) \psi A(u) dx dt. \end{aligned} \tag{3.30}$$

Accordingly,

$$\iint_{Q_T} \psi (\vec{\zeta} - b(x,t) |\nabla A(v)|^{p(x)-2} \nabla A(v)) \cdot (\nabla A(u) - \nabla A(v)) dx dt \geq 0. \tag{3.31}$$

Let $A(v) = A(u) - \lambda \varphi$, $\lambda > 0$, $\varphi \in C_0^1(Q_T)$, or equivalently $v = A^{-1}(A(u) - \lambda \varphi)$. Then

$$\iint_{Q_T} \psi (\vec{\zeta} - b(x,t) |\nabla(A(u) - \lambda \varphi)|^{p(x)-2} \nabla(A(u) - \lambda \varphi)) \cdot \nabla \varphi dx dt \geq 0.$$

If $\lambda \rightarrow 0$, then

$$\iint_{Q_T} \psi (\vec{\zeta} - b(x,t) |\nabla A(u)|^{p(x)-2} \nabla A(u)) \cdot \nabla \varphi dx dt \geq 0.$$

Moreover, if $\lambda < 0$, then we similarly get

$$\iint_{Q_T} \psi (\vec{\zeta} - b(x,t) |\nabla A(u)|^{p(x)-2} \nabla A(u)) \cdot \nabla \varphi dx dt \leq 0.$$

Thus

$$\iint_{Q_T} \psi (\vec{\zeta} - b(x,t) |\nabla A(u)|^{p(x)-2} \nabla A(u)) \cdot \nabla \varphi dx dt = 0.$$

Since $\psi = 1$ on $\text{supp } \varphi$, (3.22) holds.

Finally, let us prove that the initial condition (1.4) is satisfied in the sense of (2.3). For any $0 \leq t_1 < t_2 < T$, by (3.13) we have

$$\begin{aligned}
& \int_{\Omega} \left| \int_0^{u_{\varepsilon}(x,t_2)} \sqrt{a(s)} - \int_0^{u_{\varepsilon}(x,t_1)} \sqrt{a(s)} ds \right| dx \\
& \leq (t_2 - t_1) \int_{\Omega} \left| \int_0^1 \frac{\partial}{\partial s} \int_0^{u_{\varepsilon}(x,st_2+(1-s)t_1)} \sqrt{a(s)} ds \right| dx \\
& \leq (t_2 - t_1) \int_{\Omega} \int_0^1 \left| \frac{\partial}{\partial s} \int_0^{u_{\varepsilon}(x,st_2+(1-s)t_1)} \sqrt{a(s)} ds \right| ds dx \\
& \leq (t_2 - t_1) \int_0^T \int_{\Omega} \left| \frac{\partial}{\partial t} \int_0^{u_{\varepsilon}(x,s)} \sqrt{a(s)} ds \right| ds dx dt \\
& \leq (t_2 - t_1) \left(\int_0^T \int_{\Omega} |\sqrt{a(u_{\varepsilon})} u_{\varepsilon t}|^2 dx dt \right)^{\frac{1}{2}} \\
& \leq c(t_2 - t_1).
\end{aligned} \tag{3.32}$$

Thus u satisfies equation (1.1) in the sense of Definition 2.2. \square

4 Stability theorem

Theorem 4.1 Suppose $b(x, t)$ satisfies (1.2), $A(s)$ is a strictly increasing function, $A(0) = 0$, $b(x, t)$ satisfies, for n large enough,

$$n^{1-\frac{1}{p^+}} \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} |\nabla b|^{p(x)} dx \right)^{\frac{1}{p^+}} \leq c(T), \tag{4.1}$$

and $\alpha(x, t)$ satisfies

$$n^{\frac{1}{2}} \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} |\nabla \alpha|^2 dx \right)^{\frac{1}{2}} \leq c(T). \tag{4.2}$$

