Ceng and Shang *Journal of Inequalities and Applications* (2020) 2020:33
https://doi.org/10.1186/s13660-020-2306-1

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

# Composite extragradient implicit rule for solving a hierarch variational inequality with constraints of variational inclusion and fixed point problems

Lu-Chuan Ceng[1] and Meijuan Shang[2*]

*Correspondence: meijuanshang@163.com
[2]College of Science, Shijiazhuang University, Shijiazhuang, China
Full list of author information is available at the end of the article

**Abstract**

Let $X$ be a uniformly convex and $q$-uniformly smooth Banach space with $1 < q \le 2$. In the framework of this space, we are concerned with a composite gradient-like implicit rule for solving a hierarchical monotone variational inequality with the constraints of a system of monotone variational inequalities, a variational inclusion and a common fixed point problem of a countable family of nonlinear operators $\{S_n\}_{n=0}^{\infty}$. Our rule is based on the Korpelevich extragradient method, the perturbation mapping, and the $W$-mappings constructed by $\{S_n\}_{n=0}^{\infty}$.

**MSC:** 47H05; 47H09

**Keywords:** Gradient-like implicit rule; System of variational inequalities; Variational inclusions; $W$-mappings; Uniform convexity; Uniform smoothness

## 1 Introduction

Throughout this work, one always supposes that $C$ is a nonempty convex set in a Banach space $X$ whose dual is denoted by $X^*$. One denotes by the same notation, $\|\cdot\|$, the norms of $X$ and $X^*$. A common problem in machine learning, automatic control, and utility-based bandwidth allocation problems consists of finding a solution of some equation satisfying some constraints. This common problem is called the convex feasibility problem, which can be characterized via the following model: $x \in \bigcap_{i\in I} C_i$, where $I$ denotes some index set, $C_i$ is a convex set in $X$.

Next, one employs $J_q : X \to 2^{X^*}$, where $q > 1$ is real number, to denote the duality mapping, which is defined by $J_q(x) := \{\phi \in X^* : \langle x, \phi\rangle = \|x\|^q, \|x\|^{q-1} = \|\phi\|\}$, $\forall x \in X$. Let $A_1, A_2 : C \to X$ be two nonlinear non-self mappings. Consider the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - y^* + \mu_1 A_1 y^*, J(x - x^*)\rangle \ge 0, & \forall x \in C, \\ \langle y^* - x^* + \mu_2 A_2 x^*, J(x - y^*)\rangle \ge 0, & \forall x \in C, \end{cases} \tag{1.1}$$

with two positive real constants $\mu_1$ and $\mu_2$. This is called a system of generalized variational inequalities (SGVIs). This is a natural extension of the generalized variational inequality considered by Aoyama, Iiduka and Takahashi [1] in uniformly convex and 2-uniformly smooth Banach spaces; see [1] for more details. In Hilbert spaces, the system is reduced to the system of variational inequalities considered by Ceng et al. [2]. Problem (1.1) and its special cases are now under the spotlight of research because of their connections to other real convex and set optimization problems; see, e.g., [3–8] and the references therein. Recently, a fixed point method has been studied for solving convex and non-convex optimization problems since the equivalence between fixed point problems and zero point problems; see, e.g., [9–13] and the references therein. Indeed, one can transfer zero point problems (inclusion problems) to some fixed point problem of nonexpansive operators. The core is the resolvent of original operators. For example, one can show that the resolvent operator of m-accretive or maximally accretive operators is nonexpansive. Hence, Mann-like algorithms are applicable, however, they are only weakly convergent. Strong convergence is desirable in lots of situations, such as, image recovery, optimal control and quantum physics since they are in infinite-dimensional spaces. In this paper, we study, in the framework of Banach spaces, a convex feasibility problem with the constraints of the generalized system of monotone variational inequalities, a variational inclusion and a countable family of nonexpansive operators. Strong convergence theorems are obtained without any compact assumption on operators. Our rule is based on the Korpelevich extragradient method, the perturbation mapping, and the $W$-mappings constructed by $\{S_n\}_{n=0}^{\infty}$. The main results extend and improve some recent results in [14–17].

## 2 Preliminaries

Next, one uses $\rho_X : [0,\infty) \to [0,\infty)$ to stand for the smoothness modulus of space $X$ which is defined by $\rho_X(t) = \sup\{(\|x+y\| + \|x-y\|)/2 - 1 : x \in U, \|y\| \le t\}$. One says that $X$ is uniformly smooth if $\lim_{t\to 0^+} \rho_X(t)/t = 0$. Let $q \in (1,2]$ be a fixed real number. A Banach space $X$ is said to be $q$-uniformly smooth if $\rho_X(t) \le t^q d$, $\forall t > 0$, where $d$ is some constant. It is well known that Hilbert spaces, $L^p$ and $\ell_p$ are uniformly smooth where $p > 1$. More precisely, each Hilbert space is 2-uniformly smooth, while $L^p$ and $\ell_p$ are $\min\{p,2\}$-uniformly smooth for each $p > 1$.

Let $A : C \to 2^X$ be a set-valued operator with $Ax \neq \emptyset$, $\forall x \in C$. An operator $A$ is said to be accretive if, $\forall x, y \in C$, $\langle u - v, j_q(x-y)\rangle \ge 0$, $\forall u \in Ax$, $v \in Ay$, where $j_q(x-y) \in J_q(x-y)$. A single-valued accretive operator $A$ is said to be $\alpha$-inverse-strongly accretive of order $q$ if, $\forall x, y \in C$, there exist $\alpha > 0$ and $j_q(x-y) \in J_q(x-y)$ such that $\langle u-v, j_q(x-y)\rangle \ge \alpha\|Ax - Ay\|^q$, $\forall u \in Ax$, $v \in Ay$. Back to Hilbert spaces, $A$ is called the inverse-strongly monotone. This class of mappings is a key component in projection-based approximation methods; see, e.g., [18–22]. An accretive operator $A$ is said to be $m$-accretive if and only if $A$ is accretive and satisfies the range condition: $(I + \lambda A)C = X$ for all $\lambda > 0$. For an accretive operator $A$, we define the mapping $J_\lambda^A : (I + \lambda A)C \to C$ by $J_\lambda^A = (I + \lambda A)^{-1}$ for each $\lambda > 0$. Such $J_\lambda^A$ is called the resolvent of $A$; see, e.g., [23–25] and the references therein. Recall now that a single-valued mapping $F : C \to X$ is called $\eta$-strongly accretive if $\langle Fx - Fy, j(x-y)\rangle \ge \eta\|x-y\|^2$ for some $\eta \in (0,1)$ and $j(x-y) \in J(x-y)$. Moreover, $F$ is called $\xi$-strictly pseudocontractive if, $\forall x, y \in C$, $\langle Fx - Fy, j(x-y)\rangle \le \|x-y\|^2 - \xi\|x - y - (Fx - Fy)\|^2$ for some $\xi \in (0,1)$, where $j(x-y) \in J(x-y)$.

Let $F : C \to X$ be a mapping. Then (i) if $F : C \to X$ is $\eta$-strongly accretive and $\xi$-strictly pseudocontractive with $\eta + \xi \geq 1$, then $I - F$ is nonexpansive, and $F$ is Lipschitz continuous with constant $1 + \frac{1}{\xi}$; (ii) if $F : C \to X$ is $\eta$-strongly accretive and $\xi$-strictly pseudocontractive with $\eta + \xi \geq 1$, then, for any fixed $\tau \in (0,1)$, $I - \tau F$ is a contraction with constant $1 - \tau(1 - \sqrt{\frac{1-\eta}{\xi}})$.

From now on, one employs $\Pi$ to denote a mapping from $C$ onto its subset $D$. One says that $\Pi$ is sunny if, whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$, $\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x)$. A mapping $\Pi$ defined on $C$ is called a retraction if $\Pi = \Pi^2$. One says that subset $D$ is a sunny nonexpansive retract of the set $C$ if there exists a sunny nonexpansive retraction from $C$ onto $D$.

Let $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive mappings defined on $C$, which is a convex and closed subset of a strictly convex Banach space, and let $\{\zeta_n\}_{n=0}^{\infty}$ be a sequence in $[0,1]$. For any $n \geq 0$, define a mapping $W_n : C \to C$ as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \zeta_n S_n U_{n,n+1} + (1 - \zeta_n)I, \\ \cdots \\ U_{n,1} = \zeta_1 S_1 U_{n,2} + (1 - \zeta_1)I, \\ W_n = U_{n,0} = \zeta_0 S_0 U_{n,1} + (1 - \zeta_0)I. \end{cases} \tag{2.1}$$

**Lemma 2.1** ([25, 26]) *Suppose that $\{S_n\}_{n=0}^{\infty}$ is a countable family of nonexpansive mappings defined on a subset $C$ of a strictly convex space $X$. Suppose that $\bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \neq \emptyset$, and $\{\zeta_n\}_{n=0}^{\infty}$ is a real sequence such that $0 < \zeta_n \leq b < 1$, $\forall n \geq 0$. Then*

(i) *$W_n$ is nonexpansive and $\mathrm{Fix}(W_n) = \bigcap_{i=0}^{n} \mathrm{Fix}(S_i)$, $\forall n \geq 0$;*

(ii) *the limit $\lim_{n\to\infty} U_{n,k}x$ exists for all $x \in C$ and $k \geq 0$;*

(iii) *the mapping $W : C \to C$ defined by $Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,0}x$, $\forall x \in C$, is a nonexpansive mapping satisfying $\mathrm{Fix}(W) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$ and it is called the $W$-mapping. If $D$ is any bounded subset of $C$, then $\lim_{n\to\infty} \sup_{x\in D} \|W_n x - Wx\| = 0$.*

For our main strong convergence theorems, the following tools are also needed.

