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Nonlinear ordered variational inclusion problem involving XOR operation with fuzzy mappings

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Abstract

In the setting of real ordered positive Hilbert spaces, a nonlinear fuzzy ordered variational inclusion problem with its corresponding nonlinear fuzzy ordered resolvent equation problem involving XOR operation has been recommended and solved by employing an iterative algorithm. We establish the equivalence between nonlinear fuzzy ordered variational inclusion problem and nonlinear fuzzy ordered resolvent equation problem. The existence and convergence analysis of the solution of nonlinear fuzzy ordered variational inclusion problem involving XOR operation has been substantiated by applying a new resolvent operator method with XOR operation technique. The iterative algorithm and results demonstrated in this article have witnessed a significant improvement in many previously known results of this domain.

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1 Introduction

A number of solutions of nonlinear equations were introduced and studied by Amann [1] in 1972. In recent past, the fixed point theory and its applications have been intensively studied in real ordered Banach spaces. Therefore, it is very important and natural for generalized nonlinear ordered variational inequalities (ordered equations) to be studied and discussed, see [2–4]. In 1994, Hassouni and Moudafi [5] used the resolvent operator technique form maximal monotone mapping to study a class of mixed type variational inequalities with single-valued mappings, which was called variational inclusions, and developed a perturbed algorithm for finding approximate solutions of the mixed variational inequalities, see [6]. It has been proved that the theory of variational inequalities (inclusions) is quite application-oriented and thus it has been generalized in several different directions. This theory is used to solve efficiently many problems related to economics, optimization, transportation, elasticity, basic and applied sciences, etc., see [7–16] and the references therein.

In 2008, Li [17–19] studied the nonlinear ordered variational inequalities and proposed an algorithm to approximate the solution for a class of nonlinear ordered variational inequalities (ordered equations) in real ordered Banach spaces. Very recently, Ahmad et al. [20–22] considered some classes of ordered variational inclusions involving XOR operator in different settings.

A lot of work has been done by Li [17–19, 23–25] to approximate the solution of general nonlinear ordered variational inequalities and ordered equations in ordered Banach spaces. On the other hand, the fuzzy set theory due to Zadeh [26] was specifically designed to mathematically represent uncertainty and vagueness and to provide formalized tools for dealing with the imprecision intrinsic to many problems, see [10–12, 27–30].

In this paper, we consider a new resolvent operator and prove that it is single-valued, compression as well as Lipschitz continuous. We establish the equivalence between nonlinear fuzzy ordered variational inclusion problem and nonlinear fuzzy ordered resolvent equation problem. Then these new results are used to solve a nonlinear fuzzy ordered variational inclusion problem with its corresponding nonlinear fuzzy ordered resolvent equation problem involving XOR operation after defining an iterative algorithm by applying a new resolvent operator method with XOR operation technique. We claim that all the results of this paper, either preliminary or main, are the extension of results of Li [17–19, 23, 24].

2 Preliminaries

Throughout this paper, we assume that $\mathcal{H}_p$ is a real ordered positive Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Let $2^{\mathcal{H}_p}$ (respectively, $CB(\mathcal{H}_p)$) be a family of all nonempty (respectively, nonempty closed and bounded) subsets of $\mathcal{H}_p$.

Let $\mathcal{F}(\mathcal{H}_p)$ be a collection of all fuzzy sets over $\mathcal{H}_p$. A mapping $F:\mathcal{H}_p \to \mathcal{F}(\mathcal{H}_p)$ is said to be a fuzzy mapping on $\mathcal{H}_p$. For each $p \in \mathcal{H}_p$, $F(p)$ (in the sequel, it will be denoted by F_p) is a fuzzy set on $\mathcal{H}_p$ and $F_p(q)$ is the membership function of q in F_p.

A fuzzy mapping $F:\mathcal{H}_p \to \mathcal{F}(\mathcal{H}_p)$ is said to be closed if, for each $p \in \mathcal{H}_p$, the function $q \to F_p(q)$ is upper semi-continuous, that is, for any given net $\{q_\alpha\} \subset \mathcal{H}_p$, satisfying $q_\alpha \to q_0 \in \mathcal{H}_p$, we have

$$\limsup_{\alpha} F_p(q_\alpha) \le F_p(q_0).$$

For $R \in \mathcal{F}(\mathcal{H}_p)$ and $\lambda \in [0,1]$, the set $(R)_\lambda = \{p \in \mathcal{H}_p : R(p) \ge \lambda\}$ is called a λ-cut set of R. Let $F:\mathcal{H}_p \to \mathcal{F}(\mathcal{H}_p)$ be a closed fuzzy mapping satisfying the following condition.

Condition $(*)$: If there exists a function $a:\mathcal{H}_p \to [0,1]$ such that, for each $p \in \mathcal{H}_p$, the set $(F_p)_{a(p)} = \{q \in \mathcal{H}_p : F_p(q) \ge a(p)\}$ is a nonempty bounded subset of $\mathcal{H}_p$.

If F is a closed fuzzy mapping satisfying condition $(*)$, then for each $p \in \mathcal{H}_p$, $(F_p)_{a(p)} \in CB(\mathcal{H}_p)$. In fact, let $\{q_\alpha\} \subset (F_p)_{a(p)}$ be a net and $q_\alpha \to q_0 \in \mathcal{H}_p$, then $(F_p)_{a(p)} \ge a(p)$ for each α. Since F is closed, we have

$$F_p(q_0) \ge \limsup_{\alpha} F_p(q_\alpha) \ge a(p),$$

which implies that $q_0 \in (F_p)_{a(p)}$, and so $(F_p)_{a(p)} \in CB(\mathcal{H}_p)$.

For the presentation of the results, let us demonstrate some known definitions and results.

Definition 2.1 ([7, 31]) A nonempty closed convex subset C of $\mathcal{H}_p$ is said to be a cone if:
(i) for any $p \in C$ and any $\lambda > 0$, then $\lambda p \in C$;
(ii) if $p \in C$ and $-p \in C$, then $p = 0$.

Definition 2.2 ([31, 32]) A nonempty subset C of $\mathcal{H}_p$ is called
(i) a normal cone if there exists a constant $\delta_N > 0$ such that, for $0 \leq p \leq q$, we have $\|p\| \leq \delta_N \|q\|$ for any $p, q \in \mathcal{H}_p$;
(ii) for any $p, q \in \mathcal{H}_p$, $p \leq q$ if and only if $q - p \in C$;
(iii) p and q are said to be comparative to each other if and only if we have either $p \leq q$ or $q \leq p$, which is denoted by $p \propto q$.

Definition 2.3 ([31]) An ordered Hilbert space $\mathcal{H}$ is said to be a positive Hilbert space with a partially ordered relation "$\leq$" (denoted by $\mathcal{H}_p$) if, for any $p, q \in \mathcal{H}$, $p \geq 0$ and $q \geq 0$, then $\langle p, q \rangle \geq 0$.

Example 2.4 Let $\mathcal{H} = \mathbb{R}^2$ with the usual inner product and norm, and let $C = \{(p,q) | p, q \geq 0, p \leq q \text{ and } p, q \in \mathbb{R}\}$ be a closed convex subset, and let $\leq$ defined by a normal cone C be a partial ordered relation in $\mathbb{R}^2$. It is clear that $\mathbb{R}^2_p$ is a positive Hilbert space with partial ordered relation $\leq$. However, when letting $C_1 = \{(p,q) | q \geq 0, |p| \leq 4q, p, q \in \mathbb{R}\}$, then C_1 is a closed convex subset. Obviously, $\mathbb{R}^2$ is a nonpositive Hilbert space with $\leq$ because $\langle (-2.5p, p), (p, p) \rangle = -1.5p^2 < 0$ for $(-2.5p, p), (p, p) \in C_1$.

Definition 2.5 ([31]) For arbitrary elements $p, q \in \mathcal{H}_p$, $lub\{p, q\}$ and $glb\{p, q\}$ mean the least upper bound and the greatest upper bound of the set $\{p, q\}$. Suppose that $lub\{p, q\}$ and $glb\{p, q\}$ exist, some binary operations are defined as follows:
(i) $p \vee q = lub\{p, q\}$;
(ii) $p \wedge q = glb\{p, q\}$;
(iii) $p \oplus q = (p - q) \vee (q - p)$;
(iv) $p \odot q = (p - q) \wedge (q - p)$.
The operations $\oplus$, $\odot$, $\vee$, and $\wedge$ are called XOR, XNOR, OR, and AND operations, respectively.

