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A subgradient extragradient algorithm for solving monotone variational inequalities in Banach spaces

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Abstract

In this paper, we introduce an algorithm for solving classical variational inequalities problem with Lipschitz continuous and monotone mapping in Banach space. We modify the subgradient extragradient methods with a new and simple iterative step size, the strong convergence of algorithm is established without the knowledge of the Lipschitz constant of the mapping. Finally, a numerical experiment is presented to show the efficiency and advantage of the proposed algorithm. Our results generalize some of the work in Hilbert spaces to Banach spaces.

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1 Introduction

The variational inequality problem (VIP) which was first introduced by Hartman and Stampacchia [1] in 1966, is a very important tool in studying engineering mechanics, physics, economics, optimization theory and applied sciences in a unified and general framework (see [2, 3]). Under appropriate conditions, there are two general approaches for solving the variational inequality problem, one is the regularized method and the other is the projection method. Many projection-type algorithms for solving the variational inequalities problem have been proposed and analyzed by many authors [4–22]. The gradient method is the simplest algorithm in which only one projection on feasible set is performed, and the convergence of the method requires a strongly monotonicity. To avoid the hypothesis of the strongly monotonicity, Korpelevich [4] proposed an algorithm for solving the variational inequalities in Euclidean space, which was called the extragradient-type method. The subgradient extragradient-type algorithm was introduced by Censor et al. in [5] for solving variational inequalities in real Hilbert space. Yao et al. in [6] proposed an iterative algorithm for solving a common solution of the pseudomonotone variational inequalities and fixed point of pseudocontractive operators in Hilbert spaces.

In the past, most variational inequalities were in Euclidean or Hilbert space, recently, extragradient-type method was extended from Hilbert spaces to Banach spaces (see [23–

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27)). In [23] they used the subgradient extragradient method and Halpern method to propose an algorithm for solving variational inequalities in Banach spaces. In [24], they proposed a splitting algorithm for finding a common zero of a finite family of inclusion problems of accretive operators in Banach space. Inspired by the work mentioned, in this work, we extend subgradient extragradient algorithm proposed by [8] for solving variational inequalities from Hilbert spaces to Banach spaces. It is worth stressing that our algorithm has a simple structure and the convergence of algorithms is not required to know the Lipschitz constant of the mapping. The paper is organized as follows. In Sect. 2, we present some preliminaries that will be needed in the sequel. In Sect. 3, we propose an algorithm and analyze its convergence. Finally, in Sect. 4 we present a numerical example and comparison.

2 Mathematical preliminaries

This section we will introduce some definitions and basic results that will be used in our paper. Assume that X is a real Banach space with its dual X^* , $\|\cdot\|$ and $\|\cdot\|_*$ denote the norms of X and X^* , respectively, $\langle x, x^* \rangle$ the duality coupling in $X \times X^*$ for all $x^* \in X^*$ and $x \in X$, $x_n \rightarrow x$ strong convergence of a sequence $\{x_n\}$ of X to $x \in X$, $x_n \rightharpoonup x$ weak convergence of a sequence $\{x_n\}$ of X to $x \in X$. S_X denote the unit sphere of X , and B_X the closed unit ball of X . Let C be a nonempty closed convex subset of X , its closure be denoted by \bar{C} and $F : C \rightarrow X^*$ be a continuous mapping. Consider with the variational inequality (for short, $VI(F, C)$) which consists in finding a point $x \in C$ such that

$$\langle F(x), y - x \rangle \geq 0, \quad \forall y \in C. \tag{1}$$

Let S be the solution set of (1). Finding a solution of S is fundamental problem in optimization theory. It is well known that x is the solution of the $VI(F, C)$ if and only if x is the solution of the fixed-point equation $x = P_C(x - \lambda F(x))$, where λ is an arbitrary positive constant. Therefore, the knowledge of fixed-point algorithms can be used to solve $VI(F, C)$.

We next recall some properties of the Banach space. Let X be a real Banach space and X^* be the corresponding dual space.

