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Best approximation of κ-random operator inequalities in matrix MB-algebras

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Abstract

We introduce a class of stochastic matrix control functions and apply them to stabilize pseudo stochastic κ-random operator inequalities in matrix MB-algebras. We obtain an approximation for stochastic κ-random operator inequalities and calculate the maximum error of the estimate.

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1 Introduction

In this paper, we study distribution functions with the ranges in a class of matrix algebras [1–3] and introduce the concept of a matrix Menger normed algebra using the generalized triangular norm which is a generalization of an MB-algebra [4], i.e., a Menger normed space with algebraic structures [5–8]. This concept helps us to study intuitionistic spaces and their generalization, i.e., neutrosophic spaces introduced by Smarandache [9, 10]. We define a stochastic matrix control function and stabilize pseudo-stochastic κ-random operator inequalities, and this process leads to best approximation of a κ-random operator inequality.

2 Preliminaries

Let

$$\operatorname{diag} M_n\big([0,1]\big) = \left\{ \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix} = \operatorname{diag}[t_1,\ldots,t_n], t_1,\ldots,t_n \in [0,1] \right\}.$$

We denote $\mathbf{t} := \operatorname{diag}[t_1,\ldots,t_n] \preceq \mathbf{s} := \operatorname{diag}[s_1,\ldots,s_n]$ if and only if $t_i \le s_i$ for all $i = 1,\ldots,n$, also $\mathbf{1} = \operatorname{diag}[1,\ldots,1]$ and $\mathbf{0} = \operatorname{diag}[0,\ldots,0]$.

Now, we extend the concept of triangular norms [11, 12] on $\operatorname{diag} M_n([0,1])$.

Definition 2.1 A generalized triangular norm (GTN) on $\operatorname{diag} M_n([0,1])$ is an operation $\circledast : \operatorname{diag} M_n([0,1]) \times \operatorname{diag} M_n([0,1]) \to \operatorname{diag} M_n([0,1])$ satisfying the following conditions:

(a) $(\forall \mathbf{t} \in \operatorname{diag} M_n([0,1]))(\mathbf{t} \circledast \mathbf{1} = \mathbf{t})$ (boundary condition);
(b) $(\forall (\mathbf{t},\mathbf{s}) \in (\operatorname{diag} M_n([0,1]))^2)(\mathbf{t} \circledast \mathbf{s} = \mathbf{s} \circledast \mathbf{t})$ (commutativity);
(c) $(\forall (\mathbf{t},\mathbf{s},\mathbf{p}) \in (\operatorname{diag} M_n([0,1])^3)(\mathbf{t} \circledast (\mathbf{s} \circledast \mathbf{p})) = ((\mathbf{t} \circledast \mathbf{s}) \circledast \mathbf{p})$ (associativity);
(d) $(\forall (\mathbf{t},\mathbf{t}',\mathbf{s},\mathbf{s}') \in (\operatorname{diag} M_n([0,1]^4)(\mathbf{t} \preceq \mathbf{t}' \text{ and } \mathbf{s} \preceq \mathbf{s}' \Longrightarrow \mathbf{t} \circledast \mathbf{s} \preceq \mathbf{t}' \circledast \mathbf{s}')$ (monotonicity).

If for every $\mathbf{t},\mathbf{s} \in \operatorname{diag} M_n([0,1])$ and every sequences $\{\mathbf{t}_k\}$ and $\{\mathbf{s}_k\}$ converging to $\mathbf{t}$ and $\mathbf{s}$ we have

$$\lim_k (\mathbf{t}_k \circledast \mathbf{s}_k) = \mathbf{t} \circledast \mathbf{s},$$

then $\circledast$ on $\operatorname{diag} M_n([0,1])$ will be continuous (in short, CGTN). Now we present some examples of CGTN.

(1) Define $\circledast_P : \operatorname{diag} M_n([0,1]) \times \operatorname{diag} M_n([0,1]) \to \operatorname{diag} M_n([0,1])$ such that

$$\mathbf{t} \circledast_P \mathbf{s} = \operatorname{diag}[t_1,\dots,t_n] \circledast_P \operatorname{diag}[s_1,\dots,s_n] = \operatorname{diag}[t_1 \cdot s_1,\dots,t_n \cdot s_n],$$

then $\circledast_P$ is CGTN (product CGTN);

(2) Define $\circledast_M : \operatorname{diag} M_n([0,1]) \times \operatorname{diag} M_n([0,1]) \to \operatorname{diag} M_n([0,1])$ such that

$$\mathbf{t} \circledast_P \mathbf{s} = \operatorname{diag}[t_1,\dots,t_n] \circledast_P \operatorname{diag}[s_1,\dots,s_n] = \operatorname{diag}\big[\min\{t_1,s_1\},\dots,\min\{t_n,s_n\}\big],$$

then $\circledast_M$ is CGTN (minimum CGTN);

(3) Define $\circledast_L : \operatorname{diag} M_n([0,1]) \times \operatorname{diag} M_n([0,1]) \to \operatorname{diag} M_n([0,1])$ such that

$$\begin{aligned}\mathbf{t} \circledast_L \mathbf{s} &= \operatorname{diag}[t_1,\dots,t_n] \circledast_L \operatorname{diag}[s_1,\dots,s_n] \\ &= \operatorname{diag}\big[\max\{t_1+s_1-1,0\},\dots,\max\{t_n+s_n-1,0\}\big],\end{aligned}$$

then $\circledast_P$ is CGTN (Lukasiewicz CGTN).

Now, we present some numerical examples:

$$\begin{aligned}&\operatorname{diag}\left[\frac{1}{3},\frac{2}{5},1\right] \circledast_M \operatorname{diag}\left[\frac{1}{4},\frac{5}{7},0\right] \\ &\quad= \begin{bmatrix}\frac{1}{3} & & \\ & \frac{2}{5} & \\ & & 1\end{bmatrix} \circledast_M \begin{bmatrix}\frac{1}{4} & & \\ & \frac{5}{7} & \\ & & 0\end{bmatrix} = \begin{bmatrix}\frac{1}{4} & & \\ & \frac{5}{7} & \\ & & 0\end{bmatrix} = \operatorname{diag}\left[\frac{1}{4},\frac{5}{7},0\right], \\ &\operatorname{diag}\left[\frac{1}{3},\frac{2}{5},1\right] \circledast_P \operatorname{diag}\left[\frac{1}{4},\frac{5}{7},0\right] \\ &\quad= \begin{bmatrix}\frac{1}{3} & & \\ & \frac{2}{5} & \\ & & 1\end{bmatrix} \circledast_P \begin{bmatrix}\frac{1}{4} & & \\ & \frac{5}{7} & \\ & & 0\end{bmatrix} = \begin{bmatrix}\frac{1}{12} & & \\ & \frac{2}{7} & \\ & & 0\end{bmatrix} = \operatorname{diag}\left[\frac{1}{12},\frac{2}{7},0\right], \\ &\operatorname{diag}\left[\frac{1}{3},\frac{2}{5},1\right] \circledast_L \operatorname{diag}\left[\frac{1}{4},\frac{5}{7},0\right] \\ &\quad= \begin{bmatrix}\frac{1}{3} & & \\ & \frac{2}{5} & \\ & & 1\end{bmatrix} \circledast_L \begin{bmatrix}\frac{1}{4} & & \\ & \frac{5}{7} & \\ & & 0\end{bmatrix} = \begin{bmatrix}0 & & \\ & \frac{4}{35} & \\ & & 0\end{bmatrix} = \operatorname{diag}\left[0,\frac{4}{35},0\right].\end{aligned}$$