Lemma 2.6 ([32]) *For any natural number n, $p \propto q_n$ and $q_n \to q^*$ as $n \to \infty$, then $p \propto q^*$.*

Proposition 2.7 ([17, 19, 23–25]) *Let $\odot$ be an XNOR operation and $\oplus$ be an XOR operation. Then the following relations hold*:
(i) $p \odot p = p \oplus p = 0$, $p \odot q = q \odot p = -(p \oplus q) = -(q \oplus p)$;
(ii) *if* $p \propto 0$, *then* $-p \oplus 0 \leq p \leq p \oplus 0$;
(iii) $(\lambda p) \oplus (\lambda q) = |\lambda|(p \oplus q)$;
(iv) $0 \leq p \oplus q$ *if* $p \propto q$;
(v) *if* $p \propto q$, *then* $p \oplus q = 0$ *if and only if* $p = q$;
(vi) $(p + q) \odot (u + v) \geq (p \odot u) + (q \odot v)$;
(vii) $(p + q) \odot (u + v) \geq (p \odot v) + (q \odot u)$;
(viii) *if* p, q, *and* w *are comparative to each other, then* $(p \oplus q) \leq p \oplus w + w \oplus q$;
(ix) *if* $p \propto q$, *then* $((p \oplus 0) \oplus (q \oplus 0)) \leq (p \oplus q) \oplus 0 = p \oplus q$;
(x) $\alpha p \oplus \beta p = |\alpha - \beta| p = (\alpha \oplus \beta) p$ *if* $p \propto 0$, *for all* $p, q, u, v, w \in \mathcal{H}_p$ *and* $\alpha, \beta, \lambda \in \mathbb{R}$.

Proposition 2.8 ([32]) *Let C be a normal cone in $\mathcal{H}_p$ with normal constant δ_N, then, for each $p,q \in \mathcal{H}_p$, the following relations hold*:

(i) $\|0 \oplus 0\| = \|0\| = 0$;

(ii) $\|p \vee q\| \leq \|p\| \vee \|q\| \leq \|p\| + \|q\|$;

(iii) $\|p \oplus q\| \leq \|p - q\| \leq \delta_N \|p \oplus q\|$;

(iv) *if $p \propto q$, then* $\|p \oplus q\| = \|p - q\|$.

Definition 2.9 ([24, 25]) Let $g : \mathcal{H}_p \to \mathcal{H}_p$ be a single-valued mapping. Then

(i) g is said to be a strongly comparison mapping if g is a comparison mapping and $g(p) \propto g(q)$ if and only if $p \propto q$ for all $p,q \in \mathcal{H}_p$;

(ii) g is said to be a β-ordered compression mapping if G is a comparison mapping and

$$g(p) \oplus g(q) \leq \beta(p \oplus q) \quad \text{for } 0 < \beta < 1.$$

Definition 2.10 ([23, 24]) A mapping $N : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ is said to be (κ, ν)-ordered Lipschitz continuous if $p \propto q$, $u \propto v$, then $N(p,u) \propto N(q,v)$ and there exist constants $\kappa, \nu > 0$ such that

$$N(p,u) \oplus N(q,v) \leq \kappa(p \oplus q) + \nu(u \oplus v) \quad \text{for all } p,q,u,v \in \mathcal{H}_p.$$

Definition 2.11 A set-valued mapping $A : \mathcal{H}_p \to CB(\mathcal{H}_p)$ is said to be D-Lipschitz continuous if, for any $p,q \in \mathcal{H}_p$, $p \propto q$, there exists a constant $\lambda_{D_A} > 0$ such that

$$D\big(A(p), A(q)\big) \leq \lambda_{D_A}(p \oplus q) \quad \text{for all } p,q,u,v \in \mathcal{H}_p.$$

Definition 2.12 ([17, 25, 33]) Let $A : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping. Then

(i) A is said to be a weak comparison mapping if, for any $v_p \in A(p)$, $p \propto v_p$, and if $p \propto q$, then for any $v_p \in A(p)$ and $v_q \in A(q)$, $v_p \propto v_q$ for all $p,q \in \mathcal{H}_p$;

(ii) a weak comparison mapping A is said to be α-weak-nonordinary difference mapping if, for each $p,q \in \mathcal{H}_p$, there exist $\alpha > 0$ and $v_p \in A(p)$ and $v_q \in A(q)$ such that

$$(v_p \oplus v_q) \oplus \alpha(p \oplus q) = 0;$$

(iii) a weak comparison mapping A is said to be a λ-XOR-ordered different weak compression mapping if, for each $p,q \in \mathcal{H}_p$, there exist a constant $\lambda > 0$ and $v_p \in A(p)$, $v_q \in A(q)$ such that

$$\lambda(v_p \oplus v_q) \geq p \oplus q.$$

Now, we introduce some new definitions of an XOR-weak-NODD set-valued mapping and a resolvent operator associated with the XOR-weak-NODD set-valued mapping.

Definition 2.13 A comparison mapping $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is said to be an (α, λ)-XOR-weak-NODD set-valued mapping if A is an α-weak-nonordinary difference mapping and a λ-XOR-ordered different weak compression mapping, and $[I \oplus \lambda A](\mathcal{H}_p) = \mathcal{H}_p$ for $\lambda, \beta, \alpha > 0$.

Definition 2.14 Let $A : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an (α, λ)-XOR-weak-NODD set-valued mapping. The resolvent operator $\mathcal{J}_A^\lambda : \mathcal{H}_p \to \mathcal{H}_p$ associated with A is defined by

$$\mathcal{J}_A^\lambda(p) = [I \oplus \lambda A]^{-1}(p), \quad \forall p \in \mathcal{H}_p, \tag{2.1}$$

where $\lambda > 0$ is a constant.

Now, we show that the resolvent operator defined by (2.1) is a single-valued, comparison mapping as well as Lipschitz continuous.

Lemma 2.15 *Let $A : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an α-nonordinary difference comparison mapping with $\alpha > \frac{1}{\lambda}$. Then the resolvent operator $\mathcal{J}_A^\lambda : \mathcal{H}_p \to \mathcal{H}_p$ is single-valued for all $\lambda > 0$.*

Proof Proof is similar to Proposition 2.15 in [33]. □

Lemma 2.16 *Let $A : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an (α, λ)-XOR-weak-NODD set-valued mapping with respect to $\mathcal{J}_A^\lambda$. Then the resolvent operator $\mathcal{J}_A^\lambda : \mathcal{H}_p \to \mathcal{H}_p$ is a comparison mapping.*

Proof Let A be an (α, λ)-XOR-weak-NODD set-valued mapping with respect to $\mathcal{J}_A^\lambda$. That is, A is α-nonordinary difference and λ-XOR-ordered different weak comparison mapping with respect to $\mathcal{J}_A^\lambda$, so that $p \propto \mathcal{J}_A^\lambda(p)$. For any $p, q \in \mathcal{H}_p$, let $p \propto q$ and

$$v_p = \frac{1}{\lambda}\bigl(p \oplus \mathcal{J}_A^\lambda(p)\bigr) \in A\bigl(\mathcal{J}_A^\lambda(p)\bigr) \tag{2.2}$$

and

$$v_q = \frac{1}{\lambda}\bigl(q \oplus \mathcal{J}_A^\lambda(q)\bigr) \in A\bigl(\mathcal{J}_A^\lambda(q)\bigr). \tag{2.3}$$

