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New estimations for the Berezin number inequality

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Abstract

In this paper, by the definition of Berezin number, we present some inequalities involving the operator geometric mean. For instance, it is shown that if $X, Y, Z \in \mathcal{L}(\mathcal{H})$ such that X and Y are positive operators, then

$$\mathrm{ber}^r((X \sharp Y)Z) \leq \mathrm{ber}\Big(\frac{(Z^\star YZ)^{\frac{rq}{2}}}{q} + \frac{X^{\frac{rp}{2}}}{p}\Big) - \frac{1}{p}\inf_{\lambda\in\Omega}\big(\big[\widetilde{X}(\lambda)\big]^{\frac{rp}{4}} - \big[\widetilde{(Z^\star YZ)}(\lambda)\big]^{\frac{rq}{4}}\big)^2,$$

in which $X \sharp Y = X^{\frac{1}{2}}(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^{\frac{1}{2}}X^{\frac{1}{2}}$, $p \geq q > 1$ such that $r \geq \frac{2}{q}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

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1 Introduction and preliminaries

We denote the C^*-algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$ with $\mathcal{L}(\mathcal{H})$. An operator $X \in \mathcal{L}(\mathcal{H})$ is called positive if $\langle Xx, x\rangle \geq 0$ for every $x \in \mathcal{H}$, and in this case we write $X \geq 0$. The numerical range and numerical radius of $X \in \mathcal{L}(\mathcal{H})$ are respectively defined by $W(X) := \{\langle Xf, f\rangle : f \in \mathcal{H}, \|f\| = 1\}$ and $w(X) := \sup\{|f| : f \in W(X)\}$. We denote by $\mathcal{F}(\Omega)$ the set of all complex-valued functions on a nonempty set Ω. Let $\mathcal{H} = \mathcal{H}(\Omega) \subset \mathcal{F}(\Omega)$ be a Hilbert space. The Riesz representation theorem makes certain that a functional Hilbert space has a reproducing kernel, which is a function $k_\lambda : \Omega \times \Omega \to \mathcal{H}$, that is called the reproducing kernel enjoying the reproducing property $k_\lambda := k(\cdot, \lambda) \in \mathcal{H}$ $(\lambda \in \Omega)$ such that $f(\lambda) = \langle f, k_\lambda\rangle_{\mathcal{H}}$, in which $\lambda \in \Omega$ and $f \in \mathcal{H}$ (see [18]). For $\{\xi_n(z)\}_{n\geq 0}$, an orthonormal basis of the space $\mathcal{H}(\Omega)$, the reproducing kernel can be presented as follows:

$$k_\lambda(z) = \sum_{n=0}^{\infty} \overline{\xi_n(\lambda)}\xi_n(z)$$

(see [2, 18] and the references therein). Throughout the paper, $\mathcal{H} = \mathcal{H}(\Omega)$ for some nonempty set Ω. If $X \in \mathcal{L}(\mathcal{H})$, then the Berezin symbol of X is the function $\widetilde{X}$ with

$$\widetilde{X}(\mu) := \langle X\widehat{k}_\mu, \widehat{k}_\mu\rangle_{\mathcal{H}} \quad (\mu \in \Omega),$$

where $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ is the normalized reproducing kernel of $\mathcal{H}$ (see [7]). Karaev in [13–15] defined the Berezin set and the Berezin number for operator X as follows:

$$\mathrm{Ber}(X) := \{\widetilde{X}(\lambda) : \lambda \in \Omega\} \quad \text{and} \quad \mathrm{ber}(X) := \sup\{|\widetilde{X}(\lambda)| : \lambda \in \Omega\},$$

respectively. Moreover, the Berezin number of two operators X, Y satisfies the following properties:

(i) $\mathrm{ber}(\nu X) = |\nu| \,\mathrm{ber}(X)$ for all $\nu \in \mathcal{C}$;
(ii) $\mathrm{ber}(X + Y) \le \mathrm{ber}(X) + \mathrm{ber}(X)$.

Also, we know that

$$\mathrm{ber}(X) \le w(X) \le \|X\|$$

for all $X \in \mathcal{L}(\mathcal{H})$. In some recent papers, several Berezin number inequalities have been investigated by authors [3–6, 9, 10, 12, 21, 22].