Also, since

$$\operatorname{diag}\left[\frac{1}{4},\frac{5}{7},0\right] \succeq \operatorname{diag}\left[\frac{1}{12},\frac{2}{7},0\right] \succeq \operatorname{diag}\left[0,\frac{4}{35},0\right],$$

we get

$$\begin{aligned}
&\operatorname{diag}\left[\frac{1}{3},\frac{2}{5},1\right] \circledast_M \operatorname{diag}\left[\frac{1}{4},\frac{5}{7},0\right]\\
&\quad\succeq \operatorname{diag}\left[\frac{1}{3},\frac{2}{5},1\right] \circledast_P \operatorname{diag}\left[\frac{1}{4},\frac{5}{7},0\right]\\
&\quad\succeq \operatorname{diag}\left[\frac{1}{3},\frac{2}{5},1\right] \circledast_L \operatorname{diag}\left[\frac{1}{4},\frac{5}{7},0\right].
\end{aligned}$$

We consider the set of matrix distribution functions (MDF) Ξ^+ which are left-continuous and increasing maps $\Theta : \mathbb{R} \cup \{-\infty,\infty\} \to \operatorname{diag} M_n([0,1])$ such that $\Theta_0 = \mathbf{0}$ and $\Theta_{+\infty} = \mathbf{1}$. Now $O^+ \subseteq \Xi^+$ are all (proper) mappings $\Theta \in \Xi^+$ for which $\ell^-\Theta_{+\infty} = \mathbf{1}$ ($\ell^-\Theta_\tau = \lim_{\sigma\to\tau^-}\Theta_\sigma$). Note proper MDFs are the MDFs of real random variables (i.e., of those random variables g that a.s. take real values ($P(|g| = \infty) = 0$)).

In Ξ^+, we define "$\preceq$" as follows:

$$\Theta \preceq \Upsilon \quad\Longleftrightarrow\quad \Theta_\tau \preceq \Upsilon_\tau, \quad \forall\tau \in \mathbb{R}.$$

Also

$$\nabla^w_\varsigma = \begin{cases} \mathbf{0}, & \text{if } \varsigma \le w,\\ \mathbf{1}, & \text{if } \varsigma > w,\end{cases}$$

belongs to Ξ^+, and for every MDF Θ we have $\Theta \preceq \nabla^0$ [11, 13–16]. For example,

$$\Theta^u_\tau = \begin{cases} \mathbf{0}, & \text{if } \tau \le 0,\\ \operatorname{diag}[1-e^{-\tau}, \frac{\tau}{1+\tau}, e^{-\frac{1}{\tau}}], & \text{if } \tau > 0,\end{cases}$$

is an MDF in $\operatorname{diag} M_3([0,1])$. Note that $\Theta_\tau = \operatorname{diag}[\theta_{1,\tau},\dots,\theta_{n,\tau}]$, in which $\theta_{i,\tau}$ are distribution functions, is an MDF.

Definition 2.2 Consider the CGTN $\circledast$, a linear space W, and MDF $\Theta : W \to O^+$. In this case, we call a matrix Menger normed space (MMN-space) the triple $(W,\Theta,\circledast)$ if the following conditions are satisfied:

(MMN1) $\Theta^w_\tau = \nabla^0_\tau$ for all $\tau > 0$ if and only if $w = 0$;
(MMN2) $\Theta^{\alpha w}_\tau = \Theta^w_{\frac{\tau}{|\alpha|}}$ for all $w \in W$ and $\alpha \in \mathbb{C}$ with $\alpha \ne 0$;
(MMN3) $\Theta^{w+w'}_{\tau+\varsigma} \succeq \Theta^w_\tau \circledast \Theta^{w'}_\varsigma$ for all $w, w' \in W$ and $\tau,\varsigma \ge 0$.

For example, the MDF Θ given by

$$\Theta^w_\varrho = \begin{cases} \mathbf{0}, & \text{if } \varrho \le 0,\\ \operatorname{diag}[\exp(-\frac{\|w\|}{\varsigma}), \frac{\varrho}{\varrho+\|w\|}, \exp(-\frac{\|w\|}{\varrho})], & \text{if } \varrho > 0,\end{cases}$$

is a matrix Menger norm and $(W, \Theta, \circledast_M)$ is an MMN-space; here $(W, \|\cdot\|)$ is a normed linear space.

Note that in neutrosophic set theory we need three norms to describe an object (probability, improbability, undecidability), while in intuitionistic random normed spaces we need two norms to describe an object, so MDFs on $\operatorname{diag} M_3([0,1])$ and $\operatorname{diag} M_2([0,1])$ are suitable for these theories, respectively.

Definition 2.3 Consider the CGTN's $\circledast$, $\circledstar$ and the MMN-space $(W, \Theta, \circledast)$. If

(MMN-5) $\Theta^{ww'}_{\tau\varsigma} \succeq \Theta^{w}_{\tau} \circledstar \Theta^{w'}_{\varsigma}$ for all $w, w' \in W$ and all $\tau, \varsigma > 0$,

we say that $(V, \Theta, \circledast, \circledstar)$ is a matrix Menger normed algebra (in short, MMN-algebra).

If

$$\|ww'\| \leq \|w\| \|w'\| + \varsigma \|w'\| + \tau \|w\| \quad \bigl(w, w' \in (W, \|\cdot\|); \tau, \varsigma > 0\bigr),$$

then

$$\Theta^{w}_{\varsigma} = \begin{cases} \mathbf{0}, & \text{if } \varsigma \leq 0, \\ \operatorname{diag}[\exp(-\frac{\|w\|}{\varsigma}), \frac{\varsigma}{\varsigma + \|w\|}], & \text{if } \varsigma > 0, \end{cases}$$

is an MMN-algebra $(W, \Theta, \circledast_M, \circledast_P)$, and vice versa. A Menger Banach algebra (MMB-algebra) is a complete MMN-algebra. Consider the complete MMN-spaces U and V. Consider the probability measure space (Γ, Π, Θ) with the Borel measurable spaces $(U, \mathfrak{B}_U)$ and $(V, \mathfrak{B}_V)$. A random operator is a map $\Lambda : \Gamma \times U \to V$ such that $\{\gamma : \Lambda(\gamma, u) \in B\} \in \Pi$ for all u in U and $B \in \mathfrak{B}_V$. If

$$\Lambda(\gamma, \alpha u_1 + \beta u_2) = \alpha \Lambda(\gamma, u_1) + \beta \Lambda(\gamma, u_2), \quad \forall u_1, u_2 \in U, \alpha, \beta \in \mathbb{R}$$

then Λ is linear, and if we can find a $H(\gamma) > 0$ such that

$$\Theta^{\Lambda(\gamma,u)-\Lambda(\gamma,v)}_{H(\gamma)\tau} \succeq \Theta^{u-v}_{\tau}, \quad \forall u_1, u_2 \in U, \tau > 0,$$

then Λ is bounded.