Since A is a λ-XOR-ordered different weak comparison mapping, using (2.2) and (2.3), we have

$$\begin{aligned}
&p \oplus q \le \lambda(v_p \oplus v_q) = \bigl(p \oplus \mathcal{J}_A^\lambda(p)\bigr) \oplus \bigl(q \oplus \mathcal{J}_A^\lambda(q)\bigr),\\
&p \oplus q \le (p \oplus q) \oplus \bigl(\mathcal{J}_A^\lambda(p) \oplus \mathcal{J}_A^\lambda(q)\bigr),\\
&0 \le \mathcal{J}_A^\lambda(p) \oplus \mathcal{J}_A^\lambda(q) = \bigl[\mathcal{J}_A^\lambda(p) - \mathcal{J}_A^\lambda(q)\bigr] \vee \bigl[\mathcal{J}_A^\lambda(q) - \mathcal{J}_A^\lambda(p)\bigr],\\
&0 \le \mathcal{J}_A^\lambda(p) - \mathcal{J}_A^\lambda(q) \quad \text{or} \quad 0 \le \mathcal{J}_A^\lambda(q) - \mathcal{J}_A^\lambda(p).
\end{aligned}$$

Thus, we have

$$\mathcal{J}_A^\lambda(p) \le \mathcal{J}_A^\lambda(q) \quad \text{or} \quad \mathcal{J}_A^\lambda(q) \le \mathcal{J}_A^\lambda(p),$$

which implies that

$$\mathcal{J}_A^\lambda(p) \propto \mathcal{J}_A^\lambda(q).$$

Therefore, the resolvent operator $\mathcal{J}_A^\lambda$ is a comparison mapping. □

Lemma 2.17 *Let $A : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an (α,λ)-XOR-weak-NODD mapping with respect to $\mathcal{J}_A^\lambda$ for $\alpha\lambda > \mu$ and $\mu \geq 1$. Then the resolvent operator $\mathcal{J}_A^\lambda$ satisfies the following condition*:

$$\mathcal{J}_A^\lambda(p) \oplus \mathcal{J}_A^\lambda(q) \leq \frac{\mu}{(\alpha\lambda \oplus \mu)}(p \oplus q), \quad \forall p,q \in \mathcal{H}_p,$$

i.e., the resolvent operator $\mathcal{J}_A^\lambda$ is $\frac{\lambda}{(\alpha\lambda\oplus\mu)}$-Lipschitz type continuous mapping.

Proof Let $p,q \in \mathcal{H}_p$, $u_p = \mathcal{J}_A^\lambda(p)$, $u_q = \mathcal{J}_A^\lambda(q)$, and let

$$v_p = \frac{1}{\lambda}(p \oplus u_p) \in A(u_p) \quad \text{and} \quad v_q = \frac{1}{\lambda}(q \oplus u_q) \in A(u_q).$$

As A is an (α,λ)-XOR-weak-NODD set-valued mapping with respect to $\mathcal{J}_A^\lambda$, it follows that A is also an α-nonordinary weak difference mapping with respect to $\mathcal{J}_A^\lambda$, we have

$$(v_p \oplus v_q) \oplus \alpha(u_p \oplus u_q) = 0 \tag{2.4}$$

and

$$\begin{aligned} v_p \oplus v_q &= \frac{1}{\lambda}\big[(p \oplus u_p) \oplus (q \oplus u_q)\big] \\ &= \frac{1}{\lambda}\big[(p \oplus q) \oplus (u_p \oplus u_q)\big] \\ &\leq \frac{\mu}{\lambda}\big[(p \oplus q) \oplus (u_p \oplus u_q)\big] \quad \text{for } \mu \geq 1. \end{aligned}$$

From (2.4), we have

$$\begin{aligned} \alpha(u_p \oplus u_q) &= v_p \oplus v_q \\ &\leq \frac{\mu}{\lambda}\big[(p \oplus q) \oplus (u_p \oplus u_q)\big], \\ \left[\frac{\alpha\lambda}{\mu} \oplus 1\right]&(u_p \oplus u_q) \leq p \oplus q. \end{aligned}$$

It follows that $u_p \oplus u_q \leq (\frac{\mu}{(\alpha\lambda\oplus\mu)})(p \oplus q)$ and, consequently, we have

$$\mathcal{J}_A^\lambda(p) \oplus \mathcal{J}_A^\lambda(q) \leq \frac{\mu}{(\alpha\lambda \oplus \mu)}(p \oplus q), \quad \forall p,q \in \mathcal{H}_p.$$

Therefore, the resolvent operator $\mathcal{J}_A^\lambda$ is a $\frac{\lambda}{(\alpha\lambda\oplus\mu)}$-Lipschitz type continuous mapping. □

Proposition 2.18 *Let $g : \mathcal{H}_p \to \mathcal{H}_p$ be a strongly comparison and β-ordered compression mapping. Let $A : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an (α,λ)-XOR-weak-NODD set-valued mapping with respect to the first argument. The resolvent operator $\mathcal{J}_A^\lambda : \mathcal{H}_p \to \mathcal{H}_p$ associated with A is defined by*

$$\mathcal{J}_{A(\cdot,z)}^\lambda(p) = \big[I \oplus \lambda A(\cdot,z)\big]^{-1}(p), \quad \text{for } z \in \mathcal{H}_p. \tag{2.5}$$

Then, for any given $z \in \mathcal{H}_p$, the resolvent operator $\mathcal{J}^{\lambda}_{A(\cdot,z)} : \mathcal{H}_p \to \mathcal{H}_p$ is well-defined, single-valued, continuous, comparison, and $\frac{\mu}{(\lambda\alpha \oplus \mu)}$-nonexpansive mapping with $\lambda\alpha > \mu$ and $\mu \geq 1$, that is,

$$\mathcal{J}^{\lambda}_{A(\cdot,z)}(p) \oplus \mathcal{J}^{\lambda}_{A(\cdot,z)}(q) \leq \frac{\mu}{(\lambda\alpha \oplus \mu)}(p \oplus q), \quad \textit{for all } p,q \in \mathcal{H}_p. \tag{2.6}$$

3 Formulation of the problem and iterative algorithm

Let $\mathcal{H}_p$ be a real ordered positive Hilbert space and C be a normal cone with normal constant δ_N. Let $S,T,U,V : \mathcal{H}_p \to \mathcal{F}(\mathcal{H}_p)$ be closed fuzzy mappings satisfying the following condition $(*)$, with functions $a,b,c,d : \mathcal{H} \to [0,1]$ such that, for each $p \in \mathcal{H}_p$, we have $(S_p)_{a(p)}$, $(T_p)_{b(p)}$, $(U_p)_{c(p)}$, and $(V_p)_{d(p)}$ in $CB(\mathcal{H}_p)$, respectively, and let $G,g : \mathcal{H}_p \to \mathcal{H}_p$ be surjective single-valued mappings. Let $A : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an (α,λ)-XOR-weak-NODD set-valued mapping with respect to the first argument. For a given nonlinear mapping $N : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$, we consider a problem of finding $p,u,v,w,z \in \mathcal{H}_p$ such that $S_p(u) \geq a(p)$, $T_p(v) \geq b(p)$, $U_p(w) \geq c(w)$, and $V_p(z) \geq d(z)$, i.e., $u \in (S_p)_{a(p)}$, $v \in (T_p)_{b(p)}$, $w \in (U_p)_{c(p)}$, $z \in (V_p)_{d(p)}$,

$$0 \in G(w) \oplus N(u,v) + A\bigl(g(p),z\bigr). \tag{3.1}$$

Problem (3.1) is called nonlinear fuzzy ordered variational inclusion problem involving $\oplus$ operation.

It is clear that, for suitable choices of mappings involved in the formulation of nonlinear fuzzy ordered variational inclusion problem (3.1), one can obtain many variational inclusion problems studied in recent past, i.e., [17–19, 25].

Putting $a(p) = b(p) = c(p) = d(p) = 1$ for all $p \in \mathcal{H}_p$, problem (3.1) includes many kinds of variational inequalities and variational inclusion problems [17, 19, 22–25].

In support of our problem (3.1), we provide the following examples.

Example 3.1 The continuum of players problem can be obtained from nonlinear fuzzy ordered variational inclusion problem (3.1). For more details, see Chap. 13 and exercise 13.2 of the book "Optima and Equilibria" by Aubin [34] and Example 2.1 in [35].