**Lemma 2.2** ([27]) *Let $X$ be smooth, $D$ be a nonempty subset of $C$ and $\Pi$ be a retraction of $C$ onto $D$. Then the following are equivalent*: (i) *$\Pi$ is sunny and nonexpansive*; (ii) $\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, J(\Pi(x) - \Pi(y))\rangle$, $\forall x, y \in C$; (iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \leq 0$, $\forall x \in C$, $y \in D$.

**Lemma 2.3** ([28]) *Let $q \in (1,2]$ a given real number and let $X$ be $q$-uniformly smooth. Then $\|x + y\|^q \leq q\langle y, J_q(x)\rangle + \|x\|^q + \kappa_q \|y\|^q$, $\forall x, y \in X$, where $\kappa_q$ is the $q$-uniformly smooth constant of $X$. For any given $x, y \in X$, one has $\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y)\rangle$, $\forall j_q(x + y) \in J_q(x + y)$.*

**Lemma 2.4** ([28, 29]) *Let $X$ be a uniformly convex and $q$-uniformly, where $1 < q \leq 2$, smooth Banach space. Let $A : C \to X$ be an $\alpha$-inverse-strongly accretive mapping of order $q$ and $B : C \to 2^X$ be an $m$-accretive operator. In the sequel, we will use the notation $T_\lambda := J_\lambda^B(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A)$, $\forall \lambda > 0$. The following statements hold*:

(i) *the resolvent identity*: $J_\lambda x = J_\mu(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda x)$, $\forall \lambda, \mu > 0$, $x \in X$;

(ii) *if $J_\lambda^A$ is a resolvent of $A$ for $\lambda > 0$, then $J_\lambda^A$ is a single-valued nonexpansive mapping with $\mathrm{Fix}(J_\lambda^A) = A^{-1}0$, where $A^{-1}0 = \{x \in C : 0 \in Ax\}$;*

(iii) $\mathrm{Fix}(T_\lambda) = (A+B)^{-1}0$, $\forall \lambda > 0$;

(iv) $\|x - T_\lambda x\| \le 2\|x - T_s x\|$ *for* $0 < \lambda \le s$ *and* $x \in X$;

(v) $\|T_\lambda x - T_\lambda y\| \le \|x - y\|$;

(vi) $\|(I-\lambda A)x - (I-\lambda A)y\|^q \le \|x-y\|^q - \lambda(q\alpha - \kappa_q \lambda^{q-1})\|Ax - Ay\|^q$, $\forall x, y \in C$. *In particular, if* $0 < \lambda \le (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$, *then* $I - \lambda A$ *is nonexpansive.*

**Lemma 2.5** ([30]) *Let* $T : C \to C$ *be nonexpansive with* $\mathrm{Fix}(T) \ne \emptyset$, *and let* $f : C \to C$ *be a fixed contraction mapping, where* $C$ *is convex and closed set in a real reflexive Banach space with the uniformly Gâteaux differentiable norm and the normal structure. Let* $z_t \in C$, *where* $t \in (0,1)$, *be the unique fixed point of the contraction* $C \ni z \mapsto (1-t)Tz + tf(z)$ *on* $C$, *that is,* $z_t = (1-t)Tz_t + tf(z_t)$. *Then* $\{z_t\}$ *converges to* $x^* \in \mathrm{Fix}(T)$ *in norm. This convergent point also solves* $\langle (f-I)x^*, J(p-x^*)\rangle \le 0$, $\forall p \in \mathrm{Fix}(T)$.

**Lemma 2.6** ([14]) *Suppose that* $\Pi_C$ *is a sunny nonexpansive retraction from a q-uniformly smooth* $X$ *onto its convex closed subset* $C$. *Let the mapping* $A_i : C \to X$ *be* $\alpha_i$*-inverse-strongly accretive of order* $q$ *for* $i = 1,2$. *Let the mapping* $G : C \to C$ *be defined as* $Gx := \Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)$, $\forall x \in C$. *If* $0 < \mu_i \le (\frac{q\alpha_i}{\kappa_q})^{\frac{1}{q-1}}$ *for* $i = 1,2$, *then* $G : C \to C$ *is a Lipschitz mapping. More precisely, it is nonexpansive. Let* $A_1, A_2 : C \to X$ *be two nonlinear mappings. For given* $(x^*, y^*) \in C \times C$, $(x^*, y^*)$ *is a solution of SVIs* (1.1) *iff* $x^* = \Pi_C(y^* - \mu_1 A_1 y^*)$, *where* $y^* = \Pi_C(x^* - \mu_2 A_2 x^*)$.

**Lemma 2.7** ([31]) *Let* $\{a_n\}$ *be a sequence defined by* $a_{n+1} \le \gamma_n \lambda_n + a_n(1-\lambda_n)$, $\forall n \ge 0$, *where* $\{\lambda_n\}$ *and* $\{\gamma_n\}$ *are sequences of real numbers such that* (i) $\limsup_{n\to\infty} \gamma_n \le 0$ *or* $\sum_{n=0}^{\infty} |\lambda_n \gamma_n| < \infty$; (ii) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

**Lemma 2.8** ([28]) *Let* $B_r = \{x \in X : \|x\| \le r\}$, $r > 0$, *where* $X$ *is a uniformly convex Banach space. Then there exists a continuous, strictly increasing and convex function* $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ *such that, with* $p > 1$,

$$\|\alpha x + \beta y + \gamma z\|^p + \frac{\alpha^p \beta + \beta^p \alpha}{(\alpha+\beta)^p} g\big(\|x - y\|\big) \le \alpha\|x\|^p + \beta\|y\|^p + \gamma\|z\|^p$$

*for all* $x, y, z \in B_r$ *and* $\alpha, \beta, \gamma \in [0,1]$ *with* $\alpha + \beta + \gamma = 1$.

**Lemma 2.9** ([32]) *Suppose that* $\{x_n\}$ *is a sequence defined by* $x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n$, $\forall n \ge 0$, *where* $\{y_n\}$ *is bounded sequences in Banach space* $X$ *and let* $\{\alpha_n\}$ *be a real sequence such that* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$. *If* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$, *then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

## 3 Iterative algorithms and convergence criteria

**Theorem 3.1** *Let* $X$ *be a both uniformly convex and q-uniformly smooth space with* $1 < q \le 2$ *and let* $B : C \to 2^X$ *be an m-accretive operator. Let* $A_i : C \to X$ *be an* $\alpha_i$*-inverse-strongly accretive operator of order* $q$ *for each* $i = 1,2$ *and* $A : C \to X$ *be an* $\alpha$*-inverse-strongly accretive of order* $q$. *Assume that* $\Omega = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{SVI}(C, A_1, A_2) \cap (A+B)^{-1}0 \ne \emptyset$, *where* $\mathrm{SVI}(C, A_1, A_2)$ *is the fixed point set of* $G := \Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)$ *with* $0 < \mu_i < (\frac{q\alpha_i}{\kappa_q})^{\frac{1}{q-1}}$ *for* $i = 1,2$. *Let* $f : C \to C$ *be a* $\delta$*-contraction with constant* $\delta \in (0,1)$ *and let* $F : C \to X$ *be*

*$\eta$-strongly accretive and $\xi$-strictly pseudocontractive with $\eta + \xi \geq 1$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} v_n = \Pi_C(I - \mu_1 A_1)\Pi_C(y_n - \mu_2 A_2 y_n), \\ y_n = \beta_n x_n + \gamma_n \Pi_C(I - \sigma_n F)(t_n x_n + (1 - t_n)W_n v_n) + \alpha_n f(y_n), \\ x_{n+1} = \delta_n x_n + (1 - \delta_n)J^B_{\lambda_n}(y_n - \lambda_n A y_n), \quad n \geq 0, \end{cases} \tag{3.1}$$

*where $\Pi_C$ is the sunny nonexpansive retraction from $X$ onto $C$, $\{W_n\}$ is the sequence defined by (2.1), $\{\lambda_n\} \subset (0, (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}})$, $\{\sigma_n\} \subset [0,1)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}, \{\delta_n\}, \{t_n\} \subset (0,1)$ satisfy the following conditions*:

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\sum_{n=0}^\infty \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\lim_{n\to\infty} \frac{\sigma_n}{\alpha_n} = 0$, $\lim_{n\to\infty} |\gamma_n - \gamma_{n-1}| = 0$ *and* $\lim_{n\to\infty} |\beta_n - \beta_{n-1}| = 0$;
(iii) $\lim_{n\to\infty} |t_n - t_{n-1}| = 0$, $\limsup_{n\to\infty} \gamma_n t_n(1 - t_n) < 1$ *and* $\liminf_{n\to\infty} \gamma_n(1 - t_n) > 0$;
(iv) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, $\limsup_{n\to\infty} \delta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;
(v) $0 < \bar{\lambda} \leq \lambda_n$, $\forall n \geq 0$ *and* $\lim_{n\to\infty} \lambda_n = \lambda < (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$.

