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A stochastic Gronwall inequality in random time horizon and its application to BSDE

Hun O[1], Mun-Chol Kim[1] and Chol-Kyu Pak[1*]

*Correspondence:
pck2016217@gmail.com
[1]Faculty of Mathematics, Kim Il Sung University, Pyongyang, Democratic People's Republic of Korea

Abstract

In this paper, we introduce and prove a stochastic Gronwall inequality in an (unbounded) random time horizon. As an application, we prove a comparison theorem for backward stochastic differential equation (BSDE for short) with random terminal time under a stochastic monotonicity condition.

MSC: 60E15; 60H20

Keywords: Gronwall inequality; Stochastic; Random time horizon; Backward stochastic differential equation; Comparison

1 Introduction

Gronwall's inequality is a handy tool to derive many useful results such as uniqueness, comparison, boundedness, continuous dependence with respect to initial value, and stability in the theory of differential and integral equations. It was first introduced by Gronwall [9] as a differential form, and the integral inequality was proposed by Bellman [3]. Since then, many researchers have studied the various types of generalizations of this inequality motivated by the development of the differential and integral equations [1, 5, 7, 8, 12, 16, 19]. Among such generalizations, we are concerned with the stochastic version of Gronwall's inequality.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which a d-dimensional Brownian motion $B = (B_t)_{t\geq 0}$ is defined. Let $(\mathcal{F}_t)_{t\geq 0}$ be the right-continuous completion of the natural filtration generated by B, that is, $\mathcal{F}_t := \sigma\{B_s, s \leq t\}$, and augment it by $\mathbb{P}$-null sets of $\mathcal{F}$. The stochastic forward Gronwall inequality was first proposed by Itô [10] and was developed in several papers (see e.g. [2, 18, 20]).

Recently, Wang and Fan [17] proposed the following stochastic backward Gronwall inequality due to the development of backward stochastic differential equations.

Proposition 1 *Let $c > 0$, $T > 0$, and an $(\mathcal{F}_t)_{t\geq 0}$-progressively measurable non-negative process $a : \Omega \times [0, T] \to \mathbb{R}^+$ satisfy $\int_0^T a(t)\,dt \leq M, \mathbb{P}$-a.s. for some constant $M > 0$. If an $(\mathcal{F}_t)_{t\geq 0}$-progressively measurable process $x : \Omega \times [0, T] \to \mathbb{R}^+$ satisfies*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} x(t)\Big] < +\infty, \qquad x(t) \leq c + \mathbb{E}\Big[\int_t^T a(s)x(s)\,ds \Big| \mathcal{F}_t\Big], \quad t \in [0, T],$$

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then, for each $t \in [0, T]$,

$$x(t) \le c \cdot \mathbb{E}\big[e^{\int_t^T a(s)\,ds} | \mathcal{F}_t\big], \quad \mathbb{P}\text{-}a.s.$$

Unlike the deterministic case, the backward version of stochastic Gronwall's inequality has an essential difference to the forward one. If the backward inequality is given by a differential equation, then the diffusion term needs to be controlled to ensure the adaptivity of the stochastic processes with respect to the filtration. Indeed, the solution of the backward stochastic differential equation should be a pair of adapted processes, not one process. So the stochastic backward Gronwall inequality involves the conditional expectation. If the random processes $a(t)$ and $x(t)$ are deterministic functions in the above proposition, then we reach the well-known Gronwall inequality (more precisely deterministic backward Gronwall inequality) as follows.

Corollary 1 *Let* $c > 0$ *be a constant. If* $a(t)$ *and* $x(t)$ *are two non-negative* (*deterministic*) *functions defined on* $[0, T]$ *satisfying*

$$x(t) \le c + \int_t^T a(s)x(s)\,ds, \quad t \in [0, T],$$

then, for each $t \in [0, T]$,

$$x(t) \le c \cdot e^{\int_t^T a(s)\,ds}.$$

In this paper, we study the complete version of backward Gronwall's inequality in the stochastic sense. More precisely, in Proposition 1, the constants c and T are replaced by a random variable and an (unbounded) stopping time, respectively, and the integral of $a(t)$ is not assumed to be essentially bounded.

Our method uses the martingale representation and random time change to prove the main inequality. Due to the type of the proposed inequality, the application to the stochastic differential (or integral) equation with stochastic coefficients defined up to a random time (more precisely, stopping time) is naturally considered. We give the proof of a comparison theorem of L^p-solutions to backward stochastic differential equation (BSDE for short) with random terminal time and stochastic coefficients by using the stochastic Gronwall inequality in a random time horizon, effectively.