In MMB-algebras, we study κ-random operator inequalities

$$\Theta_{\tau}^{\Lambda(\gamma,u+v,w-r)+\Lambda(\gamma,u-v,w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r)} \succeq \Theta_{\tau}^{\kappa(2\Lambda(\gamma,\frac{u+v}{2},w-r)+2\Lambda(\gamma,\frac{u-v}{2},w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r))}, \tag{2.1}$$

$$\Theta_{\tau}^{2\Lambda(\gamma,\frac{u+v}{2},w-r)+2\Lambda(\gamma,\frac{u-v}{2},w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r)} \succeq \Theta_{\tau}^{\kappa(\Lambda(\gamma,u+v,w-r)+\Lambda(\gamma,u-v,w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r))}, \tag{2.2}$$

where $0 \neq \kappa \in \mathbb{C}$ is fixed and $|\kappa| < 1$. We stabilize the pseudo-stochastic biadditive κ-random operator in MMB-algebras by a stochastic control function. This process is said to be Hyers–Ulam–Rassias (HUR) stable for additive κ-random operator inequalities in MMB-algebras.

3 Best approximation of the κ-random operator inequality (2.1)

We improve Park et al. results [17, 18] and [15, 19–23] to get a better approximation.

Lemma 3.1 *Let $\Lambda : \Gamma \times U^2 \to V$ be a random operator satisfying (2.1) and $\Lambda(\gamma, 0, w) = \Lambda(\gamma, u, 0) = 0$ for each $u, w, r \in U$ and $\gamma \in \Gamma$. Then $\Lambda : \Gamma \times U^2 \to V$ is biadditive.*

Proof Putting $u = v$ and $r = 0$ in (2.1), we obtain (note $\Lambda(\gamma, 0, w) = \Lambda(\gamma, u, 0) = 0$) $\Lambda(\gamma, 2u, w) = 2\Lambda(\gamma, u, w)$ for all $u, w \in U$ and $\gamma \in \Gamma$. Thus

$$\begin{aligned} &\Theta_\tau^{\Lambda(\gamma,u+v,w-r)+\Lambda(\gamma,u-v,w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r)} \\ &\quad\succeq \Theta_\tau^{\kappa(2F(\gamma,\frac{t+s}{2},p-r)+2F(\gamma,\frac{t-s}{2},w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r))} \\ &\quad= \Theta_\tau^{\kappa(\Lambda(\gamma,u+v,w-r)+\Lambda(\gamma,u-v,w+r)-2\Lambda(\gamma,u,p)+2\Lambda(\gamma,v,r))} \end{aligned}$$

and

$$\Lambda(\gamma, u+v, w-r) + \Lambda(\gamma, u-v, w+r) - 2\Lambda(\gamma, u, w) + 2\Lambda(\gamma, v, r) = 0, \tag{3.1}$$

for all $u, v, w, r \in U$, $\gamma \in \Gamma$, $\tau > 0$.

Putting $r = 0$ in (3.1), we have $\Lambda(\gamma, u+v, w) + \Lambda(\gamma, u-v, w) = 2\Lambda(\gamma, u, w)$ and $\Lambda(\gamma, u_1, w) + \Lambda(\gamma, v_1, w) = 2\Lambda(\gamma, \frac{u_1+v_1}{2}, p) = \Lambda(\gamma, u_1 + v_1, p)$ for all $u_1 := u + v$, $v_1 := u - v$, $w \in U$, since $|\kappa| \leq 1$ and $\Lambda(\gamma, 0, w) = 0$ for all $w \in U$. Thus $\Lambda : \Gamma \times U^2 \to V$ is additive in the second variable.

By a similar method, we can show that $\Lambda : \Gamma \times U^2 \to V$ is additive in the last variable. Then $\Lambda : \Gamma \times U^2 \to V$ is a random biadditive operator. □

Theorem 3.2 *Let $(U, \Theta, \circledast_M, \circledast_M)$ be an MMB-algebra, let $\psi : U^4 \to O^+$ be an MDF such that there exists a $\beta < 1$ with $\psi_\tau^{\frac{u}{2},\frac{v}{2},w,0} \succeq \psi_{\frac{2\tau}{\beta}}^{u,v,w,0}$ for all $u, v, w, r \in U$ and $\tau > 0$, and*

$$\lim_{n\to\infty} \psi_{\frac{\tau}{2^n}}^{\frac{u}{2^n},\frac{v}{2^n},w,0} = \nabla_\tau^0, \tag{3.2}$$

for all $u, v \in U$, $\tau > 0$. Suppose the random operator $\Lambda : \Gamma \times U^2 \to V$ satisfies $\Lambda(\gamma, u, 0) = \Lambda(\gamma, 0, w) = 0$ for all $u, w \in U$, $\gamma \in \Gamma$ and

$$\begin{aligned} &\Theta_\tau^{\Lambda(\gamma,u+v,w-r)+\Lambda(\gamma,u-v,w+r)-2\Lambda(\gamma,u,p)+2\Lambda(\gamma,v,r)} \\ &\quad\succeq \Theta_\tau^{\kappa(2\Lambda(\gamma,\frac{u+v}{2},w-r)+2\Lambda(\gamma,\frac{u-v}{2},w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r))} \\ &\qquad\circledast_M \psi_\tau^{u,v,w,r}, \end{aligned} \tag{3.3}$$

for all $u, v, w, r \in U$, $\gamma \in \Gamma$, and $\tau > 0$. Then we can find a unique biadditive random operator $\Delta : \Gamma \times U^2 \to V$ such that

$$\Theta_\tau^{\Lambda(\gamma,u,w)-\Delta(\gamma,u,w)} \succeq \psi_{\frac{2(1-\beta)\tau}{\beta}}^{u,u,w,0}, \tag{3.4}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$.