Definition 1 Assume that $C \subseteq X$ is a nonempty set, $F : C \rightarrow X^*$ is a continuous mapping, then

(A1) The mapping F is monotone, i.e.,

$$\langle F(x) - F(y), x - y \rangle \geq 0, \quad \forall x, y \in C. \tag{2}$$

(A2) The mapping F is Lipschitz-continuous with constant $L > 0$, i.e., there exists $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{3}$$

(A3) ([28]) The mapping F is called hemicontinuous of C into X^* iff for any $x, y \in C$ and $z \in X$ the function $t \mapsto \langle z, F(tx + (1 - t)y) \rangle$ of $[0, 1]$ into \mathcal{R} is continuous.

The *normalized duality mapping* J_X (usually written J) of X into X^* is defined by

$$J(x) = \{x^* \in X^* | \langle x, x^* \rangle = \|x^*\|^2 = \|x\|^2\}$$

for all $x \in X$. Let $q \in (0, 2]$. The *generalized duality mapping* $J_q : X \rightarrow 2^{X^*}$ is defined (for the definitions and properties, see [24]) by

$$J_q(x) = \{j_q(x) \in X^* | \langle j_q(x), x \rangle = \|x\| \|j_q(x)\|, \|j_q(x)\| = \|x\|^{q-1}\}$$

for all $x \in X$.

Let $U = \{x \in X : \|x\| = 1\}$. The norm of X is said to be *Gâteaux differentiable* if, for each $x, y \in U$, the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{4}$$

exists. In this case, the space X is also called *smooth*. We know that X is smooth iff J is a single-valued mapping of X into X^* , X is reflexive iff J is surjective, and X is strictly convex iff J is one-to-one. Therefore, if X is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection, and then there exist the inverse mapping J^{-1} coincides with the duality mapping J^* on X^* . More details can be found in [29–31]. If (4) converges uniformly in $x, y \in S_X$, X is said to be uniformly smooth. It is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S_X$ and $x \neq y$. The modulus δ_X of convexity is defined by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| \mid x, y \in B_X, \|x-y\| \geq \varepsilon \right\}, \tag{5}$$

for all $\varepsilon \in [0, 2]$. A Banach space X is said to be *uniformly convex* if $\delta_X(\varepsilon) > 0$. It is well known that a Banach space X is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|y_n\| = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|x_n + y_n\| = 2, \quad \lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$$

hold. A uniformly convex Banach space is strictly convex and reflexive. By [24] we know that a Banach space X is smooth if and only if the duality mapping J_q is single valued and is uniformly smooth if and only if the duality mapping J_q is single valued and norm-to-norm uniformly continuous on bounded sets of X . Moreover, if there exists $c > 0$ such that, for all $\varepsilon \in [0, 2]$, $\delta_X(\varepsilon) > c\varepsilon^2$, then X is said to be *2-uniformly convex*. It is obvious that every 2-uniformly convex Banach space is uniformly convex and all Hilbert spaces are uniformly smooth and 2-uniformly convex, and therefore are reflexive.

Now, we recall some useful definitions and results. Firstly, let us introduce the generalized projection operator of X . Let $C \subseteq X$ be a nonempty closed convex subset of a real uniformly convex Banach space X . Then we know that, for any $z \in X$, there exists a unique element $\tilde{z} \in C$ such that $\|z - \tilde{z}\| \leq \|z - y\|$ for all $y \in C$. Putting $\tilde{z} = P_C z$, the operator $P_C : X^* \rightarrow C \subset X$ is called the *generalized projection (or metric projection) operator* of X onto C .

To avoid the hypothesis of the strongly monotonicity, Korpelevich [4] give the extragradient-type method:

$$y_n = P_C(x_n - \lambda F(x_n)), \quad x_{n+1} = P_C(x_n - \lambda F(y_n)), \tag{6}$$

where $\lambda \in (0, \frac{1}{L})$.

The subgradient extragradient-type algorithm extend (6) in which the second orthogonal projection onto some constructible set in Euclidean space for solving VI(F, C) in real Hilbert space. Their method is of the following form:

$$y_n = P_C(x_n - \lambda F(x_n)), \quad x_{n+1} = P_{T_n}(x_n - \lambda F(y_n)), \tag{7}$$

where $T_n = \{x \in X | \langle x_n - \lambda F(x_n) - y_n, x - y_n \rangle \leq 0\}$ and $\lambda \in (0, \frac{1}{L})$. Cai et al. [23] suggested the following method:

$$\begin{cases} x_0 \in X, \\ y_n = P_C(Jx_n - \lambda_n F(x_n)), \\ T_n = \{x \in X | \langle Jx_n - \lambda_n F(x_n) - Jy_n, x - y_n \rangle \leq 0\}, \\ z_n = P_{T_n}(Jx_n - \lambda_n F(y_n)), \\ x_{n+1} = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)Jz_n), \end{cases} \tag{8}$$

where J is the normalized duality mapping of X into X^* , $\lambda_n \in (0, \frac{1}{L})$, $\alpha_n \in (0, 1)$, $\alpha_n \rightarrow 0$ and $\sum_{n=1}^\infty \alpha_n = +\infty$. They proved that the sequence $\{x_n\}$ generated by (8) converges strongly to $P_S Jx_0$.