*Then $x_n \to x^* \in \Omega$, which is a unique solution to the generalized variational inequality (GVI) $\langle (I - f)x^*, J(x^* - p)\rangle \leq 0$, $\forall p \in \Omega$.*

*Proof* Put $u_n = \Pi_C(y_n - \mu_2 A_2 y_n)$. It is easy to see that scheme (3.1) can be rewritten as

$$\begin{cases} y_n = \beta_n x_n + \gamma_n \Pi_C(I - \sigma_n F)(t_n x_n + (1 - t_n)W_n G y_n) + \alpha_n f(y_n), \\ x_{n+1} = \delta_n x_n + (1 - \delta_n)T_n y_n, \quad n \geq 0, \end{cases} \tag{3.2}$$

where $T_n := J^B_{\lambda_n}(I - \lambda_n A)$. From $\eta + \xi \geq 1$, $\{\sigma_n\} \subset [0,1)$, one asserts that $\Pi_C(I - \sigma_n F) : C \to C$ is a nonexpansive mapping for each $n \geq 0$. Because of the situation $\alpha_n + \beta_n + \gamma_n = 1$, one knows that

$$\alpha_n\delta + \gamma_n(1 - t_n) + \beta_n + \gamma_n t_n = \alpha_n\delta + \gamma_n + \beta_n = 1 - \alpha_n(1 - \delta) \quad \forall n \geq 0.$$

One now shows that the sequence $\{x_n\}$ generated by (3.2) is well defined. Define a mapping $F_n : C \to C$ by $F_n(x) = \beta_n x_n + \gamma_n \Pi_C(I - \sigma_n F)(t_n x_n + (1 - t_n)W_n G x) + \alpha_n f(x)$, $\forall x \in C$. Then

$$\begin{aligned} \|F_n(x) - F_n(y)\| &\leq \gamma_n \big\| \Pi_C(I - \sigma_n F)\big(t_n x_n + (1 - t_n)W_n G x\big) \\ &\quad - \Pi_C(I - \sigma_n F)\big(t_n x_n + (1 - t_n)W_n G y\big) \big\| \\ &\quad + \alpha_n \big\|f(x) - f(y)\big\| \\ &\leq \gamma_n(1 - t_n)\|W_n G x - W_n G y\| + \alpha_n\delta\|x - y\| \\ &\leq \big(1 - \alpha_n(1 - \delta)\big)\|x - y\|. \end{aligned}$$

This guarantees the result that $F_n$ is a contraction mapping. Hence there is a unique fixed point $y_n \in C$ satisfying

$$y_n = \beta_n x_n + \gamma_n \Pi_C(I - \sigma_n F)\big((1 - t_n)W_n G y_n + t_n x_n\big) + \alpha_n f(y_n).$$

One next divides the rest of the proof into several steps.

*Step 1.* Show that $\{x_n\}$ is bounded.

From $\{\lambda_n\} \subset (0, (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}})$, one observes that $T_n : C \to C$ is a nonexpansive mapping for each $n \geq 0$. Take a fixed $p \in \Omega = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{SVI}(C, A_1, A_2) \cap (A + B)^{-1}0$ arbitrarily. From Lemmas 2.4 and 2.6, we know that $W_n p = p$, $Gp = p$ and $T_n p = p$. Moreover, using the nonexpansivity of $W_n$ and $G$ yields

$$\begin{aligned}
\|y_n - p\| &\leq \beta_n \|x_n - p\| + \gamma_n \big\| \Pi_C(I - \sigma_n F)\big(t_n x_n + (1 - t_n) W_n G y_n\big) - \Pi_C(I - \sigma_n F)p \big\| \\
&\quad + \gamma_n \big\| \Pi_C(I - \sigma_n F)p - p \big\| + \alpha_n \big( \big\| f(y_n) - f(p) \big\| + \big\| f(p) - p \big\| \big) \\
&\leq \beta_n \|x_n - p\| + \alpha_n \big( \delta \|y_n - p\| + \big\| f(p) - p \big\| \big) + \gamma_n \big[ t_n \|x_n - p\| \\
&\quad + (1 - t_n) \|W_n G y_n - p\| \big] \\
&\quad + \gamma_n \sigma_n \|Fp\| \\
&\leq (\beta_n + \gamma_n t_n) \|x_n - p\| + \alpha_n \big\| f(p) - p \big\| + \sigma_n \|Fp\| + \big( \alpha_n \delta + \gamma_n (1 - t_n) \big) \|y_n - p\|,
\end{aligned}$$

which hence implies that

$$\|y_n - p\| \leq \frac{\alpha_n \|f(p) - p\| + \sigma_n \|Fp\|}{1 - (\alpha_n \delta + \gamma_n (1 - t_n))} + \frac{1 - (\alpha_n \delta + \gamma_n (1 - t_n)) - \alpha_n (1 - \delta)}{1 - (\alpha_n \delta + \gamma_n (1 - t_n))} \|x_n - p\|. \quad (3.3)$$

Since $\lim_{n\to\infty} \frac{\sigma_n}{\alpha_n} = 0$, one may suppose $\sigma_n \leq \alpha_n$. Thus, from (3.2), (3.3) and the nonexpansivity of $T_n$, we find that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \delta_n \|x_n - p\| + (1 - \delta_n) \|T_n y_n - p\| \\
&\leq \delta_n \|x_n - p\| + (1 - \delta_n) \|y_n - p\| \\
&\leq \delta_n \|x_n - p\| + (1 - \delta_n) \bigg\{ \bigg( 1 - \frac{\alpha_n (1 - \delta)}{1 - (\alpha_n \delta + \gamma_n (1 - t_n))} \bigg) \|x_n - p\| \\
&\quad + \frac{\alpha_n \|f(p) - p\| + \alpha_n \|Fp\|}{1 - (\alpha_n \delta + \gamma_n (1 - t_n))} \bigg\} \\
&= \bigg[ 1 - \frac{(1 - \delta_n)(1 - \delta)}{1 - (\alpha_n \delta + \gamma_n (1 - t_n))} \alpha_n \bigg] \|x_n - p\| \\
&\quad + \frac{(1 - \delta_n)(1 - \delta)}{1 - (\alpha_n \delta + \gamma_n (1 - t_n))} \alpha_n \frac{\|f(p) - p\| + \|Fp\|}{1 - \delta} \\
&\leq \max \bigg\{ \frac{\|f(p) - p\| + \|Fp\|}{1 - \delta}, \|x_n - p\| \bigg\}.
\end{aligned}$$

It immediately follows that $\{x_n\}$ is a bounded vector in set $C$.

*Step 2.* One shows that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Indeed,

$$\begin{aligned}
z_n - z_{n-1} &= (t_n - t_{n-1})(x_{n-1} - W_{n-1} G y_{n-1}) + (1 - t_n)(W_n G y_n - W_{n-1} G y_{n-1}) \\
&\quad + t_n (x_n - x_{n-1})
\end{aligned}$$

and

$$\begin{aligned} y_n - y_{n-1} &= (\alpha_n - \alpha_{n-1}) f(y_{n-1}) + \beta_n (x_n - x_{n-1}) + \alpha_n \bigl(f(y_n) - f(y_{n-1})\bigr) \\ &\quad + (\beta_n - \beta_{n-1}) x_{n-1} + \gamma_n \bigl(\Pi_C (I - \sigma_n F) z_n - \Pi_C (I - \sigma_{n-1} F) z_{n-1}\bigr) \\ &\quad + (\gamma_n - \gamma_{n-1}) \Pi_C (I - \sigma_{n-1} F) z_{n-1}. \end{aligned} \tag{3.4}$$

Utilizing Lemmas 2.1 and 2.4 yields

$$\begin{aligned} &\|T_n y_n - T_{n-1} y_{n-1}\| \\ &\quad \le \|T_n y_n - T_n y_{n-1}\| + \|T_n y_{n-1} - T_{n-1} y_{n-1}\| \\ &\quad \le \bigl\|J_{\lambda_n}^B (I - \lambda_n A) y_{n-1} - J_{\lambda_{n-1}}^B (I - \lambda_n A) y_{n-1}\bigr\| + \|y_n - y_{n-1}\| \\ &\qquad + \bigl\|J_{\lambda_{n-1}}^B (I - \lambda_n A) y_{n-1} - J_{\lambda_{n-1}}^B (I - \lambda_{n-1} A) y_{n-1}\bigr\| \\ &\quad = \|y_n - y_{n-1}\| + \left\| J_{\lambda_{n-1}}^B \left( \frac{\lambda_{n-1}}{\lambda_n} I + \left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right) J_{\lambda_n}^B \right) (I - \lambda_n A) y_{n-1} \right. \\ &\qquad \left. - J_{\lambda_{n-1}}^B (I - \lambda_n A) y_{n-1} \right\| + \bigl\|J_{\lambda_{n-1}}^B (I - \lambda_n A) y_{n-1} - J_{\lambda_{n-1}}^B (I - \lambda_{n-1} A) y_{n-1}\bigr\| \\ &\quad \le \left|1 - \frac{\lambda_{n-1}}{\lambda_n}\right| \bigl\|J_{\lambda_n}^B (I - \lambda_n A) y_{n-1} - (I - \lambda_n A) y_{n-1}\bigr\| + \|y_n - y_{n-1}\| \\ &\qquad + |\lambda_n - \lambda_{n-1}| \|A y_{n-1}\| \\ &\quad \le |\lambda_n - \lambda_{n-1}| M_1 + \|y_n - y_{n-1}\|, \end{aligned} \tag{3.5}$$

where $\sup_{n\ge 1}\{\frac{1}{\lambda}\|J_{\lambda_n}^B (I - \lambda_n A) y_{n-1} - (I - \lambda_n A) y_{n-1}\| + \|A y_{n-1}\|\} \le M_1$ for some $M_1 > 0$. Also, it follows from the nonexpansivity of $\Pi_C$ and $(I - \sigma_n F)$ that

$$\begin{aligned} &\bigl\|\Pi_C (I - \sigma_n F) z_n - \Pi_C (I - \sigma_{n-1} F) z_{n-1}\bigr\| \\ &\quad \le \bigl\|\Pi_C (I - \sigma_n F) z_n - \Pi_C (I - \sigma_n F) z_{n-1}\bigr\| + \bigl\|\Pi_C (I - \sigma_n F) z_{n-1} - \Pi_C (I - \sigma_{n-1} F) z_{n-1}\bigr\| \\ &\quad \le \|z_n - z_{n-1}\| + |\sigma_n - \sigma_{n-1}| \|F z_{n-1}\| \\ &\quad \le t_n \|x_n - x_{n-1}\| + |t_n - t_{n-1}| \|x_{n-1} - W_{n-1} G y_{n-1}\| \\ &\qquad + (1 - t_n) \|W_n G y_n - W_{n-1} G y_{n-1}\| + |\sigma_n - \sigma_{n-1}| \|F z_{n-1}\| \\ &\quad \le t_n \|x_n - x_{n-1}\| + |t_n - t_{n-1}| \|x_{n-1} - W_{n-1} G y_{n-1}\| \\ &\qquad + (1 - t_n) \bigl[\|y_n - y_{n-1}\| + \|W_n G y_{n-1} - W_{n-1} G y_{n-1}\|\bigr] + |\sigma_n - \sigma_{n-1}| \|F z_{n-1}\|. \end{aligned}$$