If we can take $\mathcal{H}_p = \mathbb{R}^n_p$, $U = I$ (identity mapping) and T is a single-valued mapping, and the other functions, that is, G,S,V,A,g, are equal to zero. Define $N : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ by

$$N\bigl(p,T(p)\bigr) = \int_L Q\bigl(u,T(u)\bigr)h(u)\,du.$$

We associate each player with its action $Q(u,\cdot)$, where $Q : \mathcal{H}_p \times O \to \mathbb{R}^n_p$, O is a nonempty subset of $\mathbb{R}^n_p$, and each fuzzy coalition $h(u)$ with its action $\int_L Q(u,T(u))h(u)\,du$.

Example 3.2 Let $\mathcal{H}_p = [0,10]$ and $C = \{p \in \mathcal{H}_p : 0 \leq p \leq 4\}$ be the normal cone. Let $S,T,U,V : \mathcal{H}_p \to \mathcal{F}(\mathcal{H}_p)$ be the closed fuzzy mappings defined by, for all $p,q,u,v,w,z \in \mathcal{H}_p$:

$$S_p(u) = \begin{cases} \frac{1}{3+|u-2|} & \text{if } p \in [0,1], \\ \frac{1}{3+p|u-2|} & \text{if } p \in (1,10], \end{cases} \qquad T_p(v) = \begin{cases} \frac{1}{2+(v-2)^2} & \text{if } p \in [0,1], \\ \frac{1}{2+p(v-2)^2} & \text{if } p \in (1,10], \end{cases}$$

$$U_p(w) = \begin{cases} \frac{1}{1+|w-2|} & \text{if } p \in [0,1], \\ \frac{1}{1+p|w-2|} & \text{if } p \in (1,10], \end{cases} \quad \text{and} \quad V_p(z) = \begin{cases} \frac{1}{2+p|z-2|} & \text{if } p \in [0,1], \\ \frac{1}{2+|z-2|} & \text{if } p \in (1,10]. \end{cases}$$

We define the mappings $a, b, c, d : \mathcal{H}_p \to [0,1]$ by

$$a(p) = \begin{cases} \frac{1}{5} & \text{if } p \in [0,1], \\ \frac{1}{3+2p} & \text{if } p \in (1,10], \end{cases} \qquad b(p) = \begin{cases} \frac{1}{6} & \text{if } p \in [0,1], \\ \frac{1}{2+4p} & \text{if } p \in (1,10], \end{cases}$$

$$c(p) = \begin{cases} \frac{1}{3} & \text{if } p \in [0,1], \\ \frac{1}{1+2p} & \text{if } p \in (1,10], \end{cases} \quad \text{and} \quad d(p) = \begin{cases} \frac{1}{2(1+p)} & \text{if } p \in [0,1], \\ \frac{1}{4} & \text{if } p \in (1,10]. \end{cases}$$

For any $p \in [0,1]$, we have

$$(S_p)_{a(p)} = \left\{u : S_p(u) \geq \frac{1}{5}\right\} = \{\left\{u : \frac{1}{3+|u-2|} \geq \frac{1}{5}\right\} = [0,4],$$

$$(T_p)_{b(p)} = \left\{u : T_p(v) \geq \frac{1}{6}\right\} = \left\{u : \frac{1}{2+(v-2)^2} \geq \frac{1}{6}\right\} = [0,4],$$

$$(U_p)_{c(p)} = \left\{u : U_p(w) \geq \frac{1}{3}\right\} = \left\{u : \frac{1}{1+|w-2|} \geq \frac{1}{3}\right\} = [0,4],$$

$$(V_p)_{d(p)} = \left\{u : V_p(z) \geq \frac{1}{2(1+p)}\right\} = \left\{u : \frac{1}{2+p|z-2|} \geq \frac{1}{2(1+p)}\right\} = [0,4],$$

and for any $p \in (1,10]$, we have

$$(S_p)_{a(p)} = \left\{u : S_p(u) \geq \frac{1}{3+2p}\right\} = \left\{u : \frac{1}{3+p|u-2|} \geq \frac{1}{3+2p}\right\} = [0,4],$$

$$(T_p)_{b(p)} = \left\{u : T_p(v) \geq \frac{1}{2+4p}\right\} = \left\{u : \frac{1}{2+2p(v-2)^2} \geq \frac{1}{2+4p}\right\} = [0,4],$$

$$(U_p)_{c(p)} = \left\{u : U_p(w) \geq \frac{1}{1+2p}\right\} = \left\{u : \frac{1}{1+p|w-2|} \geq \frac{1}{1+2p}\right\} = [0,4],$$

$$(V_p)_{d(p)} = \left\{u : V_p(z) \geq \frac{1}{4}\right\} = \left\{u : \frac{1}{2+|z-2|} \geq \frac{1}{4}\right\} = [0,4].$$

Now, we define the single-valued mappings $G, g : \mathcal{H}_p \to \mathcal{H}_p$ and $N : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ by

$$G(w) = \frac{w}{2}, \qquad g(p) = \frac{p}{5} \quad \text{and} \quad N(u,v) = \frac{u}{2} + \frac{v}{3},$$

and the set-valued mapping $A : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is defined by

$$A\big(g(p), z\big) = \left\{g(p) + \frac{z}{3} : p \in [0,10] \text{ and } z \in (V_p)_{d(p)}\right\}.$$

In view of the above, it is easy to verify that $0 \in G(w) \oplus N(u,v) + A(g(p),z)$, that is, problem (3.1) is satisfied.

Related to the nonlinear fuzzy ordered variational inclusion problem (3.1), we consider the following nonlinear fuzzy ordered resolvent equation problem:

Find $p, q, u, v, w, z \in \mathcal{H}_p$ such that $(S_p)(u) \geq a(p)$, $(T_p)(v) \geq b(p)$, $(U_p)(w) \geq c(p)$, and $(V_p)(z) \geq d(z)$,

$$G(w) \oplus \lambda^{-1}\mathcal{R}_{A(\cdot,z)}(q) = N(u,v), \tag{3.2}$$

where $\lambda > 0$ is a constant and $\mathcal{R}_{A(\cdot,z)} = [I \oplus \mathcal{J}^{\lambda}_{A(\cdot,z)}]$.

Now, we establish the equivalence between nonlinear fuzzy ordered variational inclusion problem and nonlinear fuzzy ordered resolvent equation problem.

Lemma 3.3 *Assume that $S, T, U, V : \mathcal{H}_p \to \mathcal{F}(\mathcal{H}_p)$ are closed fuzzy mappings satisfying the following condition $(*)$, with functions $a, b, c, d : \mathcal{H} \to [0,1]$, respectively. Let $G, g : \mathcal{H}_p \to \mathcal{H}_p$ and $N : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be single-valued mappings. Let $A : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}}_p$ be an (α, λ)-XOR-weak-NODD set-valued mapping with respect to the first argument. Then the following are equivalent*:

(i) *(p, u, v, w, z), where $p, u, v, w, z \in \mathcal{H}_p$ such that $S_p(u) \geq a(p)$, $T_p(v) \geq b(p)$, $U_p(w) \geq c(w)$, and $V_p(z) \geq d(z)$ is a solution of problem* (3.1);

(ii) *$p \in \mathcal{H}_p$ is a fixed point of the mapping $Q : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ defined by*

$$Q(p) = G(w) \oplus N(u,v) + A\bigl(g(p), z\bigr) + p; \tag{3.3}$$

(iii) *(p, u, v, w, z), where $p, u, v, w, z \in \mathcal{H}_p$ such that $S_p(u) \geq a(p)$, $T_p(v) \geq b(p)$, $U_p(w) \geq c(w)$, and $V_p(z) \geq d(z)$, is a solution of the following equation*:

$$g(p) = \mathcal{J}^{\lambda}_{A(\cdot,z)}\bigl[g(p) \oplus \lambda\bigl(G(w) \odot N(u,v)\bigr)\bigr]; \tag{3.4}$$

(iv) *(p, q, u, v, w, z), where $p, u, v, w, z \in \mathcal{H}_p$ such that $S_p(u) \geq a(p)$, $T_p(v) \geq b(p)$, $U_p(w) \geq c(w)$, and $V_p(z) \geq d(z)$, is a solution of problem* (3.2), *where*

$$\begin{aligned} &q = g(p) \oplus \lambda\bigl(G(w) \odot N(u,v)\bigr), \\ &g(p) = \mathcal{J}^{\lambda}_{A(\cdot,z)}(q). \end{aligned} \tag{3.5}$$

Proof (i) $\Longrightarrow$ (ii) Adding p to both sides of (3.1), we have

$$\begin{aligned} &0 \in G(w) \oplus N(u,v) + A\bigl(g(p), z\bigr) \\ &\quad\Longrightarrow\quad p \in G(w) \oplus N(u,v) + A\bigl(g(p), z\bigr) + p = Q(p). \end{aligned}$$

Hence, a is a fixed point of Q.