The main drawback of algorithms (7) and (8) is a requirement to know the Lipschitz constant or to know some estimation of it. Yekini and Olaniyi [7] proposed the following subgradient extragradient method:

$$\begin{cases} \text{Given } \rho \in (0, 1), \mu \in (0, 1), \\ y_n = P_C(x_n - \lambda_n F(x_n)), \text{ where } \lambda_n = \rho^{l_n} \text{ and } l_n \text{ is the smallest nonnegative inter } l \\ \text{such that } \lambda_n \|F(x_n) - F(y_n)\| \leq \mu \|x_n - y_n\|, \\ z_n = P_{T_n}(x_n - \lambda_n F(y_n)), \text{ where } T_n = \{x \in H | \langle x_n - \lambda_n F(x_n) - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)z_n, \text{ where } f : H \rightarrow H \text{ is a contraction mapping.} \end{cases} \tag{9}$$

The algorithm of (9) does not require one to know the Lipschitz constant, but the method may involve computation of additional projections.

In [32], Alber introduced a functional $V(x^*, y) : X^* \times X \rightarrow R$ by

$$V(x^*, y) = \|x^*\|_*^2 - 2\langle x^*, y \rangle + \|y\|^2. \tag{10}$$

Clearly,

$$V(x^*, y) \geq (\|x^*\|_* - \|y\|)^2.$$

The operator $P_C : X^* \rightarrow C \subseteq X$ is said to be *generalized projection operator* if it associates to an arbitrary fixed point $x^* \in X^*$, the solution to the minimization problem

$$V(x^*, \tilde{x}^*) = \inf_{y \in C} V(x^*, y),$$

where $\tilde{x}^* = P_C x^* \in C \subset X$ is called a *generalized projection* of the point x^* . For more results about P_C , see [32]. The next lemma can describe the properties of P_C .

Lemma 1 *Let C be a nonempty closed convex set in X and $x^*, y^* \in X^*$, $\tilde{x}^* = P_C x^*$. Then*

- (i) $\langle J\tilde{x}^* - x^*, y - \tilde{x}^* \rangle \geq 0, \quad \forall y \in C.$
- (ii) $V(J\tilde{x}^*, y) \leq V(x^*, y) - V(x^*, \tilde{x}^*), \quad \forall y \in C.$
- (iii) $V(x^*, z) + 2\langle J^{-1}x^* - z, y^* \rangle \leq V(x^* + y^*, z), \quad \forall z \in X.$

In [32], Alber also introduced the *Lyapunov functional* $\varphi : X \times X \rightarrow R$ by

$$\varphi(x, y) = \|x\|^2 - 2\langle Jx, y \rangle + \|y\|^2, \quad \forall x, y \in X.$$

Then, combining (10), we obtain $V(x^*, y) = \varphi(J^{-1}x^*, y)$, for all $x^* \in X^*$, $y \in X$. Moreover, we have the following lemma (see [33]).

Lemma 2 ([33]) *Let X be a real 2-uniformly convex Banach space. Then, there exists $\mu \geq 1$ such that, for all $x, y \in X$,*

$$\frac{1}{\mu} \|x - y\|^2 \leq \varphi(x, y).$$

The following two lemmas which will be useful to our subsequent convergence analysis, and they are stated and proved in [34, 35].

Lemma 3 ([34]) *Let $\{a_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ which satisfies $a_{n_j} < a_{n_j+1}$ for all $j \in \mathcal{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : a_k < a_{k+1}\},$$

where $n_0 \in \mathcal{N}$ such that $\{k \leq n_0 : a_k < a_{k+1}\}$ is nonempty. Then the following hold:

- (i) $\tau(n) \leq \tau(n + 1) \leq \dots$, and $\tau(n) \rightarrow \infty$;
- (ii) $a_{\tau(n)} \leq a_{\tau(n)+1}$ and $a_n \leq a_{\tau(n)+1}$.