This together with (3.4) guarantees

$$\begin{aligned} \|y_n - y_{n-1}\| &\le \alpha_n \delta \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \bigl\|f(y_{n-1})\bigr\| + \beta_n \|x_n - x_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \gamma_n \bigl\{ t_n \|x_n - x_{n-1}\| + |t_n - t_{n-1}| \|x_{n-1} - W_{n-1} G y_{n-1}\| \\ &\quad + (1 - t_n) \bigl[\|y_n - y_{n-1}\| + \|W_n G y_{n-1} - W_{n-1} G y_{n-1}\|\bigr] \\ &\quad + |\sigma_n - \sigma_{n-1}| \|F z_{n-1}\| \bigr\} \\ &\quad + |\gamma_n - \gamma_{n-1}| \bigl\|\Pi_C (I - \sigma_{n-1} F) z_{n-1}\bigr\| \end{aligned}$$

$$\begin{aligned}&\le \big(\alpha_n\delta+\gamma_n(1-t_n)\big)\|y_n-y_{n-1}\|+(\beta_n+\gamma_n t_n)\|x_n-x_{n-1}\|+\big(|\alpha_n-\alpha_{n-1}|\\&\quad+|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|+|\sigma_n-\sigma_{n-1}|+|t_n-t_{n-1}|\big)M_2\\&\quad+\|W_nGy_{n-1}-W_{n-1}Gy_{n-1}\|,\end{aligned}$$

where $\sup_{n\ge 0}\{\|x_n\|+\|f(y_n)\|+\|W_nGy_n\|+\|Fz_n\|+\|\Pi_C(I-\sigma_nF)z_n\|\}\le M_2$ for some $M_2>0$. Then

$$\begin{aligned}\|y_n-y_{n-1}\|&\le \frac{\beta_n+\gamma_n t_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|x_n-x_{n-1}\|\\&\quad+\frac{1}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\big[\big(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|\\&\quad+|\gamma_n-\gamma_{n-1}|+|\sigma_n-\sigma_{n-1}|+|t_n-t_{n-1}|\big)M_2+\|W_nGy_{n-1}-W_{n-1}Gy_{n-1}\|\big]\\&=\left(1-\frac{\alpha_n(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\right)\|x_n-x_{n-1}\|\\&\quad+\frac{1}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\big[\big(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|\\&\quad+|\gamma_n-\gamma_{n-1}|+|\sigma_n-\sigma_{n-1}|+|t_n-t_{n-1}|\big)M_2+\|W_nGy_{n-1}-W_{n-1}Gy_{n-1}\|\big]\\&\le\|x_n-x_{n-1}\|\\&\quad+\frac{1}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\big[\big(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|\\&\quad+|\sigma_n-\sigma_{n-1}|+|t_n-t_{n-1}|\big)M_2+\|W_nGy_{n-1}-W_{n-1}Gy_{n-1}\|\big],\end{aligned}$$

which together with (3.5) asserts that

$$\begin{aligned}\|T_ny_n-T_{n-1}y_{n-1}\|-\|x_n-x_{n-1}\|&\le\frac{1}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\big[\big(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|\\&\quad+|\gamma_n-\gamma_{n-1}|+|\sigma_n-\sigma_{n-1}|+|t_n-t_{n-1}|\big)M_2\\&\quad+\|W_nGy_{n-1}-W_{n-1}Gy_{n-1}\|\big]+|\lambda_n-\lambda_{n-1}|M_1.\end{aligned}$$

Since $\lim_{n\to\infty}\sup_{x\in D}\|W_nx-Wx\|=0$ on bounded subset $D=\{Gy_n:n\ge 0\}$ of $C$, one knows that

$$\lim_{n\to\infty}\|W_nGy_{n-1}-W_{n-1}Gy_{n-1}\|=0.$$

Note that $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\frac{\sigma_n}{\alpha_n}=0$, $\lim_{n\to\infty}\lambda_n=\lambda$ and $\liminf_{n\to\infty}(1-(\alpha_n\delta+\gamma_n(1-t_n)))>0$. Thus, from $|\beta_n-\beta_{n-1}|\to 0$, $|\gamma_n-\gamma_{n-1}|\to 0$ and $|t_n-t_{n-1}|\to 0$ as $n\to\infty$ (due to conditions (ii), (iii)), we get

$$\limsup_{n\to\infty}\big(\|T_ny_n-T_{n-1}y_{n-1}\|-\|x_n-x_{n-1}\|\big)\le 0.$$

So it follows from condition (iv) and Lemma 2.9 that $\lim_{n\to\infty}\|T_ny_n-x_n\|=0$. Hence

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\delta_n)\|T_ny_n-x_n\|=0. \tag{3.6}$$

*Step 3.* One shows that $\|x_n - y_n\| \to 0$ and $\|x_n - Gx_n\| \to 0$ as $n \to \infty$. Indeed, for simplicity, set $\bar{p} := \Pi_C(I - \mu_2 A_2)p$. Note that $u_n = \Pi_C(I - \mu_2 A_2)y_n$ and $v_n = \Pi_C(I - \mu_1 A_1)u_n$. Then $v_n = Gy_n$. An application of Lemma 2.4 yields

$$\begin{aligned} \|u_n - \bar{p}\|^q &\le \big\|(I - \mu_2 A_2)y_n - (I - \mu_2 A_2)p\big\|^q \\ &\le \|y_n - p\|^q - \mu_2\big(q\alpha_2 - \kappa_q \mu_2^{q-1}\big)\|A_2 y_n - A_2 p\|^q. \end{aligned} \tag{3.7}$$

One also has

$$\|v_n - p\|^q \le \|u_n - \bar{p}\|^q - \mu_1\big(q\alpha_1 - \kappa_q \mu_1^{q-1}\big)\|A_1 u_n - A_1 \bar{p}\|^q. \tag{3.8}$$

By using (3.7) and (3.8), one reaches

$$\begin{aligned} \|v_n - p\|^q \le{}& \|y_n - p\|^q - \mu_2\big(q\alpha_2 - \kappa_q \mu_2^{q-1}\big)\|A_2 y_n - A_2 p\|^q \\ &- \mu_1\big(q\alpha_1 - \kappa_q \mu_1^{q-1}\big)\|A_1 u_n - A_1 \bar{p}\|^q. \end{aligned} \tag{3.9}$$

Equations (3.2) and (3.9) further guarantee that $\|z_n - p\|^q \le t_n\|x_n - p\|^q + (1 - t_n)\|v_n - p\|^q$ and

$$\begin{aligned} \big\|\Pi_C(I - \sigma_n F)z_n - p\big\|^q &\le \|z_n - p - \sigma_n F z_n\|^q \\ &\le \|z_n - p\|^q - q\sigma_n\big\langle F z_n, J_q(z_n - p - \sigma_n F z_n)\big\rangle \\ &\le \|z_n - p\|^q + q\sigma_n\|F z_n\|\|z_n - p - \sigma_n F z_n\|^{q-1}. \end{aligned}$$

Thus

$$\begin{aligned} &\|y_n - p\|^q \\ &\quad\le \beta_n\|x_n - p\|^q + \gamma_n\big\|\Pi_C(I - \sigma_n F)z_n - p\big\|^q + q\alpha_n\big\langle f(p) - p, J_q(y_n - p)\big\rangle \\ &\qquad + \alpha_n\big\|f(y_n) - f(p)\big\|^q \\ &\quad\le \beta_n\|x_n - p\|^q + \gamma_n\big[\|z_n - p\|^q + q\sigma_n\|F z_n\|\|z_n - p - \sigma_n F z_n\|^{q-1}\big] \\ &\qquad + q\alpha_n\big\langle f(p) - p, J_q(y_n - p)\big\rangle + \alpha_n\big\|f(y_n) - f(p)\big\|^q \\ &\quad\le \beta_n\|x_n - p\|^q + \alpha_n\delta\|y_n - p\|^q + \gamma_n\big[t_n\|x_n - p\|^q + (1 - t_n)\|v_n - p\|^q \\ &\qquad + q\sigma_n\|F z_n\|\|z_n - p - \sigma_n F z_n\|^{q-1}\big] + q\alpha_n\big\|f(p) - p\big\|\|y_n - p\|^{q-1} \\ &\quad\le (\beta_n + \gamma_n t_n)\|x_n - p\|^q + \alpha_n\delta\|y_n - p\|^q + \gamma_n(1 - t_n)\big[\|y_n - p\|^q \\ &\qquad - \mu_2\big(q\alpha_2 - \kappa_q \mu_2^{q-1}\big)\|A_2 y_n - A_2 p\|^q - \mu_1\big(q\alpha_1 - \kappa_q \mu_1^{q-1}\big)\|A_1 u_n - A_1 \bar{p}\|^q\big] \\ &\qquad + q\sigma_n\|F z_n\|\|z_n - p - \sigma_n F z_n\|^{q-1} + q\alpha_n\big\|f(p) - p\big\|\|y_n - p\|^{q-1}, \end{aligned}$$

which immediately yields

$$\begin{aligned} \|y_n - p\|^q \le{}& \left(1 - \frac{\alpha_n(1 - \delta)}{1 - (\alpha_n\delta + \gamma_n(1 - t_n))}\right)\|x_n - p\|^q - \frac{\gamma_n(1 - t_n)}{1 - (\alpha_n\delta + \gamma_n(1 - t_n))} \\ &\times \big[\mu_2\big(q\alpha_2 - \kappa_q \mu_2^{q-1}\big)\|A_2 y_n - A_2 p\|^q + \mu_1\big(q\alpha_1 - \kappa_q \mu_1^{q-1}\big)\|A_1 u_n - A_1 \bar{p}\|^q\big] \end{aligned}$$

$$+\frac{q\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\big[\|Fz_n\|\|z_n-p-\sigma_nFz_n\|^{q-1}$$
$$+\big\|f(p)-p\big\|\|y_n-p\|^{q-1}\big].$$