Proof Putting $r = 0$ and $v = u$ in (3.3), we get

$$\Theta_\tau^{\Lambda(\gamma,2u,w)-2\Lambda(\gamma,u,w)} \succeq \psi_\tau^{u,u,w,0}, \tag{3.5}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$. Thus

$$\Theta_\tau^{\Lambda(\gamma,u,w)-2\Lambda(\gamma,\frac{u}{2},w)} \succeq \psi_\tau^{\frac{u}{2},\frac{u}{2},w,0} \succeq \psi_{\frac{2\tau}{\beta}}^{u,u,w,0}, \tag{3.6}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$. Replacing u by $\frac{u}{2^n}$ in (3.6), we obtain

$$\begin{aligned}\Theta_\tau^{2^n\Lambda(\gamma,\frac{u}{2^n},pw)-2^{n+1}\Lambda(\gamma,\frac{u}{2^{n+1}},w)} &\succeq \psi_{\frac{\tau}{2^{n-1}\beta}}^{\frac{u}{2^n},\frac{u}{2^n},w,0} \\ &\succeq \psi_{\frac{2}{\beta}(\frac{\tau}{2^{n-1}\beta})}^{\frac{u}{2^{n-1}},\frac{u}{2^{n-1}},w,0} \\ &\succeq \cdots \\ &\succeq \psi_{\frac{2}{\beta^{n+1}}\tau}^{u,u,w,0}.\end{aligned} \tag{3.7}$$

It follows from

$$2^n\Lambda\left(\gamma, \frac{u}{2^n}, w\right) - \Lambda(\gamma, u, w) = \sum_{k=1}^{n}\left(2^k\Lambda\left(\gamma, \frac{t}{2^k}, p\right) - 2^{k-1}\Lambda\left(\gamma, \frac{u}{2^{k-1}}, w\right)\right)$$

and (3.7) that

$$\Theta_{\sum_{k=1}^n \frac{\beta^k}{2}\tau}^{\Lambda(\gamma,\frac{u}{2^n},w)-\Lambda(\gamma,u,w)} \succeq \varphi_\tau^{u,u,w,0} \circledast_M \cdots \circledast_M \psi_\tau^{u,u,w,0} = \psi_\tau^{u,u,w,0},$$

for all $u, w \in U$, $\gamma \in \Gamma$, $\tau > 0$. That is,

$$\Theta_\tau^{2^n\Lambda(\gamma,\frac{u}{2^n},w)-\Lambda(\gamma,u,w)} \succeq \psi_{\frac{\tau}{\sum_{k=1}^n \frac{\beta^k}{2}}}^{u,u,w,0}. \tag{3.8}$$

Replacing u with $\frac{u}{2^m}$ in (3.8), we get

$$\Theta_\tau^{2^{n+m}\Lambda(\gamma,\frac{u}{2^{n+m}},w)-2^m\Lambda(\gamma,\frac{u}{2^m},w)} \succeq \psi_{\frac{\tau}{\sum_{k=m+1}^{n+m} \frac{\beta^k}{2}}}^{u,u,w,0}. \tag{3.9}$$

Since $\psi_{\frac{\tau}{\sum_{k=m+1}^{n+m} \frac{\beta^k}{2}}}^{u,u,w,0}$ tends to ∇_τ^0 as $m, n \to \infty$, it follows that the sequence $\{2^n\Lambda(\gamma, \frac{u}{2^n}, w)\}$ is Cauchy for all $u, w \in U$, $\gamma \in \Gamma$. Since V is an MMB-algebra, $\{2^n\Lambda(\gamma, \frac{u}{2^n}, w)\}$ is a convergent sequence. Now, we define a random operator $\Delta : \Gamma \times U^2 \to V$ by

$$\Delta(\gamma, u, w) := \lim_{k\to\infty} 2^k\Lambda\left(\gamma, \frac{u}{2^k}, w\right),$$

for all $u, w \in U$, $\gamma \in \Gamma$. Putting $m = 0$ and letting $n \to \infty$ in (3.9), we conclude that

$$\Theta_\tau^{\Lambda(\gamma,u,w)-\Delta(\gamma,u,w)} \succeq \psi_{\frac{2(1-\beta)\tau}{\beta}}^{u,u,w,0}, \tag{3.10}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$.

Now, (3.3) implies that

$$\begin{aligned}
&\Theta_\tau^{\Delta(\gamma,u+v,w-r)+\Delta(\gamma,u-v,w+r)-2\Delta(\gamma,u,w)+2\Delta(\gamma,v,r)} \\
&= \lim_{n\to\infty} \Theta_\tau^{2^n(\Lambda(\gamma,\frac{u+v}{2^n},w-r)+\Lambda(\gamma,\frac{u-v}{2^n},w+r)-2\Lambda(\gamma,\frac{u}{2^n},w)+2\Lambda(\gamma,\frac{v}{2^n},r))} \\
&\succeq \lim_{n\to\infty} \Theta_\tau^{2^n\kappa(2\Lambda(\gamma,\frac{u+v}{2^{n+1}},w-r)+2\Lambda(\gamma,\frac{u-v}{2^{n+1}},w+r)-2\Lambda(\gamma,\frac{u}{2^n},w)+2\Lambda(\gamma,\frac{v}{2^n},r))} \\
&\qquad \circledast_M \lim_{n\to\infty} \psi_{\frac{\tau}{2^n}}^{\frac{tu}{2^n},\frac{u}{2^n},w,0} \\
&\succeq \Theta_\tau^{\kappa(2\Delta(\gamma,\frac{u+v}{2},w-r)+2\Delta(\gamma,\frac{u-v}{2},w+r)-2\Delta(\gamma,u,w)+2\Delta(\gamma,v,r))},
\end{aligned}$$

for all $u,v,w,r \in T$, $\gamma \in \Gamma$, $\tau > 0$, since $\psi_{\frac{\tau}{2^n}}^{\frac{u}{2^n},\frac{u}{2^n},w,0}$ tends to ∇_τ^0 as $n \to \infty$. Thus

$$\begin{aligned}
&\Theta_\tau^{\Delta(\gamma,u+v,w-r)+\Delta(\gamma,u-v,w+r)-2\Delta(\gamma,u,w)+2\Delta(\gamma,v,r)} \\
&\succeq \Theta_\tau^{\kappa(2\Delta(\gamma,\frac{u+v}{2},w-r)+2\Delta(\gamma,\frac{u-v}{2},w+r)-2\Delta(\gamma,u,p)+2\Delta(\gamma,v,r))},
\end{aligned}$$

for all $u,v,w,r \in U$, $\gamma \in \Gamma$, $\tau > 0$. From Lemma 3.1, the random operator $\Delta : \Gamma \times U^2 \to V$ is stochastic biadditive.