(ii) $\Longrightarrow$ (iii) Let p be a fixed point of Q, then

$$\begin{aligned} &p \in G(w) \oplus N(u,v) + A\bigl(g(p), z\bigr) + p = Q(p) \\ &\quad\Longrightarrow\quad 0 \in G(w) \oplus N(u,v) + A\bigl(g(p), z\bigr) \\ &\quad\Longrightarrow\quad 0 \in \lambda\bigl(G(w) \oplus N(u,v)\bigr) + \lambda A\bigl(g(p), z\bigr) \\ &\quad\Longrightarrow\quad \lambda\bigl(G(w) \odot N(u,v)\bigr) \in \lambda A\bigl(g(p), z\bigr) \\ &\quad\Longrightarrow\quad g(p) \oplus \lambda\bigl(G(w) \odot N(u,v)\bigr) \in g(p) \oplus \lambda A\bigl(g(p), z\bigr) \end{aligned}$$

$$\implies \quad g(p) \oplus \lambda\big(G(w) \odot N(u,v)\big) \in \big[I \oplus \lambda A(\cdot,z)\big]\big(g(p)\big).$$

Hence $g(p) = \mathcal{J}^{\lambda}_{A(\cdot,z)}[g(p) \oplus \lambda(G(w) \odot N(u,v))]$.

(iii) $\implies$ (iv) Taking $q = g(p) \oplus \lambda(G(w) \odot N(u,v))$, from (3.4), we have $g(p) = \mathcal{J}^{\lambda}_{A(\cdot,z)}(q)$, so

$$q = \mathcal{J}^{\lambda}_{A(\cdot,z)}(q) \oplus \lambda\big(G(w) \odot N(u,v)\big),$$

which implies that

$$\begin{aligned}
&q \oplus \mathcal{J}^{\lambda}_{A(\cdot,z)}(q) = \lambda\big(G(w) \odot N(u,v)\big),\\
&\quad \implies \quad \big[I \oplus \mathcal{J}^{\lambda}_{A(\cdot,z)}\big](q) = \lambda\big(G(w) \odot N(u,v)\big)\\
&\quad \implies \quad \mathcal{R}_{A(\cdot,z)}(q) = \lambda\big(G(w) \odot N(u,v)\big)\\
&\quad \implies \quad \lambda^{-1}\mathcal{R}_{A(\cdot,z)}(q) = G(w) \odot N(u,v)\\
&\quad \implies \quad G(w) \odot \lambda^{-1}\mathcal{R}_{A(\cdot,z)}(q) = N(u,v).
\end{aligned}$$

Consequently, (p,q,u,v,w,z) is a solution of the fuzzy resolvent equation problem (3.2).

(iv) $\implies$ (i), from (3.5) we have

$$\begin{aligned}
g(q) &= \mathcal{J}^{\lambda}_{A(\cdot,z)}(q)\\
&= \mathcal{J}^{\lambda}_{A(\cdot,z)}\big[g(p) \oplus \lambda\big(G(w) \odot N(u,v)\big)\big],
\end{aligned}$$

i.e.,

$$g(p) = \big(I \oplus \lambda A(\cdot,z)\big)^{-1}\big[g(p) \oplus \lambda\big(G(w) \odot N(u,v)\big)\big],$$

so

$$g(p) \oplus \lambda\big(G(w) \odot N(u,v)\big) \in \big(I \oplus \lambda A(\cdot,z)\big)g(p),$$

which implies

$$0 \in G(w) \oplus N(u,v) + A\big(g(p),z\big).$$

Therefore, (p,u,v,w,z), where $p \in \mathcal{H}_p$ such that $S_p(u) \geq a(p)$, $T_p(v) \geq b(p)$, $U_p(w) \geq c(w)$, and $V_p(z) \geq d(z)$ is a solution of problem (3.1). □

Based on Lemma 3.3, we construct an iterative algorithm for finding approximate solutions of problem (3.1).

Iterative Algorithm 3.4 Let $S,T,U,V : \mathcal{H}_p \to \mathcal{F}(\mathcal{H}_p)$ be the closed fuzzy mappings satisfying the following condition $(*)$, with functions $a,b,c,d : \mathcal{H} \to [0,1]$, respectively. Let $G,g : \mathcal{H}_p \to \mathcal{H}_p$ and $N : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be single-valued mappings. Let $A : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an (α,λ)-XOR-weak-NODD set-valued mapping with respect to the first argument.

We assume that g is surjective. For any given $p_0, q_0 \in \mathcal{H}_p$, $u_0 \in (S_{p_0})_{a(p_0)}$, $v_0 \in (T_{p_0})_{b(p_0)}$, $w_0 \in (U_{p_0})_{c(p_0)}$, and $z_0 \in (V_{p_0})_{d(p_0)}$, let

$$q_1 = g(p_0) \oplus \lambda\bigl(G(w_0) \odot N(u_0, v_0)\bigr).$$

Since g is surjective, there exists $p_1 \in \mathcal{H}_p$ such that

$$p_1 = (1-\beta)q_0 + \beta\bigl[p_0 \oplus \bigl(g(p_0) \oplus \mathcal{J}^{\lambda}_{A(\cdot,z_0)}(q_1)\bigr)\bigr].$$

On the other hand, by Nadler [36], there exist $u_1 \in (S_{p_1})_{a(p_1)}$, $v_0 \in (T_{p_1})_{b(p_1)}$, $w_0 \in (U_{p_1})_{c(p_1)}$, and $z_1 \in (V_{p_1})_{d(p_1)}$, and suppose that $p_0 \propto p_1$, $u_0 \propto u_1$, $v_0 \propto v_1$, $w_0 \propto w_1$, and $z_0 \propto z_1$ such that

$$\begin{aligned}
&u_1 \in (S_{p_1})_{a(p_1)}, \quad u_1 \oplus u_0 \leq (1+1)D\bigl((S_{p_1})_{a(p_1)}, (S_{p_0})_{a(p_0)}\bigr),\\
&v_1 \in (T_{p_1})_{b(p_1)}, \quad v_1 \oplus v_0 \leq (1+1)D\bigl((T_{p_1})_{b(p_1)}, (T_{p_0})_{b(p_0)}\bigr),\\
&w_1 \in (U_{p_1})_{c(p_1)}, \quad w_1 \oplus w_0 \leq (1+1)D\bigl((U_{p_1})_{c(p_1)}, (U_{p_0})_{c(p_0)}\bigr),\\
&z_1 \in (V_{p_1})_{d(p_1)}, \quad z_1 \oplus z_0 \leq (1+1)D\bigl((V_{p_1})_{d(p_1)}, (V_{p_0})_{d(p_0)}\bigr).
\end{aligned}$$

Let

$$q_2 = g(p_1) \oplus \lambda\bigl(G(w_1) \odot N(u_1, v_1)\bigr).$$

Since g is surjective, there exists $p_2 \in \mathcal{H}_p$ such that

$$p_2 = (1-\beta)p_1 + \beta\bigl[p_1 \oplus \bigl(g(p_1) \oplus \mathcal{J}^{\lambda}_{A(\cdot,z_1)}(q_2)\bigr)\bigr].$$