Lemma 4 ([35]) *Let $\{a_n\}$ be a nonnegative real sequence and $\exists N > 0$, such that $\forall n \geq N$, satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n,$$

where (i) $\{\alpha_n\} \subset (0, 1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$; (ii) $\{\sigma_n\}$ is a sequence such that $\limsup_{n \rightarrow \infty} b_n \leq 0$; (iii) $\gamma_n \geq 0$, $\sum_{n=0}^{\infty} \gamma_n < \infty$. Then $\lim_{n \rightarrow \infty} a_n = 0$.

The following result for proving our main result relies on certain estimate and other classical properties of the iterates which are given in [23].

Lemma 5 ([23]) *Let $x^* = P_S x_0$. Define $a_n = \varphi(x_n, x^*)$ and $b_n = 2\langle x_0 - x^*, x_{n+1} - x^* \rangle$, then*

- (i) $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n$,
- (ii) $-1 \leq \limsup_{n \rightarrow \infty} b_n < \infty$

Lemma 6 ([36]) *Let C be a nonempty convex subset of a topological vector space X and $F : C \rightarrow X^*$ be a hemicontinuous mapping, then x^* is a solution of (1) if and only if*

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{11}$$

3 Main results

In this section, we introduce a new iterative algorithms for solving monotone variational inequality problems in Banach spaces. In order to present the method and establish its convergence, we make the following assumption.

Assumption 1

- (a) The feasible set C is a nonempty closed convex subset of a real 2-uniformly convex Banach space X .
- (b) $F : X \rightarrow X^*$ is a monotone on C and L -Lipschitz continuous on X .
- (c) The solution set S of $VI(F, C)$ is nonempty.

Now, we discuss the strong convergence using the following algorithm for solving monotone variational inequality. Our algorithms are of the following forms.

Algorithm A

- (Step 0) Take $\lambda_0 > 0$, $x_0 \in X$ be a given starting point, $\mu \in (0, 1)$.
- (Step 1) Given the current iterate x_n , compute

$$y_n = P_C(Jx_n - \lambda_n F(x_n)). \tag{12}$$

If $x_n = y_n$, then stop: x_n is a solution. Otherwise, go to Step 2.

- (Step 2) Construct the set $T_n = \{x \in X \mid \langle Jx_n - \lambda_n F(x_n) - Jy_n, x - y_n \rangle \leq 0\}$ and compute

$$z_n = P_{T_n}(Jx_n - \lambda_n F(y_n)), \quad x_{n+1} = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)Jz_n). \tag{13}$$

- (Step 3) Compute

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu(\|x_n - y_n\|^2 + \|z_n - y_n\|^2)}{2\langle F(x_n) - F(y_n), z_n - y_n \rangle}, \lambda_n\}, & \text{if } \langle F(x_n) - F(y_n), z_n - y_n \rangle > 0, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{14}$$

Set $n := n + 1$ and return to step 1.

We prove the strong convergence theorem for Algorithm A. Firstly, we give the following theorem, which plays a crucial role in the proof of the main theorem.

Theorem 1 Assume that Assumption 1 holds and x_n, y_n, λ_n be the sequences generated by Algorithm A, then we have the following result:

- (1) If $x_n = y_n$ for some $n \in N$, then $x_n \in S$.
- (2) The sequence $\{\lambda_n\}$ is a monotonically decreasing sequence with lower bound $\min\{\frac{\mu}{L}, \lambda_0\}$, and therefore, the limit of $\{\lambda_n\}$ exists and is denoted $\lambda = \lim_{n \rightarrow \infty} \lambda_n$. It is obvious that $\lambda > 0$.

Proof (1) If $x_n = y_n$, then $x_n = P_C(Jx_n - \lambda_n F(x_n))$, so $x_n \in C$. By the characterization of the generalized projection P_C onto C , we have

$$\langle Jx_n - \lambda_n F(x_n) - Jx_n, x_n - x \rangle \geq 0 \quad \forall x \in C.$$

Therefore,

$$\langle -\lambda_n F(x_n), x_n - x \rangle = \lambda_n \langle F(x_n), x - x_n \rangle \geq 0 \quad \forall x \in C.$$

Since $\lambda_n \geq 0, x_n \in S$.