2 Notations

Let $p > 1$ and τ be an $(\mathcal{F}_t)_{t\ge 0}$-stopping time. That is, $\forall t \ge 0$, $\{\omega | \tau(\omega) \le t\} \in \mathcal{F}_t$. Throughout the paper, $|\cdot|$ means the standard Euclidean norm. We put $A(t) := \int_0^t a(s)\,ds$, where $a(s)$ is a non-negative progressively-measurable process. The symbols $\mathbb{E}[\cdot]$ and $\mathbb{E}[\cdot|\mathcal{F}_t]$ denote the expectation and conditional expectation (with respect to $\mathcal{F}_t$), respectively.

In the following notations, $\theta \in \mathbb{R}$ is a fixed constant.

– $L^p(\theta, a, \tau, \mathbb{R})$ is the set of real-valued $\mathcal{F}_\tau$-measurable random variables ξ such that

$$\mathbb{E}\big[e^{\frac{p}{2}\theta A(\tau)}|\xi|^p\big] < +\infty.$$

– $\mathbb{S}^p(\theta, a, [0,\tau], \mathbb{R})$ is the set of real-valued càdlàg, adapted processes Y such that

$$\mathbb{E}\Big[\sup_{0\le t\le\tau} e^{\frac{p}{2}\theta A(t)}|Y_t|^p\Big] < \infty.$$

– $\mathbb{H}^{p,a}(\theta, a, [0,\tau], \mathbb{R})$ is the set of real-valued càdlàg, adapted processes Y such that

$$\mathbb{E}\bigg[\int_0^\tau a(t)e^{\frac{p}{2}\theta A(t)}|Y_t|^p\,dt\bigg] < \infty.$$

– $\mathbb{H}^p(\theta, a, [0,\tau], \mathbb{R}^{1\times d})$ is the set of predictable processes Z with values in $\mathbb{R}^{1\times d}$ such that

$$\mathbb{E}\bigg[\bigg(\int_0^\tau e^{\theta A(t)}|Z_t|^2\,ds\bigg)^{\frac{p}{2}}\bigg] < \infty.$$

– For $m, n \in \mathbb{R}$, $m \wedge n := \min\{m, n\}$ and $m^+ := \max\{m, 0\}$.
– I_B is an indicator function of a set B, that is, $I_B(x) = 1$ if $x \in B$ and $I_B(x) = 0$ if $x \notin B$.

3 Main inequality

Theorem 1 *Let $p > 1$, $l \ge 0$ be constants and q be a constant such that $1/p + 1/q = 1$. Let τ be an $(\mathcal{F}_t)_{t\ge0}$-stopping time and ξ be a non-negative random variable. Let $a(t)$ and $x(t)$ be non-negative progressively measurable processes. Let $\rho \ge 0$ be a constant such that $\mathbb{E}[e^{\frac{q}{2}(2l-\rho)^+A(\tau)}] < \infty$, where $A(t) := \int_0^t a(s)\,ds$ and $a(t) > \epsilon$ for some constant $\epsilon > 0$. Assume that ξ belongs to $L^p(\theta, a, \tau, \mathbb{R})$ and $x(t)$ belongs to $\mathbb{H}^{p,a}(\theta, a, [0,\tau], \mathbb{R})$ for some constant $\theta > \rho$.*

If $x(t) \le \mathbb{E}[\xi + l\int_{t\wedge\tau}^\tau a(s)x(s)\,ds|\mathcal{F}_t]$, $\mathbb{P}$-a.s., then we have $\mathbb{P}$-a.s.,

$$x(t) \le \mathbb{E}\big[\xi \cdot e^{l\int_{t\wedge\tau}^\tau a(s)\,ds}|\mathcal{F}_t\big].$$

Proof Define the process $X(t) := \mathbb{E}[\xi + l\int_{t\wedge\tau}^\tau a(s)x(s)\,ds|\mathcal{F}_t]$, then it follows that $x(t) \le X(t)$ from the assumption. Let us put $\eta := \xi + l\int_0^\tau a(s)x(s)\,ds$. Using Hölder's inequality with $1/p + 1/q = 1$,

$$\begin{aligned}
\bigg|\int_0^\tau a(s)x(s)\,ds\bigg|^p &\le \bigg|\int_0^\tau e^{\frac{\rho}{2}A(s)}a(s)x(s)\,ds\bigg|^p \\
&= \bigg|\int_0^\tau a(s)^{1/p}e^{\theta A(s)/2}x(s)\cdot e^{(\rho-\theta)A(s)/2}a(s)^{1/q}\,ds\bigg|^p \\
&\le \int_0^\tau a(s)x(s)^p e^{\frac{p}{2}\theta A(s)}\,ds\cdot\bigg(\int_0^\tau e^{\frac{q}{2}(\rho-\theta)A(s)}a(s)\bigg)^{p/q} \\
&= \int_0^\tau a(s)x(s)^p e^{\frac{p}{2}\theta A(s)}\,ds\cdot\bigg(\frac{2}{q(\theta-\rho)}\big(1-e^{\frac{q}{2}(\rho-\theta)A(\tau)}\big)\bigg)^{p/q} \\
&\le \int_0^\tau a(s)x(s)^p e^{\frac{p}{2}\theta A(s)}\,ds\cdot\bigg(\frac{2}{q(\theta-\rho)}\bigg)^{p/q} < \infty. \qquad (1)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\big[|\eta|^p\big] &\le 2^{p-1}\bigg(\mathbb{E}\big[|\xi|^p\big] + l^p\mathbb{E}\bigg[\bigg(\int_0^\tau a(s)x(s)\,ds\bigg)^p\bigg]\bigg) \\
&\le 2^{p-1}\mathbb{E}\big[|\xi|^p\big] + 2^{p-1}l^p\bigg(\frac{2}{q(\theta-\rho)}\bigg)^{p/q}\cdot E\bigg[\int_0^\tau a(s)x(s)^p e^{\frac{p}{2}\theta A(s)}\,ds\bigg] < \infty.
\end{aligned}$$