On the other hand, by Nadler [36], there exist $u_2 \in (S_{p_2})_{a(p_2)}$, $v_2 \in (T_{p_2})_{b(p_2)}$, $w_2 \in (U_{p_2})_{c(p_2)}$, and $z_2 \in (V_{p_2})_{d(p_2)}$, and suppose that $p_1 \propto p_2$, $u_1 \propto u_2$, $v_1 \propto v_2$, $w_1 \propto w_2$, and $z_1 \propto z_2$ such that

$$\begin{aligned}
&u_2 \in (S_{p_2})_{a(p_2)}, \quad u_2 \oplus u_1 \leq \left(1+\frac{1}{2}\right)D\bigl((S_{p_2})_{a(p_2)}, (S_{p_1})_{a(p_1)}\bigr),\\
&v_2 \in (T_{p_2})_{b(p_2)}, \quad v_2 \oplus v_1 \leq \left(1+\frac{1}{2}\right)D\bigl((T_{p_2})_{b(p_2)}, (T_{p_1})_{b(p_1)}\bigr),\\
&w_2 \in (U_{p_2})_{c(p_2)}, \quad w_2 \oplus w_1 \leq \left(1+\frac{1}{2}\right)D\bigl((U_{p_2})_{c(p_2)}, (U_{p_1})_{c(p_1)}\bigr),\\
&z_2 \in (V_{p_2})_{d(p_2)}, \quad z_2 \oplus z_1 \leq \left(1+\frac{1}{2}\right)D\bigl((V_{p_2})_{d(p_2)}, (V_{p_1})_{d(p_1)}\bigr).
\end{aligned}$$

Continuing the above process inductively with the supposition that $p_{n+1} \propto p_n$, $u_{n+1} \propto u_n$, $v_{n+1} \propto v_n$, $w_{n+1} \propto w_n$, and $z_{n+1} \propto z_n$, for all $n = 0, 1, 2, \ldots$,

$$\begin{cases} q_{n+1} = g(p_n) \oplus \lambda(G(w_n) \odot N(u_n, v_n)), \\ p_{n+1} = (1-\beta)p_n + \beta[p_n \oplus (g(p_n) \oplus \mathcal{J}^{\lambda}_{A(\cdot,z_n)}(q_{n+1}))], \\ u_{n+1} \in (S_{p_{n+1}})_{a(p_{n+1})}, \qquad u_{n+1} \oplus u_n \leq (1+\frac{1}{n+1})D((S_{p_{n+1}})_{a(p_{n+1})}, (S_{p_n})_{a(p_n)}), \\ v_{n+1} \in (T_{p_{n+1}})_{b(p_{n+1})}, \qquad v_{n+1} \oplus v_n \leq (1+\frac{1}{n+1})D((T_{p_{n+1}})_{b(p_{n+1})}, (T_{p_n})_{b(p_n)}), \\ w_{n+1} \in (U_{p_{n+1}})_{c(p_{n+1})}, \qquad w_{n+1} \oplus w_n \leq (1+\frac{1}{n+1})D((U_{p_{n+1}})_{c(p_{n+1})}, (U_{p_n})_{c(p_n)}), \\ z_{n+1} \in (V_{p_{n+1}})_{d(p_{n+1})}, \qquad z_{n+1} \oplus z_n \leq (1+\frac{1}{n+1})D((V_{p_{n+1}})_{d(p_{n+1})}, (V_{p_n})_{d(p_n)}). \end{cases} \tag{3.6}$$

4 Main results

In this section, we prove an existence and convergence result for nonlinear fuzzy ordered variational inclusion problem (3.1) and its corresponding nonlinear fuzzy ordered resolvent equation problem (3.2).

Theorem 4.1 *Let $\mathcal{H}_p$ be a real ordered positive Hilbert space and C be a normal cone with normal constant δ_N. Let $S, T, U, V : \mathcal{H}_p \to \mathcal{F}(\mathcal{H}_p)$ be the closed fuzzy mappings satisfying condition $(*)$, with functions $a, b, c, d : \mathcal{H} \to [0,1]$, respectively. Let $G, g : \mathcal{H}_p \to \mathcal{H}_p$ and $N : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings. Let $A : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an (α, λ)-XOR-weak-NODD set-valued mapping with respect to the first argument. Suppose that the following conditions hold:*

- (i) *G is comparison and λ_G-ordered compression mapping, $\lambda_G \in (0,1)$;*
- (ii) *g is comparison and λ_g-ordered compression mapping, $\lambda_g \in (0,1)$;*
- (iii) *N is comparison and (κ, ν)-ordered Lipschitz continuous mapping;*
- (iv) *S, T, U, and V are ordered Lipschitz type continuous mappings with constants λ_{D_S}, λ_{D_T}, λ_{D_U}, and λ_{D_V}, respectively.*

If the following conditions

$$\text{(a)} \quad \mathcal{J}^{\lambda}_{A(\cdot,s)}(p) \oplus \mathcal{J}^{\lambda}_{A(\cdot,t)}(p) \leq \xi(s \oplus t) \quad \text{for all } p, s, t \in \mathcal{H}_p, \xi > 0,$$

$$\text{(b)} \quad \begin{cases} |\mu(\lambda_g \oplus \lambda(\lambda_G \lambda_{D_U} \oplus (\kappa \lambda_{D_S} + \nu \lambda_{D_T}))) + \xi \lambda_{D_V}(\lambda\alpha \oplus \mu)| < |(\lambda\alpha \oplus \mu)\lambda_g|, \\ \lambda\alpha > \mu, \quad \mu \geq 1, \end{cases} \tag{4.1}$$

are satisfied, then there exist $p, q \in \mathcal{H}_p$ such that $u \in (S_p)_{a(p)}$, $v \in (T_p)_{b(p)}$, $w \in (U_p)_{c(p)}$, and $z \in (V_p)_{d(p)}$ satisfying the nonlinear fuzzy ordered resolvent equation equation (3.2), and so (p, u, v, w, z) is a solution of the nonlinear fuzzy ordered variational inclusion problem (3.1), and the iterative sequences $\{p_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, and $\{z_n\}$ generated by Algorithm 3.4 converge strongly to p, v, u, w, and z in $\mathcal{H}_p$, respectively.

Proof Since g is a λ_g-ordered compression mapping, V is a λ_{D_V}-ordered Lipschitz continuous mapping. By Algorithm 3.4, Proposition 2.7, and Proposition 2.18, we have

$$\begin{aligned} 0 &\leq p_{n+1} \oplus p_n \\ &= \big[(1-\beta)p_n + \beta\big(p_n \oplus \big(g(p_n) \oplus \mathcal{J}^{\lambda}_{A(\cdot,z_n)}(q_{n+1})\big)\big)\big] \\ &\quad \oplus \big[(1-\beta)p_{n-1} + \beta\big(p_{n-1} \oplus \big(g(p_{n-1}) \oplus \mathcal{J}^{\lambda}_{A(\cdot,z_{n-1})}(q_n)\big)\big)\big] \end{aligned}$$

$$\begin{aligned}
&\le (1-\beta)(p_n \oplus q_{n-1}) + \beta\big[\big(p_n \oplus \big(g(p_n) \oplus \mathcal{J}^{\lambda}_{A(\cdot,z_n)}(q_{n+1})\big)\big) \\
&\quad \oplus \big(p_{n-1} \oplus \big(g(p_{n-1}) \oplus \mathcal{J}^{\lambda}_{A(\cdot,z_{n-1})}(q_n)\big)\big)\big] \\
&= (1-\beta)(p_n \oplus p_{n-1}) + \beta\big[(1 \oplus \lambda_g)(p_n \oplus p_{n-1}) \oplus \big(\mathcal{J}^{\lambda}_{A(\cdot,z_n)}(q_{n+1}) \\
&\quad \oplus \mathcal{J}^{\lambda}_{A(\cdot,z_{n-1})}(q_n)\big)\big] \\
&= (1-\beta)(p_n \oplus p_{n-1}) + \beta\big[(1 \oplus \lambda_g)(p_n \oplus p_{n-1}) \\
&\quad \oplus \big(\mathcal{J}^{\lambda}_{A(\cdot,z_n)}(q_{n+1}) \oplus \mathcal{J}^{\lambda}_{A(\cdot,z_n)}(q_n) + \mathcal{J}^{\lambda}_{A(\cdot,z_n)}(q_n) \oplus \mathcal{J}^{\lambda}_{A(\cdot,z_{n-1})}(q_n)\big)\big] \\
&= (1-\beta)(p_n \oplus p_{n-1}) + \beta\left[(1 \oplus \lambda_g)(p_n \oplus p_{n-1}) \oplus \left(\frac{\mu}{(\lambda\alpha \oplus \mu)}(q_{n+1} \oplus q_n)\right.\right. \\
&\quad \left.\left. + \xi(z_n \oplus z_{n-1})\right)\right] \\
&= (1-\beta)(p_n \oplus p_{n-1}) + \beta\left[(1 \oplus \lambda_g)(p_n \oplus p_{n-1})\right. \\
&\quad \left. \oplus \left(\frac{\mu}{(\lambda\alpha \oplus \mu)}(q_{n+1} \oplus q_n) + \xi\left(1 + \frac{1}{n+1}\right)D\big((V_{p_{n+1}})_{d(p_{n+1})}, (V_{p_{n+1}})_{d(p_{n+1})}\big)\right)\right] \\
&= (1-\beta)(p_n \oplus p_{n-1}) + \beta\left[(1 \oplus \lambda_g)(p_n \oplus p_{n-1})\right. \\
&\quad \left. \oplus \left(\frac{\mu}{(\lambda\alpha \oplus \mu)}(q_{n+1} \oplus q_n) + \xi\left(1 + \frac{1}{n+1}\right)\lambda_{D_V}(p_{n+1} \oplus p_{n+1})\right)\right].
\end{aligned} \tag{4.2}$$