(2) It is obvious that $\{\lambda_n\}$ is a monotonically decreasing sequence. Since F is a Lipschitz-continuous mapping with constant $L > 0$, in the case of $\langle F(x_n) - F(y_n), x_{n+1} - y_n \rangle > 0$, we have

$$\frac{\mu(\|x_n - y_n\|^2 + \|z_n - y_n\|^2)}{2\langle F(x_n) - F(y_n), z_n - y_n \rangle} \geq \frac{2\mu\|x_n - y_n\|\|z_n - y_n\|}{2\|F(x_n) - F(y_n)\|\|z_n - y_n\|} \geq \frac{\mu\|x_n - y_n\|}{L\|x_n - y_n\|} = \frac{\mu}{L}. \tag{15}$$

Clearly, the sequence $\{\lambda_n\}$ has the lower bound $\min\{\frac{\mu}{L}, \lambda_0\}$.

Since $\{\lambda_n\}$ is monotonically decreasing sequence and has the lower bound, the limit of $\{\lambda_n\}$ exists, and we denote $\lambda = \lim_{n \rightarrow \infty} \lambda_n$. Clearly, $\lambda > 0$. □

The following lemma plays a crucial role in the proof of Theorem 2.

Lemma 7 Assume that Assumption 1 holds. Let $\{x_n\}$ be a sequence generated by Algorithm A and $\{\alpha_n\} \subset (0, 1)$. Then the sequence $\{x_n\}$ is bounded.

Proof Let $u \in S$. By Lemma 1(ii), we have

$$\begin{aligned} V(Jz_n, u) &= V(JP_{T_n}(Jx_n - \lambda_n F(y_n)), u) \\ &\leq V(Jx_n - \lambda_n F(y_n), u) - V(Jx_n - \lambda_n F(y_n), z_n) \\ &= \|Jx_n - \lambda_n F(y_n)\|^2 - 2\langle Jx_n - \lambda_n F(y_n), u \rangle + \|u\|^2 \\ &\quad - \|Jx_n - \lambda_n F(y_n)\|^2 + 2\langle Jx_n - \lambda_n F(y_n), z_n \rangle - \|z_n\|^2 \\ &= -2\langle Jx_n, u \rangle + 2\lambda_n \langle F(y_n), u - z_n \rangle + 2\langle Jx_n, z_n \rangle + \|u\|^2 - \|z_n\|^2 \\ &= V(Jx_n, u) - V(Jx_n, z_n) + 2\lambda_n \langle F(y_n), u - z_n \rangle. \end{aligned} \tag{16}$$

Since F is monotone, i.e., $\langle F(y_n) - F(u), y_n - u \rangle \geq 0$, for all $n \in N$, combining with $u \in S$, we have

$$\langle F(y_n), y_n - u \rangle \geq \langle F(u), y_n - u \rangle \geq 0.$$

Then $0 \leq \langle F(y_n), y_n - u + z_n - z_n \rangle = \langle F(y_n), y_n - z_n \rangle - \langle F(y_n), u - z_n \rangle$. It implies that

$$\langle F(y_n), y_n - z_n \rangle \geq \langle F(y_n), u - z_n \rangle, \quad \forall n \in N. \tag{17}$$

By the definition of T_n , we have $\langle Jx_n - \lambda_n F(x_n) - Jy_n, z_n - y_n \rangle \leq 0$. Then

$$\begin{aligned} & \langle Jx_n - \lambda_n F(y_n) - Jy_n, z_n - y_n \rangle \\ &= \langle Jx_n - \lambda_n F(x_n) - Jy_n, z_n - y_n \rangle + \lambda_n \langle F(x_n) - F(y_n), z_n - y_n \rangle \\ &\leq \lambda_n \langle F(x_n) - F(y_n), z_n - y_n \rangle. \end{aligned} \tag{18}$$