Assume that $X_1, \dots, X_n \in \mathcal{L}(\mathcal{H})$ and $p \ge 1$. In [3], the generalized Euclidean Berezin number of $X_1, \dots, X_n$ is defined as follows:

$$\mathbf{ber}_p(X_1, \dots, X_n) := \sup_{\lambda \in \Omega} \left(\sum_{i=1}^{n} \left| \langle X_i \hat{k}_\lambda, \hat{k}_\lambda \rangle \right|^p \right)^{\frac{1}{p}}.$$

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the Young inequality is the inequality

$$xy \le \frac{x^p}{p} + \frac{y^q}{q}, \tag{1}$$

where x and y are positive real numbers (see [11]). A refinement of (1) was obtained by Kittaneh and Manasrah [17]

$$xy + r_0 \left(x^{\frac{p}{2}} - y^{\frac{q}{2}} \right)^2 \le \frac{x^p}{p} + \frac{y^q}{q}, \tag{2}$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$ or equivalently

$$x^\nu y^{1-\nu} + r_0 \left(x^{\frac{1}{2}} - y^{\frac{1}{2}} \right)^2 \le \nu x + (1 - \nu) y, \tag{3}$$

in which $\nu \in [0, 1]$ and $r_0 = \min\{\nu, 1 - \nu\}$.

For positive operators $X, Y \in \mathcal{L}(\mathcal{H})$, the operator geometric mean is the positive operator $X \sharp Y = X^{\frac{1}{2}} (X^{-\frac{1}{2}} Y X^{-\frac{1}{2}})^{\frac{1}{2}} X^{\frac{1}{2}}$, where it has the property $X \sharp Y = Y \sharp X$. A matrix mean inequality was established by Bhatia and Kittaneh in [8], and later this inequality was generalized in [18]. A matrix Young inequality was obtained by Ando in [1]. The matrix mean inequality and the matrix Young inequality were considered with the numerical radius norm by Salemi and Sheikhhosseini in [19, 20].

In this paper, we get some upper bounds for the Berezin number of the $(X \sharp Y)Z$ on reproducing kernel Hilbert spaces (RKHS), where $Z \in \mathcal{L}(\mathcal{H})$ is arbitrary, and give some Berezin number inequalities. We also present some inequalities for the generalized Euclidean Berezin number.

2 Main results

We need the following lemma to prove our results (see [16]).

Lemma 1 *Let $X \in \mathcal{L}(\mathcal{H})$ be a positive operator, and let $x \in \mathcal{H}$ be any unit vector. If $r \geq 1$, then*

$$\langle Xx, x\rangle^r \leq \langle X^r x, x\rangle \tag{4}$$

and if $0 \leq r \leq 1$, then

$$\langle X^r x, x\rangle \leq \langle Xx, x\rangle^r.$$

Before giving our next result, we set $\|X\|_{\text{ber}} := \sup\{|\langle X\widehat{k}_\lambda, \widehat{k}_\mu\rangle| : \lambda, \mu \in \Omega\}$ and $m(X) := \inf_{\lambda\in\Omega} |\widetilde{X}(\lambda)|$.

Theorem 2 *Let $X, Y, Z \in \mathcal{L}(\mathcal{H})$ be operators such that X, Y are positive. If $p \geq q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\text{ber}^r\big((X \sharp Y)Z\big) \leq \text{ber}\left(\frac{X^{\frac{rp}{2}}}{p} + \frac{(Z^\star YZ)^{\frac{rq}{2}}}{q}\right) - \frac{1}{p}\inf_{\lambda\in\Omega}\left(\big[\widetilde{X}(\lambda)\big]^{\frac{rp}{4}} - \big[\widetilde{(Z^\star YZ)}(\lambda)\big]^{\frac{rq}{4}}\right)^2$$

for all $r \geq \frac{2}{q}$.

Proof Using the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
\big|\widetilde{(X \sharp Y)Z}(\lambda)\big|^r &= \big|\big(X^{\frac{1}{2}}\widetilde{\big(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\big)^{\frac{1}{2}}}X^{\frac{1}{2}}Z\big)(\lambda)\big|^r \\
&= \big|\big\langle\big(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\big)^{\frac{1}{2}}X^{\frac{1}{2}}Z\widehat{k}_\lambda, X^{\frac{1}{2}}\widehat{k}_\lambda\big\rangle\big|^r \\
&\leq \big\|\big(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\big)^{\frac{1}{2}}X^{\frac{1}{2}}Z\widehat{k}_\lambda\big\|^r \cdot \big\|X^{\frac{1}{2}}\widehat{k}_\lambda\big\|^r \\
&= \big\langle\big(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\big)^{\frac{1}{2}}X^{\frac{1}{2}}Z\widehat{k}_\lambda, \big(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\big)^{\frac{1}{2}}X^{\frac{1}{2}}Z\widehat{k}_\lambda\big\rangle^{\frac{r}{2}} \times \big\langle X^{\frac{1}{2}}\widehat{k}_\lambda, X^{\frac{1}{2}}\widehat{k}_\lambda\big\rangle^{\frac{r}{2}} \\
&= \big(\widetilde{Z^\star YZ}(\lambda)\big)^{\frac{r}{2}}\big(\widetilde{X}(\lambda)\big)^{\frac{r}{2}}
\end{aligned}$$