Now, to show that the random operator Δ is unique, assume that there exists a stochastic biadditive random operator $\Omega : \Gamma \times U^2 \to V$ which satisfies (3.4). Thus,

$$\begin{aligned}
&\Theta_\tau^{\Delta(\gamma,u,w)-\Omega(\gamma,u,w)} = \lim_{n\to\infty} \Theta_\tau^{2^n\Delta(\gamma,\frac{u}{2^n},w)-2^n\Omega(\gamma,\frac{u}{2^n},w)}, \\
&\Theta_\tau^{2^n\Delta(\gamma,\frac{u}{2^n},w)-2^n\Omega(\gamma,\frac{u}{2^n},w)} \\
&\succeq \Theta_{\frac{\tau}{2}}^{2^n\Delta(\gamma,\frac{u}{2^n},w)-2^n\Lambda(\gamma,\frac{u}{2^n},w)} \circledast_M \Theta_{\frac{\tau}{2}}^{2^n\Omega(\gamma,\frac{u}{2^n},w)-2^n\Lambda(\gamma,\frac{u}{2^n},w)} \\
&\succeq \psi_{\frac{2(1-\beta)\tau}{2^n\beta}}^{\frac{u}{2^n},\frac{u}{2^n},w,0} \\
&\succeq \psi_{\frac{2(1-\beta)\tau}{\beta^{n+1}}}^{u,u,w,0}.
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{2(1-\beta)}{\beta^{n+1}} = \infty$, we get that $\psi_{\frac{2(1-\beta)\tau}{\beta^{n+1}}}^{u,u,w,0}$ tends to ∇_τ^0 as $n \to \infty$.

Therefore, it follows that $\Theta_\tau^{2^n\Delta(\gamma,\frac{u}{2^n},w)-2^n\Omega(\gamma,\frac{u}{2^n}w)} = 1$, for all $u,w \in U$, $\gamma \in \Gamma$, $\tau > 0$. Thus we can conclude that $\Delta(\gamma,u,w) = \Omega(\gamma,u,w)$, for all $u,w \in U$ and $\gamma \in \Gamma$. □

Corollary 3.3 *Let $(T,\Theta,\circledast_M,\circledast_M)$ be an MMB-algebra. Assume that $\iota > 1$, ς is a non-negative real number, and $\Lambda : \Gamma \times U^2 \to V$ is a random operator satisfying $\Lambda(\gamma,u,0) = \Lambda(\gamma,0,w) = 0$ and*

$$\begin{aligned}
&\Theta_\tau^{\Lambda(\gamma,u+v,w-r)+\Lambda(\gamma,u-v,w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r)} \qquad (3.11) \\
&\succeq \Theta_\tau^{\kappa(2\Lambda(\gamma,\frac{u+v}{2},w-r)+2\Lambda(\gamma,\frac{u-v}{2},w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r))} \\
&\qquad \circledast_M \operatorname{diag}\left[\exp(-\frac{\varsigma(\|u\|^\iota+\|v\|^\iota)(\|w\|^\iota+\|r\|^\iota)}{\tau}, \frac{\tau}{\tau+\varsigma(\|u\|^\iota+\|v\|^\iota)(\|w\|^\iota+\|r\|^\iota)}\right],
\end{aligned}$$

for all $u, v, w, r \in U$, $\gamma \in \Gamma$, and $\tau > 0$. Then we can find a unique biadditive random operator $\Delta : \Gamma \times T^2 \to S$ such that

$$\begin{aligned}&\Theta_{\tau}^{\Lambda(\gamma,u,w)-\Delta(\gamma,u,w)} \\ &\quad \succeq \operatorname{diag}\left[\exp\left(-\frac{2^{\iota+2}\varsigma\|u\|^{\iota}\|w\|^{\iota}}{2(2^{\iota}-2)\tau}\right), \frac{2(2^{\iota}-2)\tau}{2(2^{\iota}-2)\tau+2^{\iota+2}\varsigma\|u\|^{\iota}\|w\|^{\iota}}\right],\end{aligned} \tag{3.12}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$.

Proof The result follows from Theorem 3.2 by putting

$$\psi_{\tau}^{u,v,w,r} = \operatorname{diag}\left[\exp(-\frac{\varsigma(\|u\|^{\iota}+\|v\|^{\iota})(\|w\|^{\iota}+\|r\|^{\iota})}{\tau}, \frac{\tau}{\tau+\varsigma(\|u\|^{\iota}+\|v\|^{\iota})(\|w\|^{\iota}+\|r\|^{\iota})}\right],$$

for all $u, w \in U$, $\gamma \in \Gamma$, $\tau > 0$, and $\beta = 2^{1-\iota}$. □

Theorem 3.4 *Let $(U, \Theta, \circledast_M, \circledast_M)$ be an MMB-algebra, let $\psi : U^4 \to O^+$ be an MDF such that there exists a $\beta < 1$ with $\psi_{\tau}^{u,v,w,0} \succeq \psi_{\frac{\tau}{2\beta}}^{\frac{u}{2},\frac{v}{2},w,0}$ for all $u, v, w \in U$, $\lim_{n\to\infty} \psi_{2^n\tau}^{2^n u,2^n v,w,0} = 1$ for all $u, v, w \in U$, $\tau > 0$. Suppose that a random operator $\Lambda : \Gamma \times U^2 \to V$ satisfies (3.3) and $\Lambda(\gamma, u, 0) = \Lambda(\gamma, 0, w) = 0$ for all $u, v \in U$ and $\gamma \in \Gamma$. Then, there is a unique biadditive random operator $\Delta : \Gamma \times U^2 \to V$ such that*

$$\Theta_{\tau}^{\Lambda(\gamma,u,w)-\Delta(\gamma,u,w)} \succeq \psi_{2(1-\beta)\tau}^{u,u,w,0}, \tag{3.13}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$.

Proof Putting $r = 0$ and $v = u$ in (3.3), we have

$$\Theta_{\tau}^{\frac{1}{2}\Lambda(\gamma,2u,pw)-\Lambda(\gamma,u,w)} \succeq \psi_{2\tau}^{u,u,w,0}, \tag{3.14}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$. Thus

$$\Theta_{\tau}^{\frac{1}{2}\Lambda(\gamma,u,w)-\Lambda(\gamma,2u,w)} \succeq \psi_{\tau}^{2t,2t,w} \succeq \psi_{\frac{\tau}{2\beta}}^{u,u,w,0}, \tag{3.15}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$. Changing u by $2^n u$ in (3.15), we have

$$\begin{aligned}\Theta_{\tau}^{\frac{1}{2^n}\Lambda(\gamma,2^n u,w)-\frac{1}{2^{n+1}}\Lambda(\gamma,2^{n+1}u,w)} &\succeq \psi_{2\times 2^n\tau}^{2^n u,2^n u,w,0}. \\ &\succeq \psi_{\frac{2\times 2^n}{(2\beta)^n}\tau}^{u,u,w,0}.\end{aligned} \tag{3.16}$$