Let $u(x, t)$ and $v(x, t)$ be two solutions of equation (1.1) with the initial values $u_0(x)$ and $v_0(x)$, respectively. Then

$$\int_{\Omega} |u(x, t) - v(x, t)| dx \leq c \int_{\Omega} |u(x, 0) - v(x, 0)| dx. \tag{4.3}$$

Proof For any positive integer n , let $S_n(s)$ be an odd function defined for $s \geq 0$ as

$$S_n(s) = \begin{cases} 1, & s > \frac{1}{n}, \\ n^2 s^2 e^{1-n^2 s^2}, & 0 \leq s \leq \frac{1}{n}, \end{cases}$$

and let

$$H_n(s) = \int_0^s S_n(s) ds.$$

Clearly,

$$\lim_{n \rightarrow \infty} S_n(s) = \operatorname{sgn}(s), \quad s \in (-\infty, +\infty).$$

Denote $\Omega_{\lambda t} = \{x \in \Omega : b(x, t) > \lambda\}$, $t \in [0, T]$, and define

$$\phi_n(x, t) = \begin{cases} 1, & x \in \Omega_{\frac{2}{n}t}, \\ n(b(x, t) - \frac{1}{n}), & x \in \Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}, \\ 0, & x \in \Omega \setminus \Omega_{\frac{1}{n}t}. \end{cases}$$

Let $u(x, t)$ and $v(x, t)$ be two weak solutions of equation (1.1) with initial values $u_0(x)$ and $v_0(x)$, respectively. We choose $\phi_n S_n(A(u) - A(v))$ as a test function. Then

$$\begin{aligned} & \int_0^t \int_{\Omega} \phi_n(x, t) S_n(A(u) - A(v)) \frac{\partial(u - v)}{\partial t} dx dt \\ & + \int_0^t \int_{\Omega} b(x, t) (|\nabla A(u)|^{p(x)-2} \nabla A(u) - |\nabla A(v)|^{p(x)-2} \nabla A(v) \\ & \cdot \nabla(A(u) - A(v)) S'_n(A(u) - A(v)) \phi_n(x, t) dx dt \\ & + \int_0^t \int_{\Omega} b(x, t) (|\nabla A(u)|^{p(x)-2} \nabla A(u) - |\nabla A(v)|^{p(x)-2} \nabla A(v)) \\ & \cdot \nabla(A(u) - A(v)) S_n(A(u) - A(v)) \nabla \phi_n(x, t) dx dt \\ & + \int_0^t \int_{\Omega} \alpha(x, t) |\nabla(A(u) - A(v))|^2 S'_n(A(u) - A(v)) \phi_n(x, t) dx dt \\ & + \int_0^t \int_{\Omega} \alpha(x, t) [\nabla A(u) - \nabla A(v)] S_n(A(u) - A(v)) \nabla \phi_n(x, t) dx dt \\ & = \int_0^t \int_{\Omega} [f(x, t, u) - f(x, t, v)] \phi_n S_n(A(u) - A(v)) dx dt. \end{aligned} \tag{4.4}$$

First, since $A(r) \geq 0$ is an increasing function, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_0^t \int_{\Omega} \phi_n(x, t) S_n(A(u) - A(v)) \frac{\partial(u - v)}{\partial t} dx dt \\ & = \int_0^t \int_{\Omega} \operatorname{sgn}(A(u) - A(v)) \frac{\partial(u - v)}{\partial t} dx dt \\ & = \int_0^t \int_{\Omega} \operatorname{sgn}(u - v) \frac{\partial(u - v)}{\partial t} dx dt \\ & = \int_{\Omega} |u(x, t) - v(x, t)| dx - \int_{\Omega} |u_0(x) - v_0(x)| dx. \end{aligned} \tag{4.5}$$