Using the definition of λ_{n+1} and (17), (18) to (16), we get

$$\begin{aligned} V(Jz_n, u) &\leq V(Jx_n, u) - V(Jx_n, z_n) + 2\lambda_n \langle F(y_n), u - z_n \rangle \\ &\leq V(Jx_n, u) - V(Jx_n, z_n) + 2\lambda_n \langle F(y_n), y_n - z_n \rangle \\ &= V(Jx_n, u) - V(Jx_n, y_n) - V(Jy_n, z_n) + 2\langle Jx_n - \lambda_n F(y_n) - Jy_n, z_n - y_n \rangle \\ &\leq V(Jx_n, u) - V(Jx_n, y_n) - V(Jy_n, z_n) + 2\lambda_n \langle F(x_n) - F(y_n), z_n - y_n \rangle \\ &\leq V(Jx_n, u) - V(Jx_n, y_n) - V(Jy_n, z_n) + \lambda_n \frac{\mu}{\lambda_{n+1}} (\|x_n - y_n\|^2 + \|z_n - y_n\|^2). \end{aligned} \tag{19}$$

By Theorem 1(2), we get $\lim_{n \rightarrow \infty} \lambda_n \frac{\mu}{\lambda_{n+1}} = \mu$ ($0 < \mu < 1$), which means that there exists a integer number $N_0 > 0$, such that, for every $n > N_0$, we have $0 < \lambda_n \frac{\mu}{\lambda_{n+1}} < 1$, taking this result in (19), we obtain, for every $n > N_0$,

$$\begin{aligned} V(Jz_n, u) &\leq V(Jx_n, u) - V(Jx_n, y_n) - V(Jy_n, z_n) + \lambda_n \frac{\mu}{\lambda_{n+1}} (\|x_n - y_n\|^2 + \|z_n - y_n\|^2) \\ &\leq V(Jx_n, u) - (1 - \mu)(V(Jx_n, y_n) + V(Jy_n, z_n)) \\ &\leq V(Jx_n, u). \end{aligned}$$

Then, by the definition of x_{n+1} , we have, for every $n > N_0$,

$$\begin{aligned} V(Jx_{n+1}, u) &= V(\alpha_n Jx_0 + (1 - \alpha_n)Jz_n, u) \\ &= \|\alpha_n Jx_0 + (1 - \alpha_n)Jz_n\|^2 - 2\langle \alpha_n Jx_0 + (1 - \alpha_n)Jz_n, u \rangle + \|u\|^2 \\ &\leq \alpha_n \|Jx_0\|^2 - 2\alpha_n \langle Jx_0, u \rangle + \alpha_n \|u\|^2 \\ &\quad + \|(1 - \alpha_n)Jz_n\|^2 - 2(1 - \alpha_n)\langle Jz_n, u \rangle + (1 - \alpha_n)\|u\|^2 \\ &= \alpha_n V(Jx_0, u) + (1 - \alpha_n)V(Jz_n, u) \\ &\leq \alpha_n V(Jx_0, u) + (1 - \alpha_n)V(Jx_n, u) \\ &\leq \max\{V(Jx_0, u), V(Jx_n, u)\} \\ &\leq \dots \leq \max\{V(Jx_0, u), V(Jx_{N_0}, u)\}. \end{aligned}$$

Hence, $\{V(Jx_n, u)\}$ is bounded. Since $V(Jx_n, u) \geq \frac{1}{\mu} \|x_n - u\|^2$, we see that $\{x_n\}$ is bounded. □

Theorem 2 Assume that Assumption 1 holds, the sequence $\{\alpha_n\}$ satisfies $\{\alpha_n\} \subset (0, 1)$, $\sum_{n=0}^\infty \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$. Let $\{x_n\}$ be a sequence generated by Algorithm A. Then $\{x_n\}$ strongly converges to a solution $x^* = P_S Jx_0$.

Proof Let $x^* = P_S Jx_0$, by Lemma 1(i), we have

$$\langle Jx_0 - Jx^*, z - x^* \rangle \leq 0, \quad \forall z \in S.$$

By the proof of Theorem 1, we get $\exists N_0 \geq 0$, such that $\forall n \geq N_0, V(Jz_n, x^*) \leq V(Jx_n, x^*)$.