From

$$\frac{1}{2^n}\Lambda(\gamma, 2^n u, w) - \Lambda(\gamma, u, w) = \sum_{k=0}^{n-1}\left(\frac{1}{2^{k+1}}\Lambda(\gamma, 2^{k+1}u, w) - \frac{1}{2^k}\Lambda(\gamma, 2^k u, w)\right)$$

and (3.16), we get

$$\Theta_{\sum_{k=0}^{n-1}\frac{(2\beta)^k}{2\times 2^k}\tau}^{\frac{1}{2^n}\Lambda(\gamma,2^n u,w)-\Lambda(\gamma,u,w)} \succeq \psi_{\tau}^{u,u,w,0},$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$. That is,

$$\Theta_\tau^{\frac{1}{2^n}\Lambda(\gamma,2^n u,w)-\Lambda(\gamma,u,w)} \succeq \psi^{u,u,w,0}_{\frac{\tau}{\sum_{k=0}^{n-1}\frac{(2\beta)^k}{2\times 2^k}}}. \tag{3.17}$$

Replacing u with $2^m u$ in (3.17), we get

$$\Theta_\tau^{\frac{1}{2^{n+m}}\Lambda(\gamma,2^{n+m}u,w)-\frac{1}{2^m}\Lambda(\gamma,2^m u,v)} \succeq \psi^{u,u,w,0}_{\frac{\tau}{\sum_{k=m}^{n+m}\frac{(2\beta)^k}{2\times 2^k}}}. \tag{3.18}$$

Since $\psi^{u,u,w,0}_{\frac{\tau}{\sum_{k=m}^{n+m}\frac{(2\beta)^k}{2\times 2^k}}}$ tends to ∇_τ^0 as $m, n \to \infty$, it follows that $\{\frac{1}{2^n}\Lambda(\gamma, 2^n u, w)\}$ is a Cauchy sequence for all $u, w \in U$ and $\gamma \in \Gamma$. Since V is an MMB-algebra, the sequence $\{\frac{1}{2^n}\Lambda(\gamma, 2^n u, w)\}$ converges. Now, we define the random operator $\Delta : \Gamma \times U^2 \to V$ by

$$\Delta(\gamma, u, w) := \lim_{k\to\infty} \frac{1}{2^k}\Lambda\bigl(\gamma, 2^k u, w\bigr),$$

for all $u, w \in U$ and $\gamma \in \Gamma$. Putting $m = 0$ and letting $n \to \infty$ in (3.18), we have

$$\Theta_\tau^{\Lambda(\gamma,u,w)-\Delta(\gamma,u,w)} \succeq \psi^{u,u,w,0}_{2(1-\beta)\tau}, \tag{3.19}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$. The proof is finished by using Theorem 3.2. □

Corollary 3.5 *Let $(U, \Theta, \circledast_M, \circledast_M)$ be an MMB-algebra. Assume that $\iota < 1$, $\varsigma \geq 0$, and $\Lambda : \Gamma \times U^2 \to V$ is a random operator satisfying (3.11) and $\Lambda(\gamma, u, 0) = \Lambda(\gamma, 0, w) = 0$ for all $u, w \in U$ and $\gamma \in \Gamma$. Then we can find a unique biadditive random operator $\Delta : \Gamma \times U^2 \to V$ such that*

$$\Theta_\tau^{\Lambda(\gamma,u,w)-\Delta(\gamma,u,w)} \succeq \operatorname{diag}\left[\exp\left(-\frac{2\varsigma\|u\|^\iota\|w\|^\iota}{(2-2^\iota)\tau}\right), \frac{(2-2^\iota)\tau}{(2-2^\iota)\tau + 2\varsigma\|u\|^\iota\|w\|^\iota}\right], \tag{3.20}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$.

Proof The result follows from Theorem 3.4 by putting

$$\psi_\tau^{u,v,w,r} = \operatorname{diag}\left[\exp(-\frac{\varsigma(\|u\|^\iota + \|v\|^\iota)(\|w\|^\iota + \|r\|^\iota)}{\tau}, \frac{\tau}{\tau + \varsigma(\|u\|^\iota + \|v\|^\iota)(\|w\|^\iota + \|r\|^\iota)}\right],$$

for all $u, w \in U$, $\gamma \in \Gamma$, $\tau > 0$, and $\beta = 2^{\iota-1}$. □

4 Best approximation of the κ-random operator inequality (2.2)

Lemma 4.1 *Let the random operator $\Lambda : \Gamma \times U^2 \to V$ satisfy $\Lambda(\gamma, 0, w) = \Lambda(\gamma, u, 0) = 0$ and*

$$\Theta_\tau^{2\Lambda(\gamma,\frac{u+v}{2},w-r)+2\Lambda(\gamma,\frac{u-v}{2},w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r)} \succeq \Theta_\tau^{\kappa(\Lambda(\gamma,u+v,w-r)+\Lambda(\gamma,u-v,w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r))}, \tag{4.1}$$

for all $u, v, w, r \in U$, $\gamma \in \Gamma$, and $\tau > 0$. Then $\Lambda : \Gamma \times U^2 \to V$ is biadditive.

Proof Putting $v = r = 0$ in (4.1), we get $4\Lambda(\gamma, \frac{u}{2}, w) = 2\Lambda(\gamma, u, w)$ for all $u, w \in U$ and $\gamma \in \Gamma$. Thus

$$\begin{aligned}
&\Theta_\tau^{\Lambda(\gamma,u+v,w-r)+\Lambda(\gamma,u-v,w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r)} \\
&\quad = \Theta_\tau^{2\Lambda(\gamma,\frac{u+v}{2},w-r)+2\Lambda(\gamma,\frac{u-v}{2},w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r)} \\
&\quad \succeq \Theta_\tau^{\kappa(\Lambda(\gamma,u+v,w-r)+\Lambda(\gamma,u-v,w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r))}
\end{aligned}$$

and

$$\Lambda(\gamma, u+v, w-r) + \Lambda(\gamma, u-v, w+r) - 2\Lambda(\gamma, u, w) + 2\Lambda(\gamma, v, r) = 0,$$

for all $u, v, w, r \in U$ and $\gamma \in \Gamma$.

The proof is completed by using a similar method as in Lemma 3.1. □

Theorem 4.2 *Let $(U, \Theta, \circledast_M, \circledast_M)$ be an MMB-algebra. Assume that $\psi : U^4 \to O^+$ is an MDF in which there is a $\beta < 1$ with $\psi_\tau^{\frac{u}{2},\frac{v}{2},w,0} \geq \psi_{\frac{2\tau}{\beta}}^{u,v,w,0}$ for all $u, v, w \in U$. Let $\Lambda : \Gamma \times U^2 \to V$ be a random operator satisfying $\Lambda(\gamma, u, 0) = \Lambda(\gamma, 0, w) = 0$ for all $u, w \in U$, $\gamma \in \Gamma$ and*

$$\begin{aligned}
&\Theta_\tau^{2\Lambda(\gamma,\frac{u+v}{2},w-r)+2\Lambda(\gamma,\frac{u-v}{2},w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r)} \\
&\quad \succeq \Theta_\tau^{\kappa(\Lambda(\gamma,u+v,w-r)+\Lambda(\gamma,u-v,w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r))} \\
&\qquad \circledast_M \psi_\tau^{u,v,w,r},
\end{aligned} \tag{4.2}$$

for all $u, v, w, r \in U$, $\gamma \in \Gamma$, and $\tau > 0$. Then we can find a unique biadditive random operator $\Delta : \Gamma \times U^2 \to V$ such that

$$\Theta_\tau^{\Lambda(\gamma,u,w)-\Delta(\gamma,u,w)} \succeq \psi_{2(1-\beta)\tau}^{u,0,w,0}, \tag{4.3}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$.