By the martingale representation theorem (see [13], Corollary 2.44), there exists a process Z satisfying $\mathbb{E}[(\int_0^T |Z_t|^2\,ds)^{\frac{p}{2}}] < \infty$ for any $T > 0$ such that $\mathbb{P}$-a.s.

$$\mathbb{E}[\eta|\mathcal{F}_t] = \mathbb{E}[\eta] + \int_0^{t\wedge\tau} Z_s\,dB_s.$$

So,

$$X(t) = \mathbb{E}[\eta|\mathcal{F}_t] - l\mathbb{E}\left[\int_0^{t\wedge\tau} a(s)x(s)\,ds\middle|\mathcal{F}_t\right] = \mathbb{E}[\eta] + \int_0^{t\wedge\tau} Z_s\,dB_s - l\int_0^{t\wedge\tau} a(s)x(s)\,ds.$$

Moreover, we have the backward version: for all $T > 0$,

$$X(t) = X(T\wedge\tau) - \int_{t\wedge\tau}^{T\wedge\tau} Z_s\,dB_s + l\int_{t\wedge\tau}^{T\wedge\tau} a(s)x(s)\,ds$$

with $X(\tau) = \xi$.

Or equivalently,

$$X(t) = \xi - \int_{t\wedge\tau}^{\tau} Z_s\,dB_s + l\int_{t\wedge\tau}^{\tau} a(s)x(s)\,ds. \tag{2}$$

Now we shall show that $Z \in \mathbb{H}^p(\rho, a, [0,\tau], \mathbb{R}^{1\times d})$. First, we introduce a certain random time change. Since the process $A_t = \int_0^t a(s)\,ds$ is strictly increasing and continuous, we can define its inverse denoted by $C_s := A^{-1}(s)$. Then a family of stopping times $\{C_s\}, s \geq 0$, is an $(\mathcal{F}_t)_{t\geq 0}$-random time change (see [14], Chapter V, Sect. 1 for systematic study of random time changes). Set $\widetilde{\mathcal{F}}_t := \mathcal{F}_{C_t}$, then $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ is a new filtration. For any adapted process X, we assume that $\widetilde{X}$ means the time-changed process, that is, $\widetilde{X}(t) = X(C_t)$ (more precisely, $\widetilde{X} = X \circ (\mathrm{id}_\Omega \times C)$. We also define $\widetilde{\tau} := A(\tau)$ and $W_t := \int_0^t \sqrt{\widetilde{a}(s)}\,d\widetilde{B}_s$. Note that $W = (W^1, W^2, \dots, W^d)$ is a d-dimensional $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$-Brownian motion by Levy's characterization theorem. In fact, for each $i \in \{1, \dots, d\}$,

$$\begin{aligned}\langle W^i, W^i\rangle &= \int_0^t \widetilde{a}(s)\,d\langle\widetilde{B}^i, \widetilde{B}^i\rangle_s = \int_0^t \widetilde{a}(s)\,d\widetilde{\langle B^i, B^i\rangle}_s \\ &= \int_0^t \widetilde{a}(s)\,dC_s = \int_0^{C_t} a(s)\,ds = A(C_t) = t.\end{aligned}$$

From the properties of (stochastic) integral with respect to time change,

$$\begin{aligned}&\int_{t\wedge\tau}^{\tau} a(s)x(s)\,ds = \int_{A(t\wedge\tau)}^{A(\tau)} x(C_s)\,ds = \int_{A(t)\wedge\widetilde{\tau}}^{\widetilde{\tau}} \widetilde{x}(s)\,ds, \\ &\int_{t\wedge\tau}^{\tau} Z_s\,dB_s = \int_{A(t)\wedge\widetilde{\tau}}^{\widetilde{\tau}} \widetilde{Z}_s\,d\widetilde{B}_s = \int_{A(t)\wedge\widetilde{\tau}}^{\widetilde{\tau}} \widetilde{a}(s)^{-1/2}\widetilde{Z}_s\,dW_s.\end{aligned}$$