for all $\lambda \in \Omega$. By using the Young inequality and (2), we get

$$\begin{aligned}
\big(\widetilde{X}(\lambda)\big)^{\frac{r}{2}}\big(\widetilde{Z^\star YZ}(\lambda)\big)^{\frac{r}{2}} \leq{}& \frac{1}{p}\langle X\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{rp}{2}} + \frac{1}{q}\big\langle Z^\star YZ\widehat{k}_\lambda, \widehat{k}_\lambda\big\rangle^{\frac{rq}{2}} \\
&- \frac{1}{p}\big(\langle X\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{rp}{4}} - \big\langle Z^\star YZ\widehat{k}_\lambda, \widehat{k}_\lambda\big\rangle^{\frac{rq}{4}}\big)^2,
\end{aligned}$$

and it follows from inequality (4) that

$$\begin{aligned}
&\frac{1}{p}\langle X\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{rp}{2}} + \frac{1}{q}\big\langle Z^\star YZ\widehat{k}_\lambda, \widehat{k}_\lambda\big\rangle^{\frac{rq}{2}} - \frac{1}{p}\big(\langle X\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{rp}{4}} - \big\langle Z^\star YZ\widehat{k}_\lambda, \widehat{k}_\lambda\big\rangle^{\frac{rq}{4}}\big)^2 \\
&\quad\leq \frac{1}{p}\big\langle X^{\frac{rp}{2}}\widehat{k}_\lambda, \widehat{k}_\lambda\big\rangle + \frac{1}{q}\big\langle(Z^\star YZ)^{\frac{rq}{2}}\widehat{k}_\lambda, \widehat{k}_\lambda\big\rangle
\end{aligned}$$

$$-\frac{1}{p}\big(\langle X\widehat{k}_\lambda,\widehat{k}_\lambda\rangle^{\frac{rp}{4}}-\langle Z^\star YZ\widehat{k}_\lambda,\widehat{k}_\lambda\rangle^{\frac{rq}{4}}\big)^2$$
$$=\left\langle\left(\frac{X^{\frac{rp}{2}}}{p}+\frac{(Z^\star YZ)^{\frac{rq}{2}}}{q}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle-\frac{1}{p}\big(\langle X\widehat{k}_\lambda,\widehat{k}_\lambda\rangle^{\frac{rp}{4}}-\langle Z^\star YZ\widehat{k}_\lambda,\widehat{k}_\lambda\rangle^{\frac{rq}{4}}\big)^2$$

for all $\lambda\in\Omega$. Since $(\widetilde{\frac{X^{\frac{rp}{2}}}{p}+\frac{(Z^\star YZ)^{\frac{rq}{2}}}{q}})(\lambda)$ is positive, then we have

$$\sup_{\lambda\in\Omega}\big|\widetilde{(X\sharp Y)Z}(\lambda)\big|^r$$
$$\leq\sup_{\lambda\in\Omega}\widetilde{\left(\frac{X^{\frac{rp}{2}}}{p}+\frac{(Z^\star YZ)^{\frac{rq}{2}}}{q}\right)}(\lambda)-\frac{1}{p}\inf_{\lambda\in\Omega}\big(\big[\widetilde{X}(\lambda)\big]^{\frac{rp}{4}}-\big[\widetilde{(Z^\star YZ)}(\lambda)\big]^{\frac{rq}{4}}\big)^2$$

for all $\lambda\in\Omega$. This implies that

$$\operatorname{ber}^r\big((X\sharp Y)Z\big)\leq\operatorname{ber}\left(\frac{X^{\frac{rp}{2}}}{p}+\frac{(Z^\star YZ)^{\frac{rq}{2}}}{q}\right)-\frac{1}{p}\inf_{\lambda\in\Omega}\big(\big[\widetilde{X}(\lambda)\big]^{\frac{rp}{4}}-\big[\widetilde{(Z^\star YZ)}(\lambda)\big]^{\frac{rq}{4}}\big)^2. \qquad (5)$$

□

Taking the $Z=I$ in inequality (5), we have the following result.

Corollary 3 *Let $X,Y\in\mathcal{L}(\mathcal{H})$ be positive operators, and let $p\geq q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\operatorname{ber}^r(X\sharp Y)\leq\operatorname{ber}\left(\frac{X^{\frac{rp}{2}}}{p}+\frac{Y^{\frac{rq}{2}}}{q}\right)-\frac{1}{p}\inf_{\lambda\in\Omega}\big(\big[\widetilde{X}(\lambda)\big]^{\frac{rp}{4}}-\big[\widetilde{Y}(\lambda)\big]^{\frac{rq}{4}}\big)^2$$

for all $r\geq\frac{2}{q}$.