On the other hand, (3.2) implies

$$\begin{aligned}
&\|x_{n+1}-p\|^q\\
&\quad\le(1-\delta_n)\|y_n-p\|^q+\delta_n\|x_n-p\|^q\\
&\quad\le\delta_n\|x_n-p\|^q+(1-\delta_n)\biggl\{\biggl(1-\frac{\alpha_n(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\biggr)\|x_n-p\|^q\\
&\qquad-\frac{\gamma_n(1-t_n)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\\
&\qquad\times\big[\mu_2\big(q\alpha_2-\kappa_q\mu_2^{q-1}\big)\|A_2y_n-A_2p\|^q+\mu_1\big(q\alpha_1-\kappa_q\mu_1^{q-1}\big)\|A_1u_n-A_1\bar{p}\|^q\big]\\
&\qquad+\frac{q\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\big[\|Fz_n\|\|z_n-p-\sigma_nFz_n\|^{q-1}+\big\|f(p)-p\big\|\|y_n-p\|^{q-1}\big]\biggr\}\\
&\quad=\biggl(1-\frac{\alpha_n(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\biggr)\|x_n-p\|^q-\frac{\gamma_n(1-\delta_n)(1-t_n)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\big[\mu_2\big(q\alpha_2-\kappa_q\mu_2^{q-1}\big)\\
&\qquad\times\|A_2y_n-A_2p\|^q+\mu_1\big(q\alpha_1-\kappa_q\mu_1^{q-1}\big)\|A_1u_n-A_1\bar{p}\|^q\big]\\
&\qquad+\frac{q(1-\delta_n)\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\big[\|Fz_n\|\|z_n-p-\sigma_nFz_n\|^{q-1}+\big\|f(p)-p\big\|\|y_n-p\|^{q-1}\big]\\
&\quad\le\|x_n-p\|^q-\frac{(1-\delta_n)\gamma_n(1-t_n)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\big[\mu_2\big(q\alpha_2-\kappa_q\mu_2^{q-1}\big)\|A_2y_n-A_2p\|^q\\
&\qquad+\mu_1\big(q\alpha_1-\kappa_q\mu_1^{q-1}\big)\|A_1u_n-A_1\bar{p}\|^q\big]+\alpha_nM_3,
\end{aligned}\tag{3.10}$$

where

$$\sup_{n\ge0}\biggl\{\frac{q(1-\delta_n)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\big[\|Fz_n\|\|z_n-p-\sigma_nFz_n\|^{q-1}+\big\|f(p)-p\big\|\|y_n-p\|^{q-1}\big]\biggr\}\le M_3$$

for some $M_3>0$. So it follows from (3.10) that

$$\begin{aligned}
&\frac{(1-\delta_n)\gamma_n(1-t_n)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\big[\mu_2\big(q\alpha_2-\kappa_q\mu_2^{q-1}\big)\|A_2y_n-A_2p\|^q\\
&\qquad+\mu_1\big(q\alpha_1-\kappa_q\mu_1^{q-1}\big)\|A_1u_n-A_1\bar{p}\|^q\big]\\
&\quad\le\|x_n-p\|^q-\|x_{n+1}-p\|^q+\alpha_nM_3\\
&\quad\le q\|x_n-x_{n+1}\|\|x_{n+1}-p\|^{q-1}+\kappa_q\|x_n-x_{n+1}\|^q+\alpha_nM_3.
\end{aligned}$$

Thanks to $0<\mu_i<(\frac{q\alpha_i}{\kappa_q})^{\frac{1}{q-1}}$ for $i=1,2$, $\liminf_{n\to\infty}\gamma_n(1-t_n)>0$, $\liminf_{n\to\infty}(1-\delta_n)>0$ and $\lim_{n\to\infty}\alpha_n=0$, one asserts

$$\lim_{n\to\infty}\|A_2y_n-A_2p\|=0\quad\text{and}\quad\lim_{n\to\infty}\|A_1u_n-A_1\bar{p}\|=0.\tag{3.11}$$

This further implies

$$\begin{aligned}\|u_n-\bar{p}\|^2 &\le \langle (I-\mu_2A_2)y_n-(I-\mu_2A_2)p, J(u_n-\bar{p})\rangle \\ &= \langle y_n-p, J(u_n-\bar{p})\rangle + \mu_2\langle A_2p-A_2y_n, J(u_n-\bar{p})\rangle \\ &\le \mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\| \\ &\quad + \frac{1}{2}\bigl[\|y_n-p\|^2+\|u_n-\bar{p}\|^2-g_1\bigl(\|y_n-u_n-(p-\bar{p})\|\bigr)\bigr],\end{aligned}$$

from which one concludes

$$\|u_n-\bar{p}\|^2 \le \|y_n-p\|^2-g_1\bigl(\|y_n-u_n-(p-\bar{p})\|\bigr)+2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|. \tag{3.12}$$

One also derives that

$$\|v_n-p\|^2 \le \|u_n-\bar{p}\|^2-g_2\bigl(\|u_n-v_n+(p-\bar{p})\|\bigr)+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|. \tag{3.13}$$

Employing (3.12) and (3.13), one arrives at

$$\begin{aligned}\|v_n-p\|^2 &\le \|y_n-p\|^2-g_1\bigl(\|y_n-u_n-(p-\bar{p})\|\bigr)-g_2\bigl(\|u_n-v_n+(p-\bar{p})\|\bigr) \\ &\quad +2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|.\end{aligned} \tag{3.14}$$

Utilizing Lemma 2.8, we obtain from (3.2) and (3.14)

$$\begin{aligned}\|z_n-p\|^2 &\le t_n\|x_n-p\|^2+(1-t_n)\|W_nGy_n-p\|^2-t_n(1-t_n)g_3\bigl(\|x_n-W_nGy_n\|\bigr) \\ &\le t_n\|x_n-p\|^2+(1-t_n)\|v_n-p\|^2-t_n(1-t_n)g_3\bigl(\|x_n-W_nGy_n\|\bigr),\end{aligned}$$

and hence

$$\begin{aligned}\|y_n-p\|^2 &\le \beta_n\|x_n-p\|^2+\alpha_n\bigl\|f(y_n)-f(p)\bigr\|^2+\gamma_n\bigl\|\Pi_C(I-\sigma_nF)z_n-p\bigr\|^2 \\ &\quad -\beta_n\gamma_ng_4\bigl(\|x_n-\Pi_C(I-\sigma_nF)z_n\|\bigr)+2\alpha_n\langle f(p)-p, J(y_n-p)\rangle \\ &\le \beta_n\|x_n-p\|^2+\alpha_n\delta\|y_n-p\|^2+\gamma_n\bigl[t_n\|x_n-p\|^2+(1-t_n)\|v_n-p\|^2 \\ &\quad -t_n(1-t_n)g_3\bigl(\|x_n-W_nGy_n\|\bigr)+2\sigma_n\|Fz_n\|\|z_n-p-\sigma_nFz_n\|\bigr] \\ &\quad +2\alpha_n\bigl\|f(p)-p\bigr\|\|y_n-p\|-\beta_n\gamma_ng_4\bigl(\|x_n-\Pi_C(I-\sigma_nF)z_n\|\bigr) \\ &\le \beta_n\|x_n-p\|^2+\alpha_n\delta\|y_n-p\|^2+\gamma_n\bigl\{t_n\|x_n-p\|^2+(1-t_n)\bigl[\|y_n-p\|^2 \\ &\quad -g_1\bigl(\|y_n-u_n-(p-\bar{p})\|\bigr)-g_2\bigl(\|u_n-v_n+(p-\bar{p})\|\bigr) \\ &\quad +2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|\bigr] \\ &\quad -t_n(1-t_n)g_3\bigl(\|x_n-W_nGy_n\|\bigr)+2\sigma_n\|Fz_n\|\|z_n-p-\sigma_nFz_n\|\bigr\} \\ &\quad +2\alpha_n\bigl\|f(p)-p\bigr\|\|y_n-p\|-\beta_n\gamma_ng_4\bigl(\|x_n-\Pi_C(I-\sigma_nF)z_n\|\bigr) \\ &\le (\beta_n+\gamma_nt_n)\|x_n-p\|^2+\bigl(\alpha_n\delta+\gamma_n(1-t_n)\bigr)\|y_n-p\|^2 \\ &\quad -\gamma_n(1-t_n)\bigl[g_1\bigl(\|y_n-u_n-(p-\bar{p})\|\bigr)+g_2\bigl(\|u_n-v_n+(p-\bar{p})\|\bigr)\bigr]\end{aligned}$$

$$
\begin{aligned}
&+ 2\mu_2\|A_2p - A_2y_n\|\|u_n - \bar{p}\| + 2\mu_1\|A_1\bar{p} - A_1u_n\|\|v_n - p\| \\
&+ 2\sigma_n\|Fz_n\|\|z_n - p - \sigma_nFz_n\| + 2\alpha_n\|f(p) - p\|\|y_n - p\| \\
&- \gamma_nt_n(1-t_n)g_3(\|x_n - W_nGy_n\|) - \beta_n\gamma_ng_4(\|x_n - \Pi_C(I - \sigma_nF)z_n\|),
\end{aligned}
$$