So, we get the following expression with respect to $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$:

$$\widetilde{X}(t) = \xi - \int_{t\wedge\widetilde{\tau}}^{\widetilde{\tau}} \widetilde{a}(s)^{-1/2}\widetilde{Z}_s\,dW_s + l\int_{t\wedge\widetilde{\tau}}^{\widetilde{\tau}} \widetilde{x}(s)\,ds,$$

or equivalently, for all $\widetilde{T} > 0$,

$$\widetilde{X}(t) = \widetilde{X}(\widetilde{T} \wedge \widetilde{\tau}) - \int_{t\wedge\widetilde{\tau}}^{\widetilde{T}\wedge\widetilde{\tau}} \widetilde{a}(s)^{-1/2}\widetilde{Z}_s \, dW_s + l\int_{t\wedge\widetilde{\tau}}^{\widetilde{T}\wedge\widetilde{\tau}} \widetilde{x}(s)\, ds$$

with $\widetilde{X}(\widetilde{\tau}) = \xi$.

According to [6], Proposition 3.2, for all $\widetilde{T} > 0$,

$$\begin{aligned}
&\mathbb{E}\biggl[\sup_{t\in[0,\widetilde{T}\wedge\widetilde{\tau}]} e^{\frac{p}{2}\rho t}\widetilde{X}(t)^p + \biggl(\int_0^{\widetilde{T}\wedge\widetilde{\tau}} \frac{e^{\rho t}}{\widetilde{a}(t)}|\widetilde{Z}_t|^2\, dt\biggr)^{p/2}\biggr] \\
&\quad\le c(p)\cdot\mathbb{E}\biggl[e^{\frac{p}{2}\rho(\widetilde{T}\wedge\widetilde{\tau})}\widetilde{X}(\widetilde{T}\wedge\widetilde{\tau})^p + \biggl(\int_0^{\widetilde{T}\wedge\widetilde{\tau}} l e^{\rho s/2}\widetilde{x}(s)\, ds\biggr)^p\biggr].
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
&\sup_{t\in[0,\widetilde{T}\wedge\widetilde{\tau}]} e^{\frac{p}{2}\rho t}\widetilde{X}(t)^p = \sup_{t\in[0,T\wedge\tau]} e^{\frac{p}{2}\rho A(t)}X(t)^p, \\
&\int_0^{\widetilde{T}\wedge\widetilde{\tau}} e^{\rho s/2}\widetilde{x}(s)\, ds = \int_0^{T\wedge\tau} e^{\rho A(s)/2}x(s)\, dA(s) = \int_0^{T\wedge\tau} a(s)e^{\rho A(s)/2}x(s)\, ds, \\
&\int_0^{\widetilde{T}\wedge\widetilde{\tau}} \frac{e^{\rho t}}{\widetilde{a}(t)}|\widetilde{Z}_t|^2\, dt = \int_0^{T\wedge\tau} \frac{e^{\rho A(t)}}{a(t)}|Z_t|^2\, dA(t) = \int_0^{T\wedge\tau} e^{\rho A(t)}|Z_t|^2\, dt,
\end{aligned}$$

where $T = C_{\widetilde{T}}$ is an $(\mathcal{F}_t)_{t\ge 0}$-stopping time. So, we get

$$\begin{aligned}
&\mathbb{E}\biggl[\sup_{t\in[0,T\wedge\tau]} e^{\frac{p}{2}\rho A(t)}X(t)^p + \biggl(\int_0^{T\wedge\tau} e^{\rho A(t)}|Z_t|^2\, dt\biggr)^{p/2}\biggr] \\
&\quad\le c(p)\cdot\mathbb{E}\biggl[e^{\frac{p}{2}\rho A(T\wedge\tau)}X(T\wedge\tau)^p + \biggl(\int_0^{T\wedge\tau} l a(s)e^{\rho A(s)/2}x(s)\, ds\biggr)^p\biggr].
\end{aligned} \tag{3}$$

Since $X(T\wedge\tau) = \mathbb{E}[\xi + l\int_{T\wedge\tau}^{\tau} a(s)x(s)\, ds|\mathcal{F}_{T\wedge\tau}]$, we see that