From Theorem 1(1), we see that the sequence $\{x_n\}$ is bounded, consequently, $\{y_n\}$ and $\{z_n\}$ are bounded. Moreover, by (19), we see that there exists $N_0 \geq 0$, such that, for every $n \geq N_0$,

$$\begin{aligned} V(Jx_{n+1}, x^*) &= V(\alpha_n Jx_0 + (1 - \alpha_n)Jz_n, x^*) \\ &\leq \alpha_n V(Jx_0, x^*) + (1 - \alpha_n)V(Jz_n, x^*) \\ &\leq \alpha_n V(Jx_0, x^*) + (1 - \alpha_n)V(Jx_n, x^*) \\ &\quad - (1 - \alpha_n)(1 - \mu)(V(Jx_n, y_n) + V(Jy_n, z_n)). \end{aligned} \tag{20}$$

Case 1 As in Lemma 5, set $a_n = \varphi(x_n, x^*)$. By Theorem 1(1), we know that there exists $N_1 \in \mathcal{N}$ ($N_1 \geq N_0$), such that the sequence $\{\varphi(x_n, x^*)\}_{n=N_1}^\infty$ is nonincreasing. Then $\{a_n\}_{n=1}^\infty$ converges, using this in (20), we obtain, when $n > N_1 \geq N_0$,

$$\begin{aligned} &(1 - \alpha_n)(1 - \mu)(V(Jx_n, y_n) + \varphi(y_n, z_n)) \\ &\leq \alpha_n V(Jx_0, x^*) - V(Jx_{n+1}, x^*) + (1 - \alpha_n)V(Jx_n, x^*) \\ &\leq V(Jx_n, x^*) - V(Jx_{n+1}, x^*) + \alpha_n V(Jx_0 - Jx_n, x^*). \end{aligned} \tag{21}$$

By $V(Jx_0 - Jx_n, x^*)$ being bounded and because $\{a_n\}_{n=1}^\infty$ converges, we have, when $n \rightarrow \infty$,

$$(1 - \alpha_n)(1 - \mu)(\varphi(x_n, y_n) + \varphi(y_n, z_n)) \leq \varphi(x_n, x^*) - \varphi(x_{n+1}, x^*) + \alpha_n \varphi(x_0 - x_n, x^*) \rightarrow 0.$$

Notice that $\varphi(x_n, y_n) \geq 0$ and $0 < \mu, \alpha_n < 1$, we have, when $n \rightarrow \infty$,

$$\|x_n - y_n\|^2 \rightarrow 0 \quad \text{and} \quad \|y_n - z_n\|^2 \rightarrow 0. \tag{22}$$

Furthermore, by the definition of x_{n+1} , we have

$$\|Jx_{n+1} - Jz_n\| = \alpha_n \|Jx_0 - Jz_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Since J^{-1} is norm-to-norm uniformly continuous on bounded subset of X^* , we have $\|x_{n+1} - z_n\| \rightarrow 0$. Therefore, we get

$$\|x_{n+1} - x_n\| = \|x_{n+1} - z_n\| + \|z_n - y_n\| + \|y_n - x_n\| \rightarrow 0, \quad n \rightarrow \infty.$$

By Theorem 1(1), we know that $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_k}\}$ that converges weakly to some $z_0 \in X$, such that $x_{n_k} \rightharpoonup z_0$ and

$$\limsup_{n \rightarrow \infty} \langle Jx_0 - Jx^*, x_n - x^* \rangle = \lim_{k \rightarrow \infty} \langle Jx_0 - Jx^*, x_{n_k} - x^* \rangle = \langle Jx_0 - Jx^*, z_0 - x^* \rangle \leq 0. \tag{23}$$

Then $y_{n_k} \rightarrow z_0$ and $z_0 \in C$. Since F is monotone and $y_{n_k} = P_C(x_{n_k} - \lambda_{n_k}F(x_{n_k}))$, by Lemma 1(i), we have $\langle Jx_{n_k} - \lambda_{n_k}F(x_{n_k}) - Jy_{n_k}, z - y_{n_k} \rangle \leq 0, \forall z \in C$. That is, for all $z \in C$,

$$\begin{aligned} 0 &\leq \langle Jy_{n_k} - Jx_{n_k}, z - y_{n_k} \rangle + \lambda_{n_k} \langle F(x_{n_k}), z - y_{n_k} \rangle \\ &= \langle Jy_{n_k} - Jx_{n_k}, z - y_{n_k} \rangle + \lambda_{n_k} \langle F(x_{n_k}), z - x_{n_k} \rangle + \lambda_{n_k} \langle F(x_{n_k}), x_{n_k} - y_{n_k} \rangle \\ &\leq \langle Jy_{n_k} - Jx_{n_k}, z - y_{n_k} \rangle + \lambda_{n_k} \langle F(z), z - x_{n_k} \rangle + \lambda_{n_k} \langle F(x_{n_k}), x_{n_k} - y_{n_k} \rangle. \end{aligned}$$

Let $k \rightarrow \infty$, using the facts that $\lim_{k \rightarrow \infty} \|y_{n_k} - x_{n_k}\| = 0, \{y_{n_k}\}$ is bounded and $\lim_{k \rightarrow \infty} \lambda_{n_k} = \lambda > 0$, we obtain $\langle F(z), z - z_0 \rangle \geq 0, \forall z \in C$. By Lemma 6, we have $z_0 \in S$.