which immediately yields

$$
\begin{aligned}
\|y_n - p\|^2 \le{}& \left(1 - \frac{\alpha_n(1-\delta)}{1 - (\alpha_n\delta + \gamma_n(1-t_n))}\right)\|x_n - p\|^2 \\
&- \frac{\gamma_n(1-t_n)}{1 - (\alpha_n\delta + \gamma_n(1-t_n))}\big[g_1(\|y_n - u_n - (p - \bar{p})\|) \\
&+ g_2(\|u_n - v_n + (p - \bar{p})\|)\big] \\
&+ \frac{2}{1 - (\alpha_n\delta + \gamma_n(1-t_n))}\big[\mu_2\|A_2p - A_2y_n\|\|u_n - \bar{p}\| \\
&+ \mu_1\|A_1\bar{p} - A_1u_n\|\|v_n - p\| + \alpha_n\|Fz_n\|\|z_n - p - \sigma_nFz_n\| \\
&+ \alpha_n\|f(p) - p\|\|y_n - p\|\big] \\
&- \frac{1}{1 - (\alpha_n\delta + \gamma_n(1-t_n))}\big[\gamma_nt_n(1-t_n)g_3(\|x_n - W_nGy_n\|) \\
&+ \beta_n\gamma_ng_4(\|x_n - \Pi_C(I - \sigma_nF)z_n\|)\big].
\end{aligned}
$$

This guarantees

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \le{}& \delta_n\|x_n - p\|^2 + (1-\delta_n)\left\{\left(1 - \frac{\alpha_n(1-\delta)}{1 - (\alpha_n\delta + \gamma_n(1-t_n))}\right)\|x_n - p\|^2\right. \\
&- \frac{\gamma_n(1-t_n)}{1 - (\alpha_n\delta + \gamma_n(1-t_n))}\big[g_1(\|y_n - u_n - (p - \bar{p})\|) \\
&+ g_2(\|u_n - v_n + (p - \bar{p})\|)\big] \\
&+ \frac{2}{1 - (\alpha_n\delta + \gamma_n(1-t_n))}\big[\mu_2\|A_2p - A_2y_n\|\|u_n - \bar{p}\| \\
&+ \mu_1\|A_1\bar{p} - A_1u_n\|\|v_n - p\| \\
&+ \alpha_n\|Fz_n\|\|z_n - p - \sigma_nFz_n\| + \alpha_n\|f(p) - p\|\|y_n - p\|\big] \\
&- \frac{1}{1 - (\alpha_n\delta + \gamma_n(1-t_n))} \\
&\left.\times \big[\gamma_nt_n(1-t_n)g_3(\|x_n - W_nGy_n\|) + \beta_n\gamma_ng_4(\|x_n - \Pi_C(I - \sigma_nF)z_n\|)\big]\right\} \\
\le{}& \left(1 - \frac{\alpha_n(1-\delta_n)(1-\delta)}{1 - (\alpha_n\delta + \gamma_n(1-t_n))}\right)\|x_n - p\|^2 \\
&- \frac{1-\delta_n}{1 - (\alpha_n\delta + \gamma_n(1-t_n))}\big[\gamma_n(1-t_n)(g_1(\|y_n - u_n - (p - \bar{p})\|) \\
&+ g_2(\|u_n - v_n + (p - \bar{p})\|)) + \gamma_nt_n(1-t_n)g_3(\|x_n - W_nGy_n\|) \\
&+ \beta_n\gamma_ng_4(\|x_n - \Pi_C(I - \sigma_nF)z_n\|)\big] \\
&+ \frac{2}{1 - (\alpha_n\delta + \gamma_n(1-t_n))}\big[\mu_2\|A_2p - A_2y_n\|\|u_n - \bar{p}\|
\end{aligned}
$$

$$
\begin{aligned}
&+ \mu_1 \|A_1\bar{p} - A_1 u_n\| \|v_n - p\| + \alpha_n \|Fz_n\| \|z_n - p - \sigma_n F z_n\| \\
&+ \alpha_n \big\|f(p) - p\big\| \|y_n - p\|\big] \\
\le{}& \|x_n - p\|^2 - \frac{1-\delta_n}{1-(\alpha_n\delta + \gamma_n(1-t_n))}\big[\gamma_n(1-t_n)\big(g_1\big(\big\|y_n - u_n - (p-\bar{p})\big\|\big) \\
&+ g_2\big(\big\|u_n - v_n + (p-\bar{p})\big\|\big)\big) + \gamma_n t_n(1-t_n) g_3\big(\|x_n - W_n G y_n\|\big) \\
&+ \beta_n\gamma_n g_4\big(\big\|x_n - \varPi_C(I-\sigma_n F)z_n\big\|\big)\big] \\
&+ \frac{2}{1-(\alpha_n\delta + \gamma_n(1-t_n))}\big[\mu_2\|A_2 p - A_2 y_n\| \|u_n - \bar{p}\| \\
&+ \mu_1 \|A_1\bar{p} - A_1 u_n\| \|v_n - p\| + \alpha_n \|Fz_n\| \|z_n - p \\
&- \sigma_n F z_n\| + \alpha_n \big\|f(p) - p\big\| \|y_n - p\|\big],
\end{aligned}
$$

which immediately yields

$$
\begin{aligned}
&\frac{1-\delta_n}{1-(\alpha_n\delta + \gamma_n(1-t_n))}\big[\gamma_n(1-t_n)\big(g_1\big(\big\|y_n - u_n - (p-\bar{p})\big\|\big) + g_2\big(\big\|u_n - v_n + (p-\bar{p})\big\|\big)\big) \\
&\quad + \gamma_n t_n(1-t_n) g_3\big(\|x_n - W_n G y_n\|\big) + \beta_n\gamma_n g_4\big(\big\|x_n - \varPi_C(I-\sigma_n F)z_n\big\|\big)\big] \\
&\quad\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \frac{2}{1-(\alpha_n\delta + \gamma_n(1-t_n))}\big[\mu_2\|A_2 p - A_2 y_n\| \|u_n - \bar{p}\| \\
&\quad + \mu_1 \|A_1\bar{p} - A_1 u_n\| \|v_n - p\| + \alpha_n \|Fz_n\| \|z_n - p - \sigma_n F z_n\| + \alpha_n \big\|f(p) - p\big\| \|y_n - p\|\big] \\
&\quad\le \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_n - x_{n+1}\| \\
&\quad + \frac{2}{1-(\alpha_n\delta + \gamma_n(1-t_n))}\big[\mu_2\|A_2 p - A_2 y_n\| \|u_n - \bar{p}\| \\
&\quad + \mu_1 \|A_1\bar{p} - A_1 u_n\| \|v_n - p\| + \alpha_n \|Fz_n\| \|z_n - p - \sigma_n F z_n\| + \alpha_n \big\|f(p) - p\big\| \|y_n - p\|\big].
\end{aligned}
$$

Utilizing (3.6) and (3.11), we asserts from $\liminf_{n\to\infty}(1-\delta_n) > 0$, $\liminf_{n\to\infty}\gamma_n t_n(1-t_n) > 0$ and $\liminf_{n\to\infty}\beta_n\gamma_n > 0$ that $\lim_{n\to\infty} g_1(\|y_n - u_n - (p-\bar{p})\|) = 0$, $\lim_{n\to\infty} g_2(\|u_n - v_n + (p-\bar{p})\|) = 0$, $\lim_{n\to\infty} g_3(\|x_n - W_n G y_n\|) = 0$ and $\lim_{n\to\infty} g_4(\|x_n - \varPi_C(I-\sigma_n F)z_n\|) = 0$. So, $\lim_{n\to\infty}\|y_n - u_n - (p-\bar{p})\| = \lim_{n\to\infty}\|u_n - v_n + (p-\bar{p})\| = 0$ and

$$
\lim_{n\to\infty}\|x_n - W_n G y_n\| = \lim_{n\to\infty}\big\|x_n - \varPi_C(I-\sigma_n F)z_n\big\| = 0. \tag{3.15}
$$

Furthermore, one has

$$
\begin{aligned}
\|y_n - G y_n\| &= \|y_n - v_n\| \\
&\le \big\|y_n - u_n - (p-\bar{p})\big\| + \big\|u_n - v_n + (p-\bar{p})\big\| \to 0 \quad (n\to\infty).
\end{aligned} \tag{3.16}
$$

Since $y_n - x_n = \alpha_n(f(y_n) - x_n) + \gamma_n(\varPi_C(I-\sigma_n F)z_n - x_n)$, we see from (3.15) that $\|y_n - x_n\| \le \|\varPi_C(I-\sigma_n F)z_n - x_n\| + \alpha_n\|x_n - f(y_n)\| \to 0$ $(n\to\infty)$. With the aid of (3.16), one asserts

$$
\begin{aligned}
\|x_n - G x_n\| &\le \|x_n - y_n\| + \|y_n - G y_n\| + \|G y_n - G x_n\| \\
&\le 2\|x_n - y_n\| + \|y_n - G y_n\| \to 0 \quad (n\to\infty).
\end{aligned} \tag{3.17}
$$