$$\begin{aligned}
\mathbb{E}\bigl[e^{\frac{p}{2}\rho A(T\wedge\tau)}X(T\wedge\tau)^p\bigr] &= \mathbb{E}\biggl[e^{\frac{p}{2}\rho A(T\wedge\tau)}\cdot\biggl(\mathbb{E}\biggl[\xi + l\int_{T\wedge\tau}^{\tau} a(s)x(s)\, ds\Big|\mathcal{F}_{T\wedge\tau}\biggr]\biggr)^p\biggr] \\
&\le \mathbb{E}\biggl[e^{\frac{p}{2}\rho A(T\wedge\tau)}\cdot\biggl(\xi + l\int_{T\wedge\tau}^{\tau} a(s)x(s)\, ds\biggr)^p\biggr] \\
&\le 2^{p-1}\mathbb{E}\biggl[e^{\frac{p}{2}\rho A(T\wedge\tau)}\biggl(\xi^p + l^p\cdot\biggl(\int_{T\wedge\tau}^{\tau} a(s)x(s)\, ds\biggr)^p\biggr)\biggr] \\
&\le 2^{p-1}\mathbb{E}\bigl[e^{\frac{p}{2}\rho A(\tau)}\xi^p\bigr] + 2^{p-1}l^p\mathbb{E}\biggl[e^{\frac{p}{2}\rho A(T\wedge\tau)}\biggl(\int_{T\wedge\tau}^{\tau} a(s)x(s)\, ds\biggr)^p\biggr].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&e^{\frac{p}{2}\rho A(T\wedge\tau)}\biggl(\int_{T\wedge\tau}^{\tau} a(s)x(s)\, ds\biggr)^p \\
&\quad= e^{\frac{p}{2}\rho A(T\wedge\tau)}\biggl(\int_{T\wedge\tau}^{\tau} a(s)^{1/p}e^{\frac{\rho}{2}A(s)}x(s)a(s)^{1/q}e^{-\frac{\rho}{2}A(s)}\, ds\biggr)^p
\end{aligned}$$

$$\leq \left(\int_{T\wedge\tau}^{\tau} a(s)e^{\frac{p}{2}\rho A(s)}x(s)^p\,ds\right)\cdot\left(e^{\frac{\rho}{2}qA(T\wedge\tau)}\int_{T\wedge\tau}^{\tau} a(s)e^{-\frac{\rho}{2}qA(s)}\,ds\right)^{p/q}$$
$$\leq \left(\int_{T\wedge\tau}^{\tau} a(s)e^{\frac{p}{2}\rho A(s)}x(s)^p\,ds\right)\cdot\left(\frac{2}{\rho q}\right)^{p/q}\cdot\left(1-e^{-\frac{\rho}{2}q[A(\tau)-A(T\wedge\tau)]}\right)^{p/q}$$
$$\leq \left(\frac{2}{\rho q}\right)^{p/q}\cdot\int_0^{\tau} a(s)e^{\frac{p}{2}\rho A(s)}x(s)^p\,ds.$$

Therefore, we get

$$\mathbb{E}\left[e^{\frac{p}{2}\rho A(T\wedge\tau)}X(T\wedge\tau)^p\right]\leq c(p)\cdot\mathbb{E}\left[e^{\frac{p}{2}\rho A(\tau)}\xi^p+\int_0^{\tau} a(s)e^{\frac{p}{2}\rho A(s)}x(s)^p\,ds\right] \tag{4}$$

for some constant $c(p)>0$ depending on p. Using (1), (3), (4), we deduce that

$$\mathbb{E}\left[\sup_{t\in[0,T\wedge\tau]} e^{\frac{p}{2}\rho t}X(t)^p+\left(\int_0^{T\wedge\tau} e^{\rho A(t)}|Z_t|^2\,dt\right)^{p/2}\right]$$
$$\leq c(p)\cdot\mathbb{E}\left[e^{\frac{p}{2}\rho A(\tau)}\xi^p+\int_0^{\tau} a(s)e^{\frac{p}{2}\rho A(s)}x(s)^p\,ds\right]<\infty.$$

Sending $\widetilde{T}\to+\infty$ (hence $T\to\infty$), Fatou's lemma ensures that

$$\mathbb{E}\left[\sup_{t\in[0,\tau]} e^{\frac{p}{2}\rho A(t)}X(t)^p+\left(\int_0^{\tau} e^{\rho A(t)}|Z_t|^2\,dt\right)^{p/2}\right]<\infty. \tag{5}$$

So we have proved that $Z\in\mathbb{H}^p(\rho,a,[0,\tau],\mathbb{R}^{1\times d})$. Applying Ito's formula to (2), we get

$$X(t)e^{lA(t\wedge\tau)}=\xi e^{l}A(\tau)+l\int_{t\wedge\tau}^{\tau} a(s)e^{lA(s)}\left[x(s)-X(s)\right]ds-\int_{t\wedge\tau}^{\tau} e^{lA(s)}Z_s\,dB_s.$$

From $x(t)\leq X(t)$, it follows that

$$X(t)e^{lA(t\wedge\tau)}\leq\xi e^{lA(\tau)}-\int_{t\wedge\tau}^{\tau} e^{lA(s)}Z_s\,dB_s. \tag{6}$$