Since G is a λ_G-ordered compression mapping, $N(\cdot,\cdot)$ is a (κ,ν)-ordered Lipschitz continuous and g is a λ_G-ordered compression mapping and D-Lipschitz continuous of T, U, and V with constants $\lambda_{D_T}, \lambda_{D_U}$, and λ_{D_V}. Using Proposition 2.7, we have

$$\begin{aligned}
q_{n+1} \oplus q_n &= \big[g(p_n) \oplus \lambda\big(G(w_n) \odot N(u_n, v_n)\big)\big] \oplus \big[g(p_{n-1}) \oplus \lambda\big(G(w_{n-1}) \\
&\quad \odot N(u_{n-1}, v_{n-1})\big)\big] \\
&\le \big(g(p_n) \oplus g(p_{n-1})\big) \oplus \lambda\big[\big(G(w_n) \odot N(u_n, v_n)\big) \oplus \big(G(w_{n-1}) \\
&\quad \odot N(u_{n-1}, v_{n-1})\big)\big] \\
&\le \big(g(p_n) \oplus g(p_{n-1})\big) \oplus \lambda\big[-\big(G(w_n) \oplus N(u_n, v_n)\big) \oplus -\big(G(w_{n-1}) \\
&\quad \oplus N(u_{n-1}, v_{n-1})\big)\big] \\
&\le \big(g(p_n) \oplus g(p_{n-1})\big) \oplus \lambda|-1|\big[\big(G(w_n) \oplus N(u_n, v_n)\big) \oplus \big(G(w_{n-1}) \\
&\quad \oplus N(u_{n-1}, v_{n-1})\big)\big] \\
&\le \lambda_g(p_n \oplus p_{n-1}) \oplus \lambda\big[\big(G(w_n) \oplus G(w_{n-1})\big) \oplus \big(N(u_n, v_n) \\
&\quad \oplus N(u_{n-1}, v_{n-1})\big)\big] \\
&\le \lambda_g(p_n \oplus p_{n-1}) \oplus \lambda\big[\lambda_G(w_n \oplus w_{n-1}) \oplus \big(\kappa(u_n \oplus u_{n-1}) + \nu(v_n \oplus v_{n-1})\big)\big] \\
&\le \lambda_g(p_n \oplus p_{n-1}) \oplus \lambda\left[\lambda_G\left(1 + \frac{1}{n+1}\right)D\big((U_{p_{n+1}})_{c(p_{n+1})}, (U_{p_n})_{c(p_n)}\big)\right. \\
&\quad \oplus \left(\kappa\left(1 + \frac{1}{n+1}\right)D\big((S_{p_{n+1}})_{a(p_{n+1})}, (S_{p_n})_{a(p_n)}\big)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\nu\left(1+\frac{1}{n+1}\right)D\Big((T_{p_{n+1}})_{b(p_n+1)},(T_{p_n})_{b(p_n)}\Big)\Big)\Big]\\
&\leq \lambda_g(p_n\oplus p_{n-1})\oplus\lambda[\lambda_G\lambda_{D_U}\left(1+\frac{1}{n+1}\right)(p_n\oplus p_{n-1})\\
&\quad\oplus\left(\kappa\lambda_{D_S}\left(1+\frac{1}{n+1}\right)(p_n\oplus p_{n-1})+\nu\lambda_{D_T}\left(1+\frac{1}{n+1}\right)(p_n\oplus p_{n-1})\right)\\
&\leq\left[\lambda_g\oplus\lambda\big(\lambda_G\lambda_{D_U}\oplus(\kappa\lambda_{D_S}+\nu\lambda_{D_T})\big)\left(1+\frac{1}{n+1}\right)\right](p_n\oplus p_{n-1}).
\end{aligned}
\tag{4.3}
$$

Using (4.3), (4.2) becomes

$$
\begin{aligned}
0&\leq p_{n+1}\oplus p_n\\
&=\left[(1-\beta)+\beta\left((1\oplus\lambda_g)\oplus\left(\frac{\mu(\lambda_g\oplus\lambda(\lambda_G\lambda_{D_U}\oplus(\kappa\lambda_{D_S}+\nu\lambda_{D_T})))}{(\lambda\alpha\oplus\mu)}\left(1+\frac{1}{n+1}\right)\right.\right.\right.\\
&\quad\left.\left.\left.+\xi\left(1+\frac{1}{n+1}\right)\lambda_{D_V}\right)\right)\right](p_n\oplus p_{n-1})\\
&=\left[(1-\beta\left(1-\left((1\oplus\lambda_g)\oplus\left(\left(\frac{\mu(\lambda_g\oplus\lambda(\lambda_G\lambda_{D_U}\oplus(\kappa\lambda_{D_S}+\nu\lambda_{D_T})))}{(\lambda\alpha\oplus\mu)}\right.\right.\right.\right.\right.\\
&\quad\left.\left.\left.\left.\left.+\xi\lambda_{D_V}\right)\left(1+\frac{1}{n+1}\right)\right)\right)\right)\right](p_n\oplus p_{n-1})\\
&=\Omega_n(p_n\oplus p_{n-1}),
\end{aligned}
\tag{4.4}
$$

where

$$
\begin{aligned}
\Omega_n=&\left[1-\beta\left(1-(1\oplus\lambda_g)\oplus\left(\left(\frac{\mu(\lambda_g\oplus\lambda(\lambda_G\lambda_{D_U}\oplus(\kappa\lambda_{D_S}+\nu\lambda_{D_T})))}{(\lambda\alpha\oplus\mu)}\right.\right.\right.\right.\\
&\left.\left.\left.\left.+\xi\lambda_{D_V}\right)\left(1+\frac{1}{n+1}\right)\right)\right)\right].
\end{aligned}
$$

Letting

$$
\Omega=\left[1-\beta\left(1-(1\oplus\lambda_g)\oplus\left(\left(\frac{\mu(\lambda_g\oplus\lambda(\lambda_G\lambda_{D_U}\oplus(\kappa\lambda_{D_S}+\nu\lambda_{D_T})))}{(\lambda\alpha\oplus\mu)}+\xi\lambda_{D_V}\right)\right)\right)\right].
$$