Corollary 4 *Let $X,Y\in\mathcal{L}(\mathcal{H})$ be positive operators. Then*

$$\sqrt{2}\operatorname{ber}(X\sharp Y)\leq\operatorname{ber}_2(X,Y)\leq\operatorname{ber}^{\frac{1}{2}}\big(X^2+Y^2\big).$$

Proof As in the same arguments in the proof of Theorem 2, if we put $r=p=q=2$, then we get

$$\big|\widetilde{(X\sharp Y)}(\lambda)\big|\leq\frac{1}{2}\big(\big[\widetilde{X}(\lambda)\big]^2+\big[\widetilde{Y}(\lambda)\big]^2\big)$$
$$\leq\frac{1}{2}\big(\widetilde{X^2}(\lambda)+\widetilde{Y^2}(\lambda)\big)=\frac{1}{2}\big(\widetilde{X^2+Y^2}\big)(\lambda)\quad(\lambda\in\Omega).$$

Since $[\widetilde{X}(\lambda)]^2\geq 0$, $[\widetilde{Y}(\lambda)]^2\geq 0$, and $(\widetilde{X^2+Y^2})(\lambda)\geq 0$, taking the supremum over $\lambda\in\Omega$, we get that

$$\sqrt{2}\operatorname{ber}(X\sharp Y)\leq\operatorname{ber}_2(X,Y)\leq\operatorname{ber}^{\frac{1}{2}}\big(X^2+Y^2\big).$$

□

Proposition 5 *Let $X,Y,Z\in\mathcal{L}(\mathcal{H})$ such that X, Y are positive, and let $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\big\|(X\sharp Y)Z\big\|_{\mathrm{ber}}^r\leq\left\|\frac{X^{\frac{rp}{2}}}{p}\right\|_{\mathrm{ber}}+\left\|\frac{(Z^\star YZ)^{\frac{rq}{2}}}{q}\right\|_{\mathrm{ber}}-\frac{1}{p}\inf_{\mu,\lambda\in\Omega}\big(\langle X\widehat{k}_\mu,\widehat{k}_\mu\rangle^{\frac{rp}{4}}-\langle Z^\star YZ\widehat{k}_\lambda,\widehat{k}_\lambda\rangle^{\frac{rq}{4}}\big)^2$$

for all $r \geq \frac{2}{q}$.

Proof Indeed, for every $\lambda, \mu \in \Omega$, we have

$$\begin{aligned}
&\left|\langle (X \sharp Y)Z\widehat{k}_\lambda, \widehat{k}_\mu\rangle\right|^r \\
&= \left|\langle X^{\frac{1}{2}}\big(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\big)^{\frac{1}{2}}X^{\frac{1}{2}}Z\widehat{k}_\lambda, \widehat{k}_\mu\rangle\right| \\
&= \left|\langle \big(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\big)^{\frac{1}{2}}X^{\frac{1}{2}}Z\widehat{k}_\lambda, X^{\frac{1}{2}}\widehat{k}_\mu\rangle\right|^r \\
&= \left|\langle \big(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\big)^{\frac{1}{2}}X^{\frac{1}{2}}Z\widehat{k}_\lambda, X^{\frac{1}{2}}\widehat{k}_\mu\rangle\right|^r \\
&\leq \left\|\big(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\big)^{\frac{1}{2}}X^{\frac{1}{2}}Z\widehat{k}_\lambda\right\|^r \cdot \left\|X^{\frac{1}{2}}\widehat{k}_\mu\right\|^r \\
&= \langle \big(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\big)^{\frac{1}{2}}X^{\frac{1}{2}}Z\widehat{k}_\lambda, \big(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\big)^{\frac{1}{2}}X^{\frac{1}{2}}Z\widehat{k}_\lambda\rangle^{\frac{r}{2}} \times \langle X^{\frac{1}{2}}\widehat{k}_\mu, X^{\frac{1}{2}}\widehat{k}_\mu\rangle^{\frac{r}{2}} \\
&= \langle X\widehat{k}_\mu, \widehat{k}_\mu\rangle^{\frac{r}{2}}\langle Z^\star YZ\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{r}{2}} \\
&\leq \frac{1}{p}\langle X\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{rp}{2}} + \frac{1}{q}\langle Z^\star YZ\widehat{k}_\mu, \widehat{k}_\mu\rangle^{\frac{rq}{2}} \\
&\quad - \frac{1}{p}\big(\langle X\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{rp}{4}} - \langle Z^\star YZ\widehat{k}_\mu, \widehat{k}_\mu\rangle^{\frac{rq}{4}}\big)^2 \quad \text{(by (1) and (2))} \\
&\leq \frac{1}{p}\langle X^{\frac{rp}{2}}\widehat{k}_\mu, \widehat{k}_\mu\rangle + \frac{1}{q}\langle (Z^\star YZ)^{\frac{rq}{2}}\widehat{k}_\lambda, \widehat{k}_\lambda\rangle \\
&\quad - \frac{1}{p}\big(\langle X\widehat{k}_\mu, \widehat{k}_\mu\rangle^{\frac{rp}{4}} - \langle Z^\star YZ\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{rq}{4}}\big)^2 \quad \text{(by (4))} \\
&\leq \frac{1}{p}\langle X^{\frac{rp}{2}}\widehat{k}_\mu, \widehat{k}_\mu\rangle + \frac{1}{q}\langle (Z^\star YZ)^{\frac{rq}{2}}\widehat{k}_\lambda, \widehat{k}_\lambda\rangle \\
&\quad - \frac{1}{p}\inf_{\mu,\lambda\in\Omega}\big(\langle X\widehat{k}_\mu, \widehat{k}_\mu\rangle^{\frac{rp}{4}} - \langle Z^\star YZ\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{rq}{4}}\big)^2
\end{aligned} \tag{6}$$