Second,

$$\begin{aligned} & \int_0^t \int_{\Omega} b(x, t) (|\nabla A(u)|^{p(x)-2} \nabla A(u) - |\nabla A(v)|^{p(x)-2} \nabla A(v)) \\ & \cdot \nabla(A(u) - A(v)) S'_n(A(u) - A(v)) \phi_n(x, t) dx dt \\ & \geq 0, \end{aligned} \tag{4.6}$$

and

$$\int_0^t \int_{\Omega} \alpha(x, t) |\nabla A(u) - \nabla A(v)|^2 S'_n(A(u) - A(v)) \phi_n(x, t) dx dt \geq 0. \tag{4.7}$$

Third, for any $t \in [0, T]$, $|\nabla \phi_n(x, t)| = \frac{1}{\lambda} \nabla b(x, t)$ for $x \in \Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}$ and equals zero otherwise. Thus by (4.1) we have

$$\begin{aligned}
& \left| \int_0^t \int_{\Omega} b(x, t) \left(|\nabla A(u)|^{p(x)-2} \nabla A(u) - |\nabla A(v)|^{p(x)-2} \nabla A(v) \right) \right. \\
& \quad \cdot \nabla \phi_n(x, t) S_n(A(u) - A(v)) dx dt \Big| \\
&= \left| \int_0^t \int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x, t) \left(|\nabla A(u)|^{p(x)-2} \nabla A(u) - |\nabla A(v)|^{p(x)-2} \nabla A(v) \right) \right. \\
& \quad \cdot \nabla \phi_n g_n(Au - Av) dx dt \Big| \\
&\leq \int_0^t n \int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x, t) |\nabla A(u)|^{p(x)-1} + |\nabla A(v)|^{p(x)-1} |\nabla b S_n(A(u) - A(v))| dx \\
&\leq c \int_0^t \left[\left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x, t) |\nabla A(u)|^{p(x)} \right)^{\frac{1}{q^+}} \right. \\
& \quad \left. + \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x, t) |\nabla A(v)|^{p(x)} \right)^{\frac{1}{q^+}} \right] dt \\
&\quad \cdot \int_0^t n \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x, t) |\nabla b(x, t)|^{p(x)} dx \right)^{\frac{1}{p^+}} dt \\
&\leq c \int_0^t \left[\left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x, t) |\nabla A(u)|^{p(x)} \right)^{\frac{1}{q^+}} \right. \\
& \quad \left. + \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x, t) |\nabla A(v)|^{p(x)} \right)^{\frac{1}{q^+}} \right] dt \\
&\quad \cdot \int_0^t n^{1-\frac{1}{p^+}} \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} |\nabla b(x, t)|^{p(x)} dx \right)^{\frac{1}{p^+}} dt \\
&\leq c \int_0^t \left[\left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x, t) |\nabla A(u)|^{p(x)} \right)^{\frac{1}{q^+}} \right. \\
& \quad \left. + \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x, t) |\nabla A(v)|^{p(x)} \right)^{\frac{1}{q^+}} \right] dt,
\end{aligned} \tag{4.8}$$

which tends to 0 as $n \rightarrow \infty$, where we denote $q(x) = \frac{p(x)}{p(x)-1}$ as usual and $q^+ = \max_{x \in \overline{\Omega}} q(x)$. Since $\alpha(x, t)$ satisfies (4.2), we have

$$\begin{aligned}
& \left| \int_0^t \int_{\Omega} \alpha(x, t) (\nabla A(u) - \nabla A(v)) \cdot \nabla \phi_n(x, t) S_n(A(u) - A(v)) dx dt \right| \\
&= \left| \int_0^t \int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) (\nabla A(u) - \nabla A(v)) \cdot \nabla \phi_n g_n(A(u) - A(v)) dx dt \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^t n \int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) |\nabla A(u)| + |\nabla A(v)| |\nabla \alpha S_n(A(u) - A(v))| dx \\
&\leq c \int_0^t \left[\left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) |\nabla A(u)|^2 \right)^{\frac{1}{2}} + \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) |\nabla A(v)|^2 \right)^{\frac{1}{2}} \right] dt \\
&\quad \cdot \int_0^t n \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) |\nabla \alpha(x, t)|^2 dx \right)^{\frac{1}{2}} dt \\
&\leq c \int_0^t \left[\left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) |\nabla A(u)|^2 \right)^{\frac{1}{2}} + \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) |\nabla A(v)|^2 \right)^{\frac{1}{2}} \right] dt \\
&\quad \cdot \int_0^t n^{\frac{1}{2}} \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} |\nabla \alpha(x, t)|^2 dx \right)^{\frac{1}{2}} dt \\
&\leq c \int_0^t \left[\left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) |\nabla A(u)|^2 \right)^{\frac{1}{2}} + \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) |\nabla A(v)|^2 \right)^{\frac{1}{2}} \right] dt,
\end{aligned} \tag{4.9}$$