*Step 4.* One shows that $\|x_n - Wx_n\| \to 0$, $\|x_n - T_\lambda x_n\| \to 0$ and $\|x_n - \Gamma x_n\| \to 0$ as $n \to \infty$, where $Wx = \lim_{n\to\infty} W_n x$, $\forall x \in C$, $T_\lambda = J_\lambda^B(I - \lambda A)$ and $\Gamma x = \theta_1 Wx + \theta_2 Gx + \theta_3 T_\lambda x$, $\forall x \in C$ for constants $\theta_1, \theta_2, \theta_3 \in (0,1)$ satisfying $\theta_1 + \theta_2 + \theta_3 = 1$. Indeed, utilizing (3.15) and (3.17), one deduces that

$$\begin{aligned}\|Wx_n - x_n\| &\le \|Wx_n - WGx_n\| + \|WGx_n - W_nGx_n\| + \|W_nGx_n - W_nGy_n\| \\ &\quad + \|W_nGy_n - x_n\| \\ &\le \|x_n - Gx_n\| + \|WGx_n - W_nGx_n\| + \|x_n - y_n\| \\ &\quad + \|W_nGy_n - x_n\| \to 0 \quad (n \to \infty). \end{aligned} \tag{3.18}$$

Furthermore, since $x_{n+1} - x_n + x_n - y_n = \delta_n(x_n - y_n) + (1 - \delta_n)(T_n y_n - y_n)$, from $x_n - x_{n+1} \to 0$ and $x_n - y_n \to 0$, we have

$$\begin{aligned}\|T_n y_n - y_n\| &= \frac{1}{1-\delta_n}\big\|x_{n+1} - x_n + (1-\delta_n)(x_n - y_n)\big\| \\ &\le \frac{\|x_{n+1} - x_n\| + \|x_n - y_n\|}{1-\delta_n} \to 0 \quad (n \to \infty).\end{aligned}$$

Also, utilizing similar arguments to those of (3.5), we obtain

$$\begin{aligned}\|T_n y_n - T_\lambda y_n\| &\le \left|1 - \frac{\lambda}{\lambda_n}\right| \big\|J_{\lambda_n}^B(I - \lambda_n A)y_n - (I - \lambda_n A)y_n\big\| + |\lambda_n - \lambda|\|Ay_n\| \\ &= \left|1 - \frac{\lambda}{\lambda_n}\right| \big\|T_n y_n - (I - \lambda_n A)y_n\big\| + |\lambda_n - \lambda|\|Ay_n\|.\end{aligned}$$

Since $\lim_{n\to\infty} \lambda_n = \lambda$ and the sequences $\{y_n\}$, $\{T_n y_n\}$, $\{Ay_n\}$ are bounded, we get

$$\lim_{n\to\infty} \|T_n y_n - T_\lambda y_n\| = 0. \tag{3.19}$$

Taking into account condition (v), i.e., $0 < \bar{\lambda} \le \lambda_n$, $\forall n \ge 0$ and $\lim_{n\to\infty} \lambda_n = \lambda$, where $\kappa_q \lambda^{q-1} < q\alpha$, we know that $0 < \kappa_q \bar{\lambda}^{q-1} \le \kappa_q \lambda^{q-1} < q\alpha$. So $\mathrm{Fix}(T_\lambda) = (A+B)^{-1}0$ and $T_\lambda : C \to C$ is nonexpansive. Therefore, we infer from (3.19) and $x_n - y_n \to 0$ that

$$\begin{aligned}\|T_\lambda x_n - x_n\| &\le \|T_\lambda x_n - T_\lambda y_n\| + \|T_\lambda y_n - T_n y_n\| + \|T_n y_n - y_n\| + \|y_n - x_n\| \\ &\le 2\|x_n - y_n\| + \|T_\lambda y_n - T_n y_n\| + \|T_n y_n - y_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{3.20}$$

One now defines the mapping $\Gamma x = \theta_1 Wx + \theta_2 Gx + \theta_3 T_\lambda x$, $\forall x \in C$ with constants $\theta_1, \theta_2, \theta_3 \in (0,1)$ satisfying $\theta_1 + \theta_2 + \theta_3 = 1$. One gets $\mathrm{Fix}(\Gamma) = \mathrm{Fix}(W) \cap \mathrm{Fix}(G) \cap \mathrm{Fix}(T_\lambda) = \Omega$. Observe that

$$\|\Gamma x_n - x_n\| \le \theta_1\|x_n - Wx_n\| + \theta_2\|x_n - Gx_n\| + \theta_3\|x_n - T_\lambda x_n\|. \tag{3.21}$$

From (3.17), (3.18), (3.20) and (3.21), one gets

$$\lim_{n\to\infty} \|\Gamma x_n - x_n\| = 0. \tag{3.22}$$

*Step 5.* Letting $x_t$ is the unique fixed point of $x \mapsto (1-t)\Gamma x + tf(x)$ for each $t \in (0,1)$, one shows that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, J(x_n - x^*)\rangle \le 0, \tag{3.23}$$

where $x^* = \text{s-}\lim_{n\to\infty} x_t$. By Lemmas 2.3 and 2.5, one asserts

$$\begin{aligned} \|x_t - x_n\|^2 &\le 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle + (1-t)^2\|\Gamma x_t - x_n\|^2 \\ &\le \|\Gamma x_n - x_n\|)^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle + (1-t)^2(\|\Gamma x_t - \Gamma x_n\| \\ &\le (1 - 2t + t^2)\|x_t - x_n\|^2 + f_n(t) + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle \\ &\quad + 2t\|x_t - x_n\|^2, \end{aligned} \tag{3.24}$$

where

$$f_n(t) = (1-t)^2\|x_n - \Gamma x_n\|\bigl(2\|x_t - x_n\| + \|x_n - \Gamma x_n\|\bigr) \to 0 \quad (n \to \infty). \tag{3.25}$$

It follows from (3.24) that

$$2\langle x_t - f(x_t), J(x_t - x_n)\rangle \le t\|x_t - x_n\|^2 + \frac{f_n(t)}{t}. \tag{3.26}$$

Letting $n \to \infty$ and employing (3.25), one derives

$$2\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le tM_4, \tag{3.27}$$

where $\sup\{\|x_t - x_n\|^2 : t \in (0,1) \text{ and } n \ge 0\} \le M_4$ for some $M_4 > 0$. Taking $t \to 0$ in (3.27), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0.$$

On the other hand, we have

$$\begin{aligned} &\langle f(x^*) - x^*, J(x_n - x^*)\rangle \\ &\quad = \langle f(x^*) - x^*, J(x_n - x^*)\rangle - \langle f(x^*) - x^*, J(x_n - x_t)\rangle \\ &\qquad + \langle f(x^*) - x^*, J(x_n - x_t)\rangle - \langle f(x^*) - x_t, J(x_n - x_t)\rangle \\ &\qquad + \langle f(x^*) - x_t, J(x_n - x_t)\rangle \\ &\qquad - \langle f(x_t) - x_t, J(x_n - x_t)\rangle + \langle f(x_t) - x_t, J(x_n - x_t)\rangle \\ &\quad = \langle f(x^*) - x^*, J(x_n - x^*) - J(x_n - x_t)\rangle + \langle x_t - x^*, J(x_n - x_t)\rangle \\ &\qquad + \langle f(x^*) - f(x_t), J(x_n - x_t)\rangle + \langle f(x_t) - x_t, J(x_n - x_t)\rangle. \end{aligned}$$

So, it follows that

$$\begin{aligned}&\limsup_{n\to\infty}\langle f(x^*)-x^*,J(x_n-x^*)\rangle\\&\quad\le\limsup_{n\to\infty}\langle f(x^*)-x^*,J(x_n-x^*)-J(x_n-x_t)\rangle\\&\qquad+(1+\delta)\|x_t-x^*\|\limsup_{n\to\infty}\|x_n-x_t\|+\limsup_{n\to\infty}\langle f(x_t)-x_t,J(x_n-x_t)\rangle.\end{aligned}$$

Taking into account that $x_t\to x^*$ as $t\to 0$, we have

$$\begin{aligned}&\limsup_{n\to\infty}\langle f(x^*)-x^*,J(x_n-x^*)\rangle\\&\quad=\limsup_{t\to 0}\limsup_{n\to\infty}\langle f(x^*)-x^*,J(x_n-x^*)\rangle\\&\quad\le\limsup_{t\to 0}\limsup_{n\to\infty}\langle f(x^*)-x^*,J(x_n-x^*)-J(x_n-x_t)\rangle.\end{aligned}\tag{3.28}$$

Thanks to the space ($q$-uniformly smooth), one knows that the two limits can be interchangeable. Equation (3.23) therefore holds. Note that $x_n-y_n\to 0$ implies $J(y_n-x^*)-J(x_n-x^*)\to 0$. Thus, we conclude from (3.23) that

$$\begin{aligned}&\limsup_{n\to\infty}\langle f(x^*)-x^*,J(y_n-x^*)\rangle\\&\quad=\limsup_{n\to\infty}\{\langle f(x^*)-x^*,J(x_n-x^*)\rangle+\langle f(x^*)-x^*,J(y_n-x^*)-J(x_n-x^*)\rangle\}\\&\quad=\limsup_{n\to\infty}\langle f(x^*)-x^*,J(x_n-x^*)\rangle\le 0.\end{aligned}\tag{3.29}$$

*Step 6.* One shows $\|x_n-x^*\|\to 0$ as $n\to\infty$.

$$\begin{aligned}\|y_n-x^*\|^2&=\|\alpha_n(f(y_n)-f(x^*))+\beta_n(x_n-x^*)+\gamma_n(\Pi_C(I-\sigma_nF)z_n-x^*)\\&\quad+\alpha_n(f(x^*)-x^*)\|^2\\&\le\alpha_n\|f(y_n)-f(x^*)\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n[\|z_n-x^*\|^2\\&\quad+2\sigma_n\|Fz_n\|\|z_n-x^*-\sigma_nFz_n\|]\\&\quad+2\alpha_n\langle f(x^*)-x^*,J(y_n-x^*)\rangle\\&\le\alpha_n\delta\|y_n-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n(t_n\|x_n-x^*\|^2+(1-t_n)\|y_n-x^*\|^2)\\&\quad+2\sigma_n\|Fz_n\|\|z_n-x^*-\sigma_nFz_n\|+2\alpha_n\langle f(x^*)-x^*,J(y_n-x^*)\rangle,\end{aligned}$$

which hence yields

$$\begin{aligned}\|y_n-x^*\|^2&\le\left(1-\frac{\alpha_n(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\right)\|x_n-x^*\|^2+\frac{2\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\\&\quad\times\left[\frac{\sigma_n}{\alpha_n}\|Fz_n\|\|z_n-x^*-\sigma_nFz_n\|+\langle f(x^*)-x^*,J(y_n-x^*)\rangle\right].\end{aligned}$$