Proof Putting $v = r = 0$ in (4.2), we have

$$\Theta_\tau^{4\Lambda(\gamma,\frac{u}{2},w)-2\Lambda(\gamma,u,w)} \succeq \psi_\tau^{u,0,w,0}, \tag{4.4}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$. Thus

$$\Theta_\tau^{\Lambda(\gamma,u,w)-2\Lambda(\gamma,\frac{u}{2},w)} \succeq \psi_{2\tau}^{u,0,w,0}, \tag{4.5}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$. Replacing u by $\frac{u}{2^n}$ in (4.5), we get

$$\begin{aligned}
\Theta_\tau^{2^n\Lambda(\gamma,\frac{u}{2^n},w)-2^{n+1}\Lambda(\gamma,\frac{u}{2^{n+1}},w)} &\succeq \psi_{\frac{\tau}{2^{n-1}}}^{\frac{u}{2^n},0,w,0} \\
&\succeq \psi_{\frac{2}{\beta^n}\tau}^{u,0,w,0}.
\end{aligned} \tag{4.6}$$

It follows from

$$2^n\Lambda\left(\gamma,\frac{u}{2^n},p\right)-\Lambda(\gamma,u,w)=\sum_{k=0}^{n-1}\left(2^k\Lambda\left(\gamma,\frac{u}{2^k},w\right)-2^{k+1}\Lambda\left(\gamma,\frac{u}{2^{k+1}},w\right)\right)$$

and (4.6) that

$$\Theta^{2^n\Lambda(\gamma,\frac{u}{2^n},w)-\Lambda(\gamma,u,w)}_{\sum_{k=0}^{n-1}\frac{\beta^k}{2}\tau}\succeq\psi^{u,0,w,0}_{\tau},$$

for all $u,w\in U$, $\gamma\in\Gamma$, and $\tau>0$. That is,

$$\Theta^{2^n\Lambda(\gamma,\frac{u}{2^n},w)-\Lambda(\gamma,u,w)}_{\tau}\succeq\psi^{u,0,w,0}_{\frac{\tau}{\sum_{k=0}^{n-1}\frac{\beta^k}{2}}}. \tag{4.7}$$

Replacing u with $\frac{u}{2^m}$ in (4.7), we get

$$\Theta^{2^{n+m}\Lambda(\gamma,\frac{u}{2^{n+m}},w)-2^m\Lambda(\gamma,\frac{u}{2^m},w)}_{\tau}\succeq\psi^{u,0,w,0}_{\frac{\tau}{\sum_{k=m}^{n+m}\frac{\beta^k}{2}}}. \tag{4.8}$$

Since $\psi^{u,0,w,0}_{\frac{\tau}{\sum_{k=m}^{n+m}\frac{\beta^k}{2}}}$ tends to ∇^0_τ as $m,n\to\infty$, it follows that the sequence $\{2^n\Lambda(\gamma,\frac{u}{2^n},w)\}$ is Cauchy for all $u,w\in U$ and $\gamma\in\Gamma$. Since V is an MMB-algebra, $\{2^n\Lambda(\gamma,\frac{u}{2^n},w)\}$ is a convergent sequence. Now, we define the random operator $\Delta:\Gamma\times U^2\to V$ by

$$\Delta(\gamma,u,w):=\lim_{k\to\infty}2^k\Lambda\left(\gamma,\frac{u}{2^k},w\right),$$

for all $u,w\in U$ and $\gamma\in\Gamma$. Putting $m=0$ and letting $n\to\infty$ in (4.8), we have

$$\Theta^{\Lambda(\gamma,u,w)-\Delta(\gamma,u,w)}_{\tau}\succeq\psi^{u,0,w,0}_{2(1-\beta)\tau},$$

for all $u,w\in U$, $\gamma\in\Gamma$, and $\tau>0$. The proof is completed by a similar method as in Theorem 3.2. □

Corollary 4.3 *Let $(U,\Theta,\circledast_M,\circledast_M)$ be an MMB-algebra. Assume that $\iota>1$, $\varsigma\geq0$, and $\Lambda:\Gamma\times U^2\to V$ is a random operator satisfying $\Lambda(\gamma,u,0)=\Lambda(\gamma,0,w)=0$ and*

$$\begin{aligned}&\Theta^{2\Lambda(\gamma,\frac{u+v}{2},w-r)+2\Lambda(\gamma,\frac{u-v}{2},w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r)}_{\tau}\\&\quad\succeq\Theta^{\kappa(\Lambda(\gamma,u+v,w-r)+\Lambda(\gamma,u-v,w+r)-2\Lambda(\gamma,u,w)+2\Lambda(\gamma,v,r))}_{\tau}\\&\qquad\circledast_M\operatorname{diag}\left[\exp(-\frac{\varsigma(\|u\|^\iota+\|v\|^\iota)(\|w\|^\iota+\|r\|^\iota)}{\tau},\frac{\tau}{\tau+\varsigma(\|u\|^\iota+\|v\|^\iota)(\|w\|^\iota+\|r\|^\iota)}\right],\end{aligned} \tag{4.9}$$

for all $u,v,w,r\in U$, $\gamma\in\Gamma$, and $\tau>0$. Then we can find a unique biadditive random operator $\Delta:\Gamma\times U^2\to V$ such that

$$\Theta^{\Lambda(\gamma,u,w)-\Delta(\gamma,u,w)}_{\tau}\succeq\operatorname{diag}\left[\exp\left(-\frac{2^{\iota-1}\varsigma\|u\|^\iota\|w\|^\iota}{(2^\iota-2)\tau}\right),\frac{2^{\iota-1}\tau}{(2^\iota-2)\tau+\varsigma\|u\|^\iota\|w\|^\iota}\right], \tag{4.10}$$

for all $u,w\in U$, $\gamma\in\Gamma$. and $\tau>0$.