which tends to 0 as $n \rightarrow \infty$.

Fourth, we have

$$\begin{aligned}
&\left| \int_0^t \int_{\Omega} [f(x, , t, u) - f(x, t, v)] \phi_n(x, t) S_n(A(u) - A(v)) dx dt \right| \\
&\leq \int_0^t \int_{\Omega} |u(x, t) - v(x, t)| dx dt.
\end{aligned} \tag{4.10}$$

Now let $n \rightarrow \infty$ in (4.4). By (4.5)–(4.9) we have

$$\begin{aligned}
&\int_{\Omega} |u(x, t) - v(x, t)| dx \\
&\leq \int_{\Omega} |u_0(x) - v_0(x)| dx + \int_0^t \int_{\Omega} |u(x, t) - v(x, t)| dx dt, \quad \forall t \in [0, T].
\end{aligned}$$

By the Gronwall inequality we have the conclusion. \square

Proof of Theorem 2.4 We only need to show (4.9) in another way. Since $\alpha(x, t)$ satisfies (2.7), that is,

$$\nabla \alpha(x, t) = 0, \quad (x, t) \in \partial \Omega \times [0, T],$$

by the definition of the trace on the boundary we have

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \left| \int_0^t \int_{\Omega} \alpha(x, t) (\nabla A(u) - \nabla A(v)) \cdot \nabla \phi_n(x, t) S_n(A(u) - A(v)) dx dt \right| \\
&= \lim_{n \rightarrow \infty} \left| \int_0^t \int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) (\nabla A(u) - \nabla A(v)) \cdot \nabla \phi_n g_n(A(u) - A(v)) dx dt \right| \\
&\leq \lim_{n \rightarrow \infty} \int_0^t n \int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) |\nabla A(u)| + |\nabla A(v)| |\nabla \alpha S_n(A(u) - A(v))| dx
\end{aligned}$$

$$\begin{aligned}
&\leq c \lim_{n \rightarrow \infty} \int_0^t \left[\left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) |\nabla A(u)|^2 \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) |\nabla A(v)|^2 \right)^{\frac{1}{2}} \right] dt \\
&\quad \cdot \int_0^t n \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) |\nabla \alpha(x, t)|^2 dx \right)^{\frac{1}{2}} dt \\
&\leq c \lim_{n \rightarrow \infty} \int_0^t \left[\left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) |\nabla A(u)|^2 \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \alpha(x, t) |\nabla A(v)|^2 \right)^{\frac{1}{2}} \right] dt \\
&\quad \cdot \int_0^t n^{\frac{1}{2}} \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} |\nabla \alpha(x, t)|^2 dx \right)^{\frac{1}{2}} dt \\
&\leq c \int_0^t \lim_{n \rightarrow \infty} \left(\frac{1}{n} \int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} |\nabla \alpha(x, t)|^2 dx \right)^{\frac{1}{2}} dt \\
&= c \int_0^t \left(\int_{\partial \Omega} |\nabla \alpha(x, t)|^2 d\Sigma \right)^{\frac{1}{2}} dt \\
&= 0.
\end{aligned} \tag{4.11}$$

The other parts of the proof of Theorem 4.1 are valid. Thus we have completed the proof of Theorem 2.4. \square

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