By condition (4.1), we have $0<\Omega<1$, thus $\{p_n\}$ is a Cauchy sequence in $\mathcal{H}_p$ and as $\mathcal{H}_p$ is complete, there exists $p\in\mathcal{H}_p$ such that $p_n\to p$ as $n\to\infty$. From (3.6) of Algorithm 3.4 and D-Lipschitz continuity of S, T, U, and V, we have

$$
\begin{aligned}
u_{n+1}\oplus u_n&\leq\left(1+\frac{1}{n+1}\right)D\big((S_{p_{n+1}})_{a(p_{n+1})},(S_{p_n})_{a(p_n)}\big)\\
&\leq\left(1+\frac{1}{n+1}\right)\lambda_{D_S}(p_{n+1}\oplus p_n),
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
v_{n+1}\oplus v_n&\leq\left(1+\frac{1}{n+1}\right)D\big((T_{p_{n+1}})_{b(p_{n+1})},(T_{p_n})_{b(p_n)}\big)\\
&\leq\left(1+\frac{1}{n+1}\right)\lambda_{D_T}(p_{n+1}\oplus p_n),
\end{aligned}
\tag{4.6}
$$

$$w_{n+1} \oplus w_n \leq \left(1 + \frac{1}{n+1}\right) D\big((U_{p_{n+1}})_{c(p_{n+1})}, (U_{p_n})_{c(p_n)}\big)$$
$$\leq \left(1 + \frac{1}{n+1}\right) \lambda_{D_U} (p_{n+1} \oplus p_n), \tag{4.7}$$
$$z_{n+1} \oplus z_n \leq \left(1 + \frac{1}{n+1}\right) D\big((V_{p_{n+1}})_{d(p_{n+1})}, (V_{p_n})_{d(p_n)}\big)$$
$$\leq \left(1 + \frac{1}{n+1}\right) \lambda_{D_V} (p_{n+1} \oplus p_n). \tag{4.8}$$

It is clear from (4.5)–(4.8) that $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, and $\{z_n\}$ are also Cauchy sequences in $\mathcal{H}_p$, so there exist u, v, w, and z in $\mathcal{H}_p$ such that $u_n \to u$, $v_n \to v$, $w_n \to w$, and $z_n \to z$ as $n \to \infty$. By using the continuity of the operators S, T, U, V, $\mathcal{J}^{\lambda}_{A(\cdot,z)}$ and iterative Algorithm 3.4, we have

$$p = (1-\beta)p + \beta\big[p \oplus \big(g(p) \oplus \mathcal{J}^{\lambda}_{A(\cdot,z)}\big(g(p) \oplus \lambda\big(G(w) \odot N(u,v)\big)\big)\big)\big],$$

which implies that

$$g(p) = \mathcal{J}^{\lambda}_{A(\cdot,z)}\big(g(p) \oplus \lambda\big(G(w) \odot N(u,v)\big)\big).$$

By Lemma 3.3, we conclude that (p,u,v,w,z) is a solution of problem (3.1). It remains to show that $u \in (S_p)_{a(p)}$, $v \in (T_p)_{b(p)}$, $w \in (U_p)_{c(p)}$, and $z \in (V_p)_{d(p)}$. Using Proposition 2.8, in fact

$$d\big(u, (S_p)_{a(p)}\big) \leq \|u \oplus u_n\| + d\big(u_n, (S_p)_{a(p)}\big)$$
$$\leq \|u \oplus u_n\| + D\big((S_{p_n})_{a(p_n)}, (S_p)_{a(p)}\big)$$
$$\leq \|u - u_n\| + \lambda_{D_S}(p_n \oplus p) \to 0, \quad \text{as } n \to \infty.$$

Hence $u \in (S_p)_{a(p)}$. Similarly, we can show that $v \in (T_p)_{b(p)}$, $w \in (U_p)_{c(p)}$, and $z \in (V_p)_{d(p)}$. This completes the proof. □

Taking $\beta = 1$ in Algorithm 3.4, we can also prove the existence and convergence result for nonlinear fuzzy ordered variational inclusion problem (3.1) and nonlinear fuzzy ordered resolvent equation problem (3.2).

Theorem 4.2 *Let $S, T, U, V : \mathcal{H}_p \to \mathcal{F}(\mathcal{H}_p)$ be the closed fuzzy mappings satisfying the following condition $(*)$, with functions $a, b, c, d : \mathcal{H} \to [0,1]$, respectively. Let $G, g : \mathcal{H}_p \to \mathcal{H}_p$ and $N : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings. Let $A : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an (α,λ)-XOR-weak-NODD set-valued mapping with respect to the first argument. Suppose that the following conditions hold*:

- (i) *G is comparison and λ_G-ordered compression mapping, $\lambda_G \in (0,1)$;*
- (ii) *g is comparison and λ_g-ordered compression mapping, $\lambda_g \in (0,1)$;*
- (iii) *N is comparison and (κ,ν)-ordered Lipschitz continuous mapping;*
- (iv) *S, T, U, and V are ordered Lipschitz type continuous mappings with constants λ_{D_S}, λ_{D_T}, λ_{D_U}, and λ_{D_V}, respectively.*

For given $p_0 \in \mathcal{H}$, let the sequences p_n, u_n, v_n, w_n, and z_n defined by the following schemes:

$$\begin{cases} q_{n+1} = g(p_n) \oplus \lambda(G(w_n) \odot N(u_n, v_n)), \\ p_{n+1} = p_n \oplus (g(p_n) \oplus \mathcal{J}^{\lambda}_{A(\cdot,z_n)}(q_{n+1})), \\ u_{n+1} \in (S_{p_{n+1}})_{a(p_{n+1})}, \qquad u_{n+1} \oplus u_n \le (1 + \frac{1}{n+1})D((S_{p_{n+1}})_{a(p_{n+1})}, (S_{p_n})_{a(p_n)}), \\ v_{n+1} \in (T_{p_{n+1}})_{b(p_{n+1})}, \qquad v_{n+1} \oplus v_n \le (1 + \frac{1}{n+1})D((T_{p_{n+1}})_{b(p_n+1)}, (T_{p_n})_{b(p_n)}), \\ w_{n+1} \in (U_{p_{n+1}})_{c(p_{n+1})}, \qquad w_{n+1} \oplus w_n \le (1 + \frac{1}{n+1})D((U_{p_{n+1}})_{c(p_{n+1})}, (U_{p_n})_{c(p_n)}), \\ z_{n+1} \in (V_{p_{n+1}})_{d(p_{n+1})}, \qquad z_{n+1} \oplus z_n \le (1 + \frac{1}{n+1})D((V_{p_{n+1}})_{d(p_{n+1})}, (V_{p_n})_{d(p_n)}). \end{cases} \tag{4.9}$$

If the following conditions

$$\begin{aligned} &\text{(a)} \quad \mathcal{J}^{\lambda}_{A(\cdot,s)}(p) \oplus \mathcal{J}^{\lambda}_{A(\cdot,t)}(p) \le \xi(s \oplus t), \quad \textit{for all } p,s,t \in \mathcal{H}_p, \xi > 0, \\ &\text{(b)} \quad \begin{cases} |\frac{\mu(\lambda_g \oplus \lambda(\lambda_G \lambda_{D_U} \oplus (\kappa\lambda_{D_S} + \nu\lambda_{D_T})))}{(\lambda\alpha \oplus \mu)\lambda_g} + \frac{\xi\lambda_{D_V}}{\lambda_g}| < 1, \\ \lambda\alpha > \mu, \mu \ge 1, \end{cases} \end{aligned} \tag{4.10}$$

are satisfied, then there exist $p,q \in \mathcal{H}_p$ such that $u \in (S_p)_{a(p)}$, $v \in (T_p)_{b(p)}$, $w \in (U_p)_{c(p)}$, and $z \in (V_p)_{d(p)}$ satisfying the generalized nonlinear mixed ordered fuzzy resolvent equation (3.2), and so (p,u,v,w,z) is a solutions of the generalized nonlinear mixed ordered fuzzy variational inclusion problem (3.1), and the iterative sequences $\{p_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, and $\{z_n\}$ generated by Algorithm 3.4 converge strongly to p,v,u,w and z in $\mathcal{H}_p$, respectively.

5 Conclusion

The aim of this paper is to introduce a resolvent operator, and we demonstrate some of its properties. The resolvent operator is used to define an iterative algorithm for solving a nonlinear fuzzy ordered variational inclusion problem and its corresponding nonlinear fuzzy ordered resolvent equation problem based on XOR operator in real ordered positive Hilbert spaces. Some preliminary results are proved to obtain the main result. We prove the convergence analysis of our proposed iterative algorithm which assumes that the suggested algorithm converges to a unique solution of our considered problem with some consequence. Our results extend and generalize most of the results involving fuzzy mappings of different authors existing in the literature.

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