so that if we take the supremum over $\lambda, \mu \in \Omega$ in inequality (6), we get

$$\begin{aligned}
\left\|(X \sharp Y)Z\right\|_{\text{ber}}^r &\leq \left\|\frac{X^{\frac{rp}{2}}}{p}\right\|_{\text{ber}} + \left\|\frac{(Z^\star YZ)^{\frac{rq}{2}}}{q}\right\|_{\text{ber}} \\
&\quad - \frac{1}{p}\inf_{\mu,\lambda\in\Omega}\big(\langle X\widehat{k}_\mu, \widehat{k}_\mu\rangle^{\frac{rp}{4}} - \langle Z^\star YZ\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{rq}{4}}\big)^2.
\end{aligned}$$

□

Remark 6 It follows from inequality

$$\begin{aligned}
&\inf_{\mu,\lambda\in\Omega}\big(\langle X\widehat{k}_\mu, \widehat{k}_\mu\rangle^{\frac{rp}{4}} - \langle Z^\star YZ\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{rq}{4}}\big)^2 \\
&= \inf_{\mu,\lambda\in\Omega}\big(\langle X\widehat{k}_\mu, \widehat{k}_\mu\rangle^{\frac{rp}{2}} + \langle Z^\star YZ\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{rq}{2}} - 2\langle X\widehat{k}_\mu, \widehat{k}_\mu\rangle^{\frac{rp}{4}}\langle Z^\star YZ\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{rq}{4}}\big) \\
&\geq \inf_{\mu\in\Omega}\langle X\widehat{k}_\mu, \widehat{k}_\mu\rangle^{\frac{rp}{2}} + \inf_{\lambda\in\Omega}\langle Z^\star YZ\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{rq}{2}} - 2\sup_{\mu\in\Omega}\langle X\widehat{k}_\mu, \widehat{k}_\mu\rangle^{\frac{rp}{4}}\sup_{\lambda\in\Omega}\langle Z^\star YZ\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{\frac{rq}{4}} \\
&= m(X)^{\frac{rp}{2}} + m\big(Z^\star YZ\big)^{\frac{rq}{2}} - 2\left\|Z^\star YZ\right\|_{\text{ber}}^{\frac{rq}{4}}\|X\|_{\text{ber}}^{\frac{rp}{4}}
\end{aligned}$$

and inequality (6) that

$$\begin{aligned}\|(X \sharp Y)Z\|_{\text{ber}}^{r} \le{}& \left\|\frac{X^{\frac{rp}{2}}}{p}\right\|_{\text{ber}} + \left\|\frac{(Z^{\star}YZ)^{\frac{rq}{2}}}{q}\right\|_{\text{ber}} \\ &- \left(m(X)^{\frac{rp}{2}} + m(Z^{\star}YZ)^{\frac{rq}{2}} - 2\|Z^{\star}YZ\|_{\text{ber}}^{\frac{rq}{4}}\|X\|_{\text{ber}}^{\frac{rp}{4}}\right).\end{aligned}$$

Proposition 7 *Let $X, Y, Z \in \mathcal{L}(\mathcal{H})$ such that X, Y are positive, and let $p \ge q > 1$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned}\left(\|X\|_{\text{ber}}\|Z^{\star}YZ\|_{\text{ber}}\right)^{\frac{r}{2}} \le{}& \left\|\frac{X^{\frac{rp}{2}}}{p}\right\|_{\text{ber}} + \left\|\frac{(Z^{\star}YZ)^{\frac{rq}{2}}}{q}\right\|_{\text{ber}} \\ &- \frac{1}{p}\inf_{\lambda\in\Omega}\left(\left[\langle X\widehat{k}_{\mu},\widehat{k}_{\mu}\rangle^{\frac{rp}{4}}\right] - \langle Z^{\star}YZ\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle^{\frac{rq}{4}}\right)^{2}\end{aligned}$$

for all $r \ge \frac{2}{q}$.