Due to the convexity of $\|\cdot\|^2$, and the nonexpansivity of $T_n$, one asserts

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \delta_n\|x_n-x^*\|^2+(1-\delta_n)\|y_n-x^*\|^2 \\
&\le \delta_n\|x_n-x^*\|^2+(1-\delta_n)\biggl\{\biggl(1-\frac{\alpha_n(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\biggr)\|x_n-x^*\|^2 \\
&\quad+\frac{2\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))} \\
&\quad\times\biggl[\frac{\sigma_n}{\alpha_n}\|Fz_n\|\|z_n-x^*-\sigma_nFz_n\|+\langle f(x^*)-x^*,J(y_n-x^*)\rangle\biggr]\biggr\} \\
&=\biggl[1-\frac{\alpha_n(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\biggr]\|x_n-x^*\|^2+\frac{\alpha_n(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))} \\
&\quad\times\frac{2[\frac{\sigma_n}{\alpha_n}\|Fz_n\|\|z_n-x^*-\sigma_nFz_n\|+\langle f(x^*)-x^*,J(y_n-x^*)\rangle]}{1-\delta}.
\end{aligned} \tag{3.30}$$

Since $\liminf_{n\to\infty}\frac{(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}>0$, $\{\frac{\alpha_n(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\}\subset(0,1)$ and $\sum_{n=0}^\infty\alpha_n=\infty$, we know

$$\biggl\{\frac{\alpha_n(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\biggr\}\subset(0,1)$$

and

$$\sum_{n=0}^\infty\frac{\alpha_n(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}=\infty.$$

Utilizing (3.29) and Lemma 2.7, we conclude from (3.30) that $\|x_n-x^*\|\to 0$ as $n\to\infty$. This completes the proof. □

*Remark* 3.1 Comparing with the corresponding results in and Chang et al. [8], we have the following aspects. The problem of solving a HVI with the constraints of SGVIs (1.1) and a countable family of nonexpansive mappings in [8, Theorem 3.1] is extended to our problem of solving a HVI with the constraints of SGVIs (1.1), a variational inclusion (VI) and a countable family of nonexpansive mappings. The modified relaxed extragradient method in[8, Theorem 3.1] is extended to our composite extragradient implicit rule (3.1). That is, two iterative steps $y_n=(1-\beta_n)x_n+\beta_nGx_n$ and $x_{n+1}=\Pi_C[\gamma_nx_n+((1-\gamma_n)I-\alpha_n\rho F)S_ny_n+\alpha_n\gamma f(x_n)]$ in [8, Theorem 3.1] are extended to our two iterative steps $y_n=\beta_nx_n+\gamma_n\Pi_C(I-\sigma_nF)(t_nx_n+(1-t_n)W_nGy_n)+\alpha_nf(y_n)$ and $x_{n+1}=\delta_nx_n+(1-\delta_n)T_ny_n$, where $T_n=J_{\lambda_n}^B(I-\lambda_nA)$.

**Acknowledgements**
The authors are grateful to the referees for useful suggestions which improved the contents of this paper.

**Funding**
This paper was supported by the National Natural Science Foundation of China under Grant 11601348.

**Availability of data and materials**
All data generated or analyzed during this study are included in this published article.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally to the manuscript and approved the final manuscript.

**Author details**

[1]Department of Mathematics, Shanghai Normal University, Shanghai, China. [2]College of Science, Shijiazhuang University, Shijiazhuang, China.

## Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 23 September 2019 Accepted: 3 February 2020 Published online: 11 February 2020

**References**

1. Aoyama, K., Iiduka, H., Takahashi, W.: Weak convergence of an iterative sequence for accretive operators in Banach spaces. Fixed Point Theory Appl. **2006**, 35390 (2006)
2. Ceng, L.C., et al.: Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities. Math. Methods Oper. Res. **67**, 375–390 (2008)
3. Cho, S.Y., Kang, S.M.: Approximation of common solutions of variational inequalities via strict pseudocontractions. Acta Math. Sci. **32**, 1607–1618 (2012)
4. An, N.T., Nam, N.M.: Solving k-center problems involving sets based on optimization techniques. J. Glob. Optim. (2019). https://doi.org/10.1007/s10898-019-00834-6
5. Qin, X., Yao, J.C.: A viscosity iterative method for a split feasibility problem. J. Nonlinear Convex Anal. **20**, 1497–1506 (2019)
6. Nguyen, L.V., Ansari, Q.H., Qin, X.: Linear conditioning, weak sharpness and finite convergence for equilibrium problems. J. Glob. Optim. (2020). https://doi.org/10.1007/s10898-019-00869-9
7. Qin, X., An, N.T.: Smoothing algorithms for computing the projection onto a Minkowski sum of convex sets. Comput. Optim. Appl. **74**, 821–850 (2019)
8. Chang, S.S., Wen, C.F., Yao, J.C.: Generalized viscosity implicit rules for solving quasi-inclusion problems of accretive operators in Banach spaces. Optimization **66**, 1105–1117 (2017)
9. Chang, S.S., Wen, C.F., Yao, J.C.: Common zero point for a finite family of inclusion problems of accretive mappings in Banach spaces. Optimization **67**, 1183–1196 (2018)
10. Qin, X., Cho, S.Y., Wang, L.: A regularization method for treating zero points of the sum of two monotone operators. Fixed Point Theory Appl. **2014**, Article ID 75 (2014)
11. Chang, S.S., Lee, H.W.J., Chan, C.K.: A new method for solving equilibrium problem, fixed point problem and variational inequality problem with application to optimization. Nonlinear Anal. **70**, 3307–3319 (2009)
12. Takahahsi, W., Yao, J.C.: The split common fixed point problem for two finite families of nonlinear mappings in Hilbert spaces. J. Nonlinear Convex Anal. **20**, 173–195 (2019)
13. Cho, S.Y.: Generalized mixed equilibrium and fixed point problems in a Banach space. J. Nonlinear Sci. Appl. **9**, 1083–1092 (2016)
14. Song, Y., Ceng, L.C.: A general iteration scheme for variational inequality problem and common fixed point problems of nonexpansive mappings in $q$-uniformly smooth Banach spaces. J. Glob. Optim. **57**, 1327–1348 (2013)
15. Ceng, L.C., et al.: Variational inequalities approaches to minimization problems with constraints of generalized mixed equilibria and variational inclusions. Mathematics **7**, 270 (2019)
16. Qin, X., Cho, S.Y., Yao, J.C.: Strong convergence of an iterative algorithm involving nonlinear mappings of nonexpansive and accretive type. Optimization **67**, 1377–1388 (2018)
17. Chang, S.S., Wen, C.F., Yao, J.C.: Zero point problem of accretive operators in Banach spaces. Bull. Malays. Math. Sci. Soc. **42**, 105–118 (2019)
18. Zhao, X., et al.: Linear regularity and linear convergence of projection-based methods for solving convex feasibility problems. Appl. Math. Optim. **78**, 613–641 (2018)
19. Cho, S.Y., Bin Dehaish, B.A.: Weak convergence of a splitting algorithm in Hilbert spaces. J. Appl. Anal. Comput. **7**, 427–438 (2017)
20. Cho, S.Y.: Convergence analysis of a hybrid algorithm for nonlinear operators in a Banach space. J. Appl. Anal. Comput. **8**, 19–31 (2018)
21. Ceng, L.C., et al.: Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions. Fixed Point Theory **20**, 113–134 (2019)
22. Ceng, L.C., et al.: Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces. Fixed Point Theory **19**, 487–502 (2018)
23. Qin, X., Cho, S.Y.: Convergence analysis of a monotone projection algorithm in reflexive Banach spaces. Acta Math. Sci. **37**, 488–502 (2017)
24. Takahashi, W., Wen, C.F.: The shrinking projection method for a finite family of demimetric mappings with variational inequality problems in a Hilbert space. Fixed Point Theory **19**, 407–419 (2018)
25. Chang, S.S., Lee, H.W.J., Chan, C.K.: A new method for solving equilibrium problem, fixed point problem and variational inequality problem with application to optimization. Nonlinear Anal. **70**, 3307–3319 (2009)
26. Shimoji, K., Takahashi, W.: Strong convergence to common fixed points of infinite nonexpansive mappings and applications. Taiwan. J. Math. **5**, 387–404 (2001)
27. Reich, S.: Weak convergence theorems for nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. **67**, 274–276 (1979)
28. Qin, X., Cho, S.Y., Yao, J.C.: Weak and strong convergence of splitting algorithms in Banach spaces. Optimization (2019). https://doi.org/10.1080/02331934.2019.1654475
29. Barbu, V.: Nonlinear Semigroups and Differential Equations in Banach Spaces. Noordhoff, Leiden (1976)

30. Qin, X., Cho, S.Y., Wang, L.: Iterative algorithms with errors for zero points of m-accretive operators. Fixed Point Theory Appl. **2013**, Article ID 148 (2013)
31. Xue, Z., Zhou, H., Cho, Y.J.: Iterative solutions of nonlinear equations for m-accretive operators in Banach spaces. J. Nonlinear Convex Anal. **1**, 313–320 (2000)
32. Suzuki, T.: Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces. Fixed Point Theory Appl. **2005**, 103–123 (2005)