From expression (5) and the Burkholder–Davis–Gundy inequality (see Chapter IV, Sect. 4 in [14] or Corollary 2.9 in [13]), we deduce

$$\mathbb{E}\left[\sup_{t\geq 0}\left|\int_0^{t\wedge\tau} e^{lA(s)}Z_s\,dB_s\right|\right]\leq c\cdot\mathbb{E}\left[\left(\int_0^{\tau} e^{2lA(s)}|Z_s|^2\,ds\right)^{1/2}\right]$$
$$=c\cdot\mathbb{E}\left[\left(\int_0^{\tau} e^{\theta A(s)}|Z_s|^2\cdot e^{(2l-\rho)A(s)}\,ds\right)^{1/2}\right]$$
$$\leq c\cdot\mathbb{E}\left[e^{(l-\rho/2)A(\tau)\vee 0}\left(\int_0^{\tau}|Z_s|^2\cdot e^{\theta A(s)}\,ds\right)^{1/2}\right]$$
$$\leq\frac{c}{p}\mathbb{E}\left[\left(\int_0^{\tau}|Z_s|^2\cdot e^{\rho A(s)}\,ds\right)^{p/2}\right]+\frac{c}{q}\mathbb{E}\left[e^{\frac{q}{2}(2l-\rho)^+A(\tau)}\right]<+\infty.$$

Thus, $M(t) = \int_0^{t\wedge\tau} e^{lA(s)} z_s\, dB_s$ is a uniformly integrable martingale. Taking conditional expectations with respect to $\mathcal{F}_t$ on both sides of (6), we get

$$X(t)e^{lA(t\wedge\tau)} = \mathbb{E}\big[X(t)e^{lA(t\wedge\tau)}|\mathcal{F}_t\big] \leq \mathbb{E}\big[\xi e^{lA(\tau)}|\mathcal{F}_t\big].$$

So, we have $x(t) \leq X(t) \leq \mathbb{E}[\xi e^{l\int_{t\wedge\tau}^{\tau} a(s)\, ds}|\mathcal{F}_t]$, which is the desired result. □

Remark 1 If $l = 1$ and $\rho = 2$ (hence $\theta > 2$), then $\mathbb{E}[e^{\frac{q}{2}(2l-\rho)^+ A(\tau)}] = 0$. So, we do not have to assume that $\mathbb{E}[e^{\frac{q}{2}(2l-\rho)^+ A(\tau)}] < \infty$.

In many cases, $x(t)$ belongs to $\mathbb{S}^p(\theta, a, [0,\tau], \mathbb{R})$ as well as $\mathbb{H}^{p,a}(\theta, a, [0,\tau], \mathbb{R})$. For example, if the process $x(t)$ is an L^p-solution of the backward stochastic differential equation, then it belongs to $\mathbb{S}^p(\theta, a, [0,\tau], \mathbb{R}) \cap \mathbb{H}^{p,a}(\theta, a, [0,\tau], \mathbb{R})$ (see e.g. [15]).

4 Application

In this section, we show a comparison principle of L^p-solutions to BSDEs with random terminal time under stochastic monotonicity condition on generator. The existence and uniqueness of L^2-solutions (that is, $p = 2$) to BSDEs under the Lipschitz condition was studied in [4]. Wang et al. [15] first established the existence and uniqueness result of L^p-solutions under the stochastic Lipschitz condition. Recently, Pardoux and Rășcanu [13] studied the existence and uniqueness of L^p-solutions to BSDEs with random terminal time under a stochastic monotonicity condition. The comparisons theorems of L^p-solutions were studied in [13, 17] under the stochastic Lipschitz (or monotonicity) conditions, but those were restricted to the case of deterministic terminal time. We shall prove the comparison theorem of L^p-solutions in an unbounded random time horizon. Let us consider the following one-dimensional BSDE with random terminal time:

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)\, ds - \int_{t\wedge\tau}^{\tau} Z_s\, dB_s, \tag{7}$$

where the terminal time τ is an $(\mathcal{F}_t)_{t\geq 0}$-stopping time, the terminal value ξ is an $\mathcal{F}_\tau$-measurable random variable, and the generator $f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{1\times d} \to \mathbb{R}$ is $(\mathcal{F}_t)_{t\geq 0}$-progressively measurable. The solution of BSDE (7) is a pair $(Y_t, Z_t)_{t\geq 0}$ of progressively measurable processes such that $Y_t = \xi$ and $Z_t = 0$, $\mathbb{P}$-a.s. for $t \geq \tau$, and for all $0 \leq T < \infty$,

$$Y_{t\wedge\tau} = Y_{T\wedge\tau} + \int_{t\wedge\tau}^{T\wedge\tau} f(s, Y_s, Z_s)\, ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s\, dB_s, \quad t \in [0, T].$$

For convenience, we characterize BSDE (7) by a triple (τ, ξ, f).