Proof By inequality (2), we have

$$\begin{aligned}&\langle X\widehat{k}_{\mu},\widehat{k}_{\mu}\rangle^{\frac{r}{2}}\langle Z^{\star}YZ\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle^{\frac{r}{2}} \\ &\quad\le \frac{1}{p}\langle X\widehat{k}_{\mu},\widehat{k}_{\mu}\rangle^{\frac{rp}{2}} + \frac{1}{q}\langle Z^{\star}YZ\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle^{\frac{rq}{2}} \\ &\qquad - \frac{1}{p}\left(\langle X\widehat{k}_{\mu},\widehat{k}_{\mu}\rangle^{\frac{rp}{4}} - \langle Z^{\star}YZ\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle^{\frac{rq}{4}}\right)^{2} \\ &\quad\le \frac{1}{p}\langle X^{\frac{rp}{2}}\widehat{k}_{\mu},\widehat{k}_{\mu}\rangle + \frac{1}{q}\langle (Z^{\star}YZ)^{\frac{rq}{2}}\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle \\ &\qquad - \frac{1}{p}\left(\langle X\widehat{k}_{\mu},\widehat{k}_{\mu}\rangle^{\frac{rp}{4}} - \langle Z^{\star}YZ\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle^{\frac{rq}{4}}\right)^{2}\end{aligned}$$

for all $\lambda, \mu \in \Omega$ and taking supremum over $\lambda, \mu \in \Omega$ in the above inequality, we get

$$\begin{aligned}\left(\|X\|_{\text{ber}}\|Z^{\star}YZ\|_{\text{ber}}\right)^{\frac{r}{2}} \le{}& \left\|\frac{X^{\frac{rp}{2}}}{p}\right\|_{\text{ber}} + \left\|\frac{(Z^{\star}YZ)^{\frac{rq}{2}}}{q}\right\|_{\text{ber}} \\ &- \frac{1}{p}\inf_{\lambda\in\Omega}\left(\left[\langle X\widehat{k}_{\mu},\widehat{k}_{\mu}\rangle^{\frac{rp}{4}}\right] - \langle Z^{\star}YZ\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle^{\frac{rq}{4}}\right)^{2}.\end{aligned}$$ □

Now, we present the next lemma to obtain our last results.

Lemma 8 ([16]) *If $f, g : [0,\infty) \longrightarrow \mathcal{R}$ are nonnegative continuous such that $f(t)g(t) = t$ ($t \in [0,\infty)$), then*

$$\left|\langle Xx, y\rangle\right| \le \left\|f(|X|)x\right\| \left\|g(|X^{*}|)x\right\|,$$

where $X \in \mathcal{L}(\mathcal{H})$ and $x, y \in \mathcal{H}$.

In the next theorem we show an upper bound for the generalized Euclidean Berezin number.

Theorem 9 *Let $X_i, Y_i, Z_i, \in \mathcal{L}(\mathcal{H})$ $(1 \le i \le n)$. Then*

$$\begin{aligned}
&\mathrm{ber}_p^p\big(X_1^* Z_1 Y_1, \ldots, X_n^* Z_n Y_n\big) \\
&\quad \le \frac{n^{1-\frac{1}{r}}}{2^{\frac{1}{r}}} \mathrm{ber}^{\frac{1}{r}}\left(\sum_{i=1}^n \big[Y_i^* f^2\big(|Z_i|\big)Y_i\big]^{rp} + \big[X_i^* g^2\big(\big|Z_i^*\big|\big)X_i\big]^{rp}\right) \\
&\qquad - \frac{1}{2}\inf_{\lambda \in \Omega}\left(\sum_{i=1}^n \Big(\sqrt{\big\langle \big(X_i^* g^2\big(\big|Z_i^*\big|\big)X_i\big)^p \hat{k}_\lambda, \hat{k}_\lambda\big\rangle} - \sqrt{\big\langle \big(Y_i^* f^2\big(|Z_i|\big)Y_i\big)^p \hat{k}_\lambda, \hat{k}_\lambda\big\rangle}\Big)^2\right),
\end{aligned} \tag{7}$$

where $f, g : [0,\infty) \longrightarrow \mathcal{R}$ are nonnegative continuous such that $f(t)g(t) = t$ $(t \in [0,\infty))$ and $p, r \ge 1$.