Proof The result follows from Theorem 4.2 by putting

$$\psi_{\tau}^{u,v,w,r} = \operatorname{diag}\left[\exp(-\frac{\varsigma(\|u\|^{\iota} + \|v\|^{\iota})(\|w\|^{\iota} + \|r\|^{\iota})}{\tau}, \frac{\tau}{\tau + \varsigma(\|u\|^{\iota} + \|v\|^{\iota})(\|w\|^{\iota} + \|r\|^{\iota})}\right],$$

for all $u, w \in U$, $\gamma \in \Gamma$, $\tau > 0$, and $\beta = 2^{1-\iota}$. □

Theorem 4.4 *Let $(U, \Theta, \circledast_M, \circledast_M)$ be an MMB-algebra. Assume that $\psi : U^4 \to O^+$ is an MDF such that there exists a $\beta < 1$ with $\varphi_{\tau}^{u,v,w,0} \succeq \psi_{\frac{\tau}{2\beta}}^{\frac{u}{2},\frac{v}{2},w,0}$ for all $u, v, w \in U$. Let $\Lambda : \Gamma \times U^2 \to V$ be a random operator satisfying (4.2) and $\Lambda(\gamma, u, 0) = \Lambda(\gamma, 0, w) = 0$ for all $u, w \in U$ and $\gamma \in \Gamma$. Then we can find a unique biadditive random operator $\Delta : \Gamma \times U^2 \to V$ such that*

$$\Theta_{\tau}^{\Lambda(\gamma,u,w)-\Delta(\gamma,u,w)} \succeq \psi_{\frac{2(1-\beta)}{\beta}\tau}^{u,0,w,0}, \tag{4.11}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$.

Proof Letting $v = r = 0$ in (4.3), we have

$$\Theta_{\tau}^{4\Lambda(\gamma,\frac{u}{2},w)-2\Lambda(\gamma,u,w)} \succeq \psi_{4\tau}^{u,0,w,0},$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$. Thus

$$\Theta_{\tau}^{\Lambda(\gamma,u,w)-\frac{1}{2}\Lambda(\gamma,2u,w)} \succeq \varphi_{4\tau}^{2u,0,w,0} \succeq \varphi_{\frac{2\tau}{\beta}}^{u,0,w,0}, \tag{4.12}$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$. Replacing u by $2^n u$ in (4.12), we get

$$\begin{aligned} \Theta_{\tau}^{\frac{1}{2^n}\Lambda(\gamma,2^n u,w)-\frac{1}{2^{n+1}}\Lambda(\gamma,2^{n+1}u,w)} &\succeq \psi_{\frac{1}{(2\beta)^n}\frac{2\times 2^n}{\beta}\tau}^{u,0,w,0} \\ &= \varphi_{\frac{2}{(\beta)^{n+1}}\tau}^{u,0,w,0}. \end{aligned} \tag{4.13}$$

From

$$\frac{1}{2^n}\Lambda(\gamma, 2^n u, w) - \Lambda(\gamma, u, w) = \sum_{k=0}^{n-1}\left(\frac{1}{2^{k+1}}\Lambda(\gamma, 2^{k+1}u, w) - \frac{1}{2^k}\Lambda(\gamma, 2^k u, w)\right)$$

and (4.13), we conclude that

$$\Theta_{\sum_{k=1}^{n}\frac{\beta^k}{2}\tau}^{\frac{1}{2^n}\Lambda(\gamma,2^n u,w)-F(\gamma,u,w)} \succeq \psi_{\tau}^{u,0,w,0},$$

for all $u, w \in U$, $\gamma \in \Gamma$, and $\tau > 0$. That is,

$$\Theta_{\tau}^{\frac{1}{2^n}\Lambda(\gamma,2^n u,w)-\Lambda(\gamma,u,w)} \succeq \psi_{\frac{\tau}{\sum_{k=1}^{n}\frac{\beta^k}{2}}}^{u,0,w,0}. \tag{4.14}$$

Replacing u with $2^m u$ in (4.14), we get

$$\Theta_{\tau}^{\frac{1}{2^{n+m}}\Lambda(\gamma,2^{n+m}u,w)-\frac{1}{2^m}\Lambda(\gamma,2^m u,w)} \succeq \psi^{u,0,w,0}_{\frac{\tau}{\sum_{k=m+1}^{n+m}\frac{\beta^k}{2}}}. \tag{4.15}$$

Since $\psi^{u,0,u,0}_{\frac{\tau}{\sum_{k=m+1}^{n+m}\frac{\beta^k}{2}}}$ tends to ∇^0_τ as $m,n\to\infty$, it follows that the sequence $\{\frac{1}{2^n}\Lambda(\gamma,2^n u,w)\}$ is Cauchy for all $u,w\in U$, $\gamma\in\Gamma$. Since V is an MMB-algebra, $\{\frac{1}{2^n}F(\gamma,2^n u,w)\}$ is a convergent sequence. Now, we define the random operator $\Delta:\Gamma\times U^2\to V$ by

$$\Delta(\gamma,u,w):=\lim_{k\to\infty}\frac{1}{2^k}\Lambda\bigl(\gamma,2^k u,w\bigr),$$

for all $u,w\in U$ and $\gamma\in\Gamma$. Putting $m=0$ and letting $n\to\infty$ in (4.15), we get

$$\Theta_{\tau}^{\Lambda(\gamma,u,w)-\Delta(\gamma,u,w)} \succeq \psi^{u,0,w,0}_{\frac{2(1-\beta)}{\beta}\tau}, \tag{4.16}$$

for all $u,w\in U$, $\gamma\in\Gamma$, and $\tau>0$. The proof is completed by a similar method as in Theorem 4.2. □

Corollary 4.5 *Let $(U,\Theta,\circledast_M,\circledast_M)$ be an MMB-algebra. Assume that $\iota<1$, $\varsigma\geq 0$, and $\Lambda:\Gamma\times U^2\to V$ is a random operator satisfying (4.9) and $\Lambda(\gamma,u,0)=\Lambda(\gamma,0,w)=0$ for all $u,w\in U$ and $\gamma\in\Gamma$. Then we can find a unique biadditive random operator $\Delta:\Gamma\times U^2\to V$ such that*

$$\Theta_{\tau}^{\Lambda(\gamma,u,w)-\Delta(\gamma,u,w)} \succeq \operatorname{diag}\biggl[\exp\biggl(-\frac{2^\iota\varsigma\|u\|^\iota\|v\|^\iota}{2(2-2^\iota)\tau}\biggr),\frac{2(2-2^\iota)\tau}{2(2-2^\iota)\tau+2^\iota\varsigma\|u\|^\iota\|v\|^\iota}\biggr], \tag{4.17}$$

for all $u,w\in T$, $\gamma\in\Gamma$, and $\tau>0$.

Proof The result follows from Theorem 4.4 by putting

$$\psi_{\tau}^{u,v,wp,r}=\operatorname{diag}\biggl[\exp(-\frac{\varsigma(\|u\|^\iota+\|v\|^\iota)(\|w\|^\iota+\|r\|^\iota)}{\tau},\frac{\tau}{\tau+\varsigma(\|u\|^\iota+\|v\|^\iota)(\|w\|^\iota+\|r\|^\iota)}\biggr],$$

for all $u,w\in U$, $\gamma\in\Gamma$, $\tau>0$, and $\beta=2^{\iota-1}$. □

5 Conclusions

In this paper, we introduce distribution functions and a triangular norm with the ranges in a class of matrix algebras, and we introduce the concept of a matrix Menger normed algebra. We apply the HUR stability process to get best approximation of stochastic κ-random operator inequalities.

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