Theorem 2 *Let $p > 1$ and consider two BSDEs with data (τ, ξ^1, f^1) and (τ, ξ^2, f^2). Let (Y^1, Z^1) and (Y^2, Z^2) be two solutions of BSDE (7) corresponding to (τ, ξ^1, f^1) and (τ, ξ^2, f^2), respectively. Suppose that f is stochastic monotone in y and stochastic Lipschitz in z: that is, there exist progressively processes r_t, u_t such that, for all $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^{1\times d}$,*

1. $(y - y')(f(t, y, z) - f(t, y', z)) \leq r_t (y - y')^2$,
2. $|f(t, y, z) - f(t, y, z')| \leq u_t |z - z'|$.

Set $a(t) := r_t + \frac{1}{2(p-1)} u_t^2$, $A(t) := \int_0^t a(s)\, ds$. Assume that $a(t) > \epsilon$ for some $\epsilon > 0$.

We further assume that (Y^i, Z^i), $i = 1, 2$, belongs to $(\mathbb{S}^p(\theta, a, [0,\tau], \mathbb{R}) \cap \mathbb{H}^{p,a}(\theta, a, [0,\tau], \mathbb{R})) \times \mathbb{H}^p(\theta, a, [0,\tau], \mathbb{R}^{1\times d})$ for some $\theta > 2$.

If $\xi^1 \leq \xi^2$ and $f(t, Y_t^2, Z_t^2) \leq f'(t, Y_t^2, Z_t^2)$, then $Y_t^1 \leq Y_t^2$, $\mathbb{P}$-a.s.

Proof Define $\bar{Y} := Y^1 - Y^2$, $\bar{Z} := Z^1 - Z^2$, $\bar{\xi} := \xi^1 - \xi^2$, then

$$\bar{Y}_t = \bar{\xi} + \int_{t\wedge\tau}^{\tau} \left[f^1\left(s, Y_s^1, Z_s^1\right) - f^2\left(s, Y_s^2, Z_s^2\right)\right] ds - \int_{t\wedge\tau}^{\tau} \bar{Z}_s \, dB_s.$$

Let θ be an arbitrary number such that $\theta > \theta' > 2$. We set $p' := \frac{\theta'}{\theta}(p-1) + 1$. Then p' satisfies $1 < p' < p$ and

$$\frac{1}{2(p'-1)} = \frac{\theta}{\theta'} \cdot \frac{1}{2(p-1)}. \tag{8}$$

By the virtue of Itô–Tanaka's formula (see Exercise VI.1.25 in [14] or p. 220 in [11] for details), we have

$$\begin{aligned}
&\left(\bar{Y}_t^+\right)^{p'} + \frac{p'(p'-1)}{2} \int_{t\wedge\tau}^{\tau} |\bar{Y}_s|^{p'-2} I_{\bar{Y}_s>0} |\bar{Z}_s|^2 \, ds \\
&\quad = \left(\bar{\xi}^+\right)^{p'} + p' \int_{t\wedge\tau}^{\tau} |\bar{Y}_s|^{p'-1} I_{\bar{Y}_s>0} \left[f^1\left(s, Y_s^1, Z_s^1\right) - f^2\left(s, Y_s^2, Z_s^2\right)\right] ds \\
&\qquad - p' \int_{t\wedge\tau}^{\tau} |\bar{Y}_s|^{p'-1} I_{\bar{Y}_s>0} \bar{Z}_s \, dB_s.
\end{aligned}$$

From assumptions,

$$\begin{aligned}
&p' |\bar{Y}_s|^{p'-1} I_{\bar{Y}_s>0} \left[f^1\left(s, Y_s^1, Z_s^1\right) - f^2\left(s, Y_s^2, Z_s^2\right)\right] \\
&\quad = p' |\bar{Y}_s|^{p'-1} I_{\bar{Y}_s>0} \left[f^1\left(s, Y_s^1, Z_s^1\right) - f^1\left(s, Y_s^2, Z_s^1\right)\right] \\
&\qquad + p' |\bar{Y}_s|^{p'-1} I_{\bar{Y}_s>0} \left[f^1\left(s, Y_s^2, Z_s^1\right) - f^1\left(s, Y_s^2, Z_s^2\right)\right] \\
&\qquad + p' |\bar{Y}_s|^{p'-1} I_{\bar{Y}_s>0} \left[f^1\left(s, Y_s^2, Z_s^2\right) - f^2\left(s, Y_s^2, Z_s^2\right)\right] \\
&\quad \le p' r_t \cdot \left(\bar{Y}_s^+\right)^{p'-2} |\bar{Y}_s|^2 + p' u_t \left(\bar{Y}_s^+\right)^{p'-1} |\bar{Z}_s| \\
&\quad = p' r_t \cdot \left(\bar{Y}_s^+\right)^{p'} + p' \left(\sqrt{p'-1} \cdot \left(\bar{Y}_s^+\right)^{\frac{p'-2}{2}} |\bar{Z}_s| \cdot \frac{1}{\sqrt{p'-1}} \left(\bar{Y}_s^+\right)^{\frac{p'}{2}} u_t\right) \\
&\quad \le p' r_t \cdot \left(\bar{Y}_s^+\right)^{p'} + \frac{p'(p'-1)}{2} \left(Y_s^+\right)^{p'-2} |\bar{Z}_s|^2 + \left(\bar{Y}_s^+\right)^{p'} \frac{p'}{2(p'-1)} u_t^2 \\
&\quad = p' \cdot \left(r_t + \frac{1}{2(p'-1)} u_t^2\right) \cdot \left(\bar{Y}_s^+\right)^{p'} + \frac{p'(p'-1)}{2} \left(\bar{Y}_s^+\right)^{p'-2} |\bar{Z}_s|^2.
\end{aligned}$$