Proof For any $\hat{k}_\lambda \in \mathcal{H}(\Omega)$, we have

$$\begin{aligned}
&\sum_{i=1}^n \big|\big\langle X_i^* Z_i Y_i \hat{k}_\lambda, \hat{k}_\lambda\big\rangle\big|^p \\
&\quad = \sum_{i=1}^n \big|\langle Z_i Y_i \hat{k}_\lambda, X_i \hat{k}_\lambda\rangle\big|^p \\
&\quad \le \sum_{i=1}^n \big\|f\big(|Z_i|\big)Y_i \hat{k}_\lambda\big\|^p \big\|g\big(\big|Z_i^*\big|\big)X_i \hat{k}_\lambda\big\|^p \quad \text{(by Lemma 8)} \\
&\quad = \sum_{i=1}^n \big\langle f\big(|Z_i|\big)Y_i \hat{k}_\lambda, f\big(|Z_i|\big)Y_i \hat{k}_\lambda\big\rangle^{\frac{p}{2}} \big\langle g\big(\big|Z_i^*\big|\big)X_i \hat{k}_\lambda, g\big(\big|Z_i^*\big|\big)X_i \hat{k}_\lambda\big\rangle^{\frac{p}{2}} \\
&\quad = \sum_{i=1}^n \big\langle Y_i^* f^2\big(|Z_i|\big)Y_i \hat{k}_\lambda, \hat{k}_\lambda\big\rangle^{\frac{p}{2}} \big\langle X_i^* g^2\big(\big|Z_i^*\big|\big)Z_i \hat{k}_\lambda, \hat{k}_\lambda\big\rangle^{\frac{p}{2}} \\
&\quad \le \sum_{i=1}^n \big\langle \big(Y_i^* f^2\big(|Z_i|\big)Y_i\big)^p \hat{k}_\lambda, \hat{k}_\lambda\big\rangle^{\frac{1}{2}} \big\langle \big(X_i^* g^2\big(\big|Z_i^*\big|\big)Z_i\big)^p \hat{k}_\lambda, \hat{k}_\lambda\big\rangle^{\frac{1}{2}} \quad \text{(by (4))} \\
&\quad \le \sum_{i=1}^n \left[\left(\frac{1}{2}\big\langle \big(Y_i^* f^2\big(|Z_i|\big)Y_i\big)^{pr} \hat{k}_\lambda, \hat{k}_\lambda\big\rangle + \frac{1}{2}\big\langle \big(X_i^* g^2\big(\big|Z_i^*\big|\big)X_i\big)^{pr} \hat{k}_\lambda, \hat{k}_\lambda\big\rangle\right)^{\frac{1}{r}}\right] \quad \text{(by (2))} \\
&\qquad - \frac{1}{2}\sum_{i=1}^n \Big(\sqrt{\big\langle \big(X_i^* g^2\big(\big|Z_i^*\big|\big)X_i\big)^p \hat{k}_\lambda, \hat{k}_\lambda\big\rangle} - \sqrt{\big\langle \big(Y_i^* f^2\big(|Z_i|\big)Y_i\big)^p, \hat{k}_\lambda\big\rangle}\Big)^2 \\
&\quad \le \frac{n^{1-\frac{1}{r}}}{2^{\frac{1}{r}}} \left\langle \left(\sum_{i=1}^n \Big(\big[Y_i^* f^2\big(|Z_i|\big)Y_i\big]^{rp} + \big[X_i^* g^2\big(\big|Z_i^*\big|\big)X_i\big]^{rp}\Big)\right) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{\frac{1}{r}} \\
&\qquad - \frac{1}{2}\sum_{i=1}^n \Big(\sqrt{\big\langle \big(X_i^* g^2\big(\big|Z_i^*\big|\big)X_i\big)^p \hat{k}_\lambda, \hat{k}_\lambda\big\rangle} - \sqrt{\big\langle \big(Y_i^* f^2\big(|Z_i|\big)Y_i\big)^p \hat{k}_\lambda, \hat{k}_\lambda\big\rangle}\Big)^2.
\end{aligned}$$

By taking the supremum on $\hat{k}_\lambda \in \mathcal{H}$ with $\|\hat{k}_\lambda\| = 1$, we reach the desired inequality. □

Selecting $X_i = Y_i = I$ for $i = 1, 2, \ldots, n$ in Theorem 9, we get the next result.

Corollary 10 *Let $Z_i \in \mathcal{L}(\mathcal{H})$ $(1 \leq i \leq n)$ and $r, p \geq 1$. Then*

$$\begin{aligned}\operatorname{ber}_p^p(Z_1,\dots,Z_n) \leq{}& \frac{n^{1-\frac{1}{r}}}{2^{\frac{1}{r}}}\operatorname{ber}^{\frac{1}{r}}\left(\sum_{i=1}^n [f^2(|Z_i|)^{rp} + g^2(|Z_i^*|)^{rp}\right)\\ &- \frac{1}{2}\sum_{i=1}^n \left(\sqrt{\langle g^{2p}(|Z_i^*|)\hat{k}_\lambda,\hat{k}_\lambda\rangle} - \sqrt{\langle f^{2p}(|Z_i|)\hat{k}_\lambda,\hat{k}_\lambda\rangle}\right)^2,\end{aligned}$$

where $f, g : [0,\infty) \longrightarrow \mathcal{R}$ are nonnegative continuous such that $f(t)g(t) = t$ $(t \in [0,\infty))$.

In particular, if $X, Y \in \mathcal{L}(\mathcal{H})$, then for all $p \geq 1$ and $0 \leq \nu \leq 1$

$$\operatorname{ber}_p^p(X,Y) \leq \frac{1}{2}\operatorname{ber}\left(|X|^{2\nu p} + \left|X^*\right|^{2(1-\nu)p} + |Y|^{2\nu p} + \left|Y^*\right|^{2(1-\nu)p}\right) - \inf_{\lambda\in\Omega}\delta(\hat{k}_\lambda),$$

where

$$\begin{aligned}\delta(\hat{k}_\lambda) = \frac{1}{2}\Big[&\left(\langle|X|^{2\nu p}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{1}{2}} - \langle\left|X^*\right|^{2(1-\nu)p}\hat{k}_\lambda,\hat{k}_\lambda x\rangle^{\frac{1}{2}}\right)^2\\ &+ \left(\langle|Y|^{2\nu p}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{1}{2}} - \langle\left|Y^*\right|^{2(1-\nu)p}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{1}{2}}\right)^2\Big].\end{aligned}$$

In the last theorem, we show another upper bound for $\operatorname{ber}_p(T_1,\dots,T_n)$.

Theorem 11 *Let $Z_i \in \mathcal{L}(\mathcal{H})$ $(1 \leq i \leq n)$. Then*

$$\operatorname{ber}_p(Z_1,\dots,Z_n) \leq \frac{1}{2}\left[\sum_{i=}^n \left(\operatorname{ber}\left(|Z_i|^{2\nu} + \left|Z_i^*\right|^{2(1-\nu)}\right) - 2\inf_{\|x\|=1}\delta(\hat{k}_\lambda)\right)^p\right]^{\frac{1}{p}}, \tag{8}$$

where $p \geq 1$, $0 \leq \nu \leq 1$, and $\delta(\hat{k}_\lambda) = (\sqrt{\langle|Z_i^*|^{2(1-\nu)}\hat{k}_\lambda,\hat{k}_\lambda\rangle} - \sqrt{\langle|Z_i|^{2\nu}\hat{k}_\lambda,\hat{k}_\lambda\rangle})^2$.

Proof Let $\hat{k}_\lambda \in \mathcal{H}(\Omega)$. Then, by using Lemma 8 and inequality (3), we have

$$\begin{aligned}&\sum_{i=1}^n \left|\langle Z_i\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|^p\\ &\quad\leq \sum_{i=1}^n \left(\langle|Z_i|^{2\nu}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{1}{2}}\langle\left|Z_i^*\right|^{2(1-\nu)}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{1}{2}}\right)^p \quad \text{(by Lemma 8)}\\ &\quad\leq \frac{1}{2^p}\sum_{i=1}^n [\langle|Z_i|^{2\nu}\hat{k}_\lambda,\hat{k}_\lambda\rangle + \langle\left|Z_i^*\right|^{2(1-\nu)}\hat{k}_\lambda,\hat{k}_\lambda\rangle\\ &\quad\quad - \left(\sqrt{\langle\left|Z_i^*\right|^{2(1-\nu)}\hat{k}_\lambda,\hat{k}_\lambda\rangle} - \sqrt{\langle|Z_i|^{2\nu}\hat{k}_\lambda,\hat{k}_\lambda\rangle}\right)^2]^p \quad \text{(by (3))}\\ &\quad= \frac{1}{2^p}\sum_{i=1}^n \left[\langle|Z_i|^{2\nu} + \left|Z_i^*\right|^{2(1-\nu)}\hat{k}_\lambda,\hat{k}_\lambda\rangle - \left(\sqrt{\langle\left|Z_i^*\right|^{2(1-\nu)}\hat{k}_\lambda,\hat{k}_\lambda\rangle} - \sqrt{\langle|Z_i|^{2\nu}\hat{k}_\lambda,\hat{k}_\lambda\rangle}\right)^2\right]^p.\end{aligned}$$

Thus

$$\left(\sum_{i=1}^n \left|\langle Z_i\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|^p\right)^{\frac{1}{p}} \leq \frac{1}{2}\left[\sum_{i=1}^n \left(\langle|Z_i|^{2\nu} + \left|Z_i^*\right|^{2(1-\nu)}\hat{k}_\lambda,\hat{k}_\lambda\rangle\right.\right.$$

$$- \left(\sqrt{\left\langle \left|Z_i^*\right|^{2(1-\nu)} \hat{k}_\lambda, \hat{k}_\lambda \right\rangle} - \sqrt{\left\langle |Z_i|^{2\nu} \hat{k}_\lambda, \hat{k}_\lambda \right\rangle}\right)^2\right)^p \Bigg]^{\frac{1}{p}}$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n}\left(\left\langle |Z_i|^{2\nu} + \left|Z_i^*\right|^{2(1-\nu)} \hat{k}_\lambda, \hat{k}_\lambda\right\rangle - 2\delta(\hat{k}_\lambda)\right)^p\right]^{\frac{1}{p}}.$$

If we get the supremum over all $\hat{k}_\lambda \in \mathcal{H}(\Omega)$ with $\|\hat{k}_\lambda\| = 1$, then we reach the desired result. □

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