Using the Burkholder–Davis–Gundy inequality (see Chapter IV, Sect. 4 in [14] or Corollary 2.9 in [13]), we see that $\int_0^{t\wedge\tau} |\bar{Y}_s|^{p'-1} I_{\bar{Y}_s>0} \bar{Z}_s \, dB_s$ is a uniformly integrable martingale. In fact,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\ge 0} \left| \int_0^{t\wedge\tau} |\bar{Y}_s|^{p'-1} I_{\bar{Y}_s>0} \bar{Z}_s \, dB_s \right|\right] &\le \mathbb{E}\left[\left(\int_0^{\tau} |\bar{Y}_s|^{2(p'-1)} |\bar{Z}_s|^2 \, ds\right)^{1/2}\right] \\
&\le \mathbb{E}\left[\sup_{s\in[0,\tau]} |\bar{Y}_s|^{p'-1} \left(\int_0^{\tau} |\bar{Z}_s|^2 \, ds\right)^{1/2}\right]
\end{aligned}$$

$$\le \frac{p'-1}{p'}\mathbb{E}\Big[\sup_{s\in[0,\tau]}|\bar{Y}_s|^{p'}\Big] + \frac{1}{p'}\mathbb{E}\bigg[\bigg(\int_0^\tau |\bar{Z}_s|^2\,ds\bigg)^{p'/2}\bigg]$$
$$< \infty.$$

Consequently, we obtain

$$\begin{aligned}
(\bar{Y}_t^+)^{p'} &\le \mathbb{E}\bigg[(\bar{\xi}^+)^{p'} + \int_{t\wedge\tau}^\tau p'\bigg(r_s + \frac{1}{2(p'-1)}u_s^2\bigg)(\bar{Y}_s^+)^{p'}\,ds\Big|\mathcal{F}_t\bigg] \\
&= \mathbb{E}\bigg[(\bar{\xi}^+)^{p'} + \int_{t\wedge\tau}^\tau p'\bigg(r_s + \frac{\theta}{\theta'}\cdot\frac{1}{2(p-1)}u_s^2\bigg)(\bar{Y}_s^+)^{p'}\,ds\Big|\mathcal{F}_t\bigg] \\
&\le \mathbb{E}\bigg[(\bar{\xi}^+)^{p'} + \int_{t\wedge\tau}^\tau p'\cdot\frac{\theta}{\theta'}\bigg(r_s + \frac{1}{2(p-1)}u_s^2\bigg)(\bar{Y}_s^+)^{p'}\,ds\Big|\mathcal{F}_t\bigg],
\end{aligned}$$

where we used expression (8). From $\xi^1 \le \xi^2$, it follows that $(\bar{\xi}^+)^{p'} = 0$.

Set $\bar{p} := p/p'$, $\bar{a}(s) := p'\cdot\frac{\theta}{\theta'}a(s)$, $\bar{A}(s) := \int_0^s \bar{a}(r)\,dr$, and $x(t) := (\bar{Y}_t^+)^{p'}$. Then we have

$$\mathbb{E}\bigg[\int_0^\tau \bar{a}(s)e^{\frac{\theta'}{2}\bar{p}\cdot\bar{A}(s)}x(s)^{\bar{p}}\,ds\bigg] = p'\cdot\frac{\theta'}{\theta}\mathbb{E}\bigg[\int_0^\tau a(s)e^{\frac{\theta}{2}pA(s)}(\bar{Y}_s^+)^p\,ds\bigg] < \infty.$$

This implies $x(t) \in \mathbb{H}^{\bar{p},\bar{a}}(\theta',\bar{a},[0,\tau],\mathbb{R})$. Theorem 1 yields that $x(t) = 0$. Hence, $Y_t^1 \le Y_t^2$. □

Remark 2 If the generator f does not depend on z, we can prove the comparison theorem more easily, by applying Itô–Tanaka's formula to $\bar{Y}_t^+$.

Remark 3 As a direct consequence of Theorem 2, we see that the solution of BSDE (7) must be unique.

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