By Lemma 1(iii) and (19), we have

$$\begin{aligned} \varphi(x_{n+1}, x^*) &= V(Jx_{n+1}, x^*) = V(\alpha_n Jx_0 + (1 - \alpha_n)Jz_n, x^*) \\ &\leq V(\alpha_n Jx_0 + (1 - \alpha_n)Jz_n - \alpha_n(Jx_0 - Jx^*), x^*) + 2\alpha_n \langle Jx_0 - Jx^*, x_{n+1} - x^* \rangle \\ &\leq \alpha_n V(Jz, x^*) + (1 - \alpha_n)V(Jz_n, x^*) + 2\alpha_n \langle Jx_0 - Jx^*, x_{n+1} - x^* \rangle \\ &= (1 - \alpha_n)V(Jz_n, x^*) + 2\alpha_n \langle Jx_0 - Jx^*, x_{n+1} - x^* \rangle \\ &\leq (1 - \alpha_n)\varphi(x_n, x^*) + 2\alpha_n \langle Jx_0 - Jx^*, x_{n+1} - x^* \rangle. \end{aligned}$$

It follows from Lemma 5 and Lemma 4 that $\lim_{n \rightarrow \infty} \varphi(x_n, x^*) = 0$, which means

$$\lim_{n \rightarrow \infty} x_n = x^*.$$

Case 2 Suppose that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\varphi(x_{m_j}, x^*) < \varphi(x_{m_j+1}, x^*)$ for all $j \in \mathcal{N}$. From Lemma 3, there exists a nondecreasing sequence $m_k \in \mathcal{N}$ such that $\lim_{k \rightarrow \infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathcal{N}$:

$$\varphi(x_{m_k}, x^*) < \varphi(x_{m_k+1}, x^*) \quad \text{and} \quad \varphi(x_k, x^*) < \varphi(x_{m_k+1}, x^*). \tag{24}$$

By (21), we know

$$\begin{aligned} (1 - \alpha_{m_k}) \left(1 - \lambda_{m_k} \frac{\mu}{\lambda_{m_k+1}} \right) (\varphi(x_{m_k}, y_{m_k}) + \varphi(y_{m_k}, z_{m_k})) \\ \leq \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + \alpha_{m_k} \|x_0 - x^*\|^2. \end{aligned} \tag{25}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{m_k}\}$ of $\{x_{n_k}\}$ which converges weakly to $z_0 \in X$. Using the same argument as in the proof of Case 1, and Combining (25) and $\lim_{k \rightarrow \infty} (1 - \lambda_{m_k} \frac{\mu}{\lambda_{m_k+1}}) = 1 - \mu > 0$, we obtain

$$\lim_{k \rightarrow \infty} \|x_{m_k} - y_{m_k}\| = 0, \quad \lim_{k \rightarrow \infty} \|z_{m_k} - y_{m_k}\| = 0, \quad \lim_{k \rightarrow \infty} \|x_{m_k+1} - x_{m_k}\| = 0.$$

Similarly we can conclude that

$$\limsup_{k \rightarrow \infty} \langle Jx_0 - Jx^*, x_{m_k+1} - x^* \rangle = \limsup_{k \rightarrow \infty} \langle Jx_0 - Jx^*, x_{m_k} - x^* \rangle \leq 0. \tag{26}$$

It follows from (25) and the proof of case 1 that, for all $m_k \geq N_0$, we have

$$\begin{aligned} \|x_{m_k+1} - x^*\|^2 &\leq (1 - \alpha_{m_k}) \|x_{m_k} - x^*\|^2 + 2\alpha_{m_k} \langle x_0 - x^*, x_{m_k+1} - x^* \rangle \\ &\leq (1 - \alpha_{m_k}) \|x_{m_k+1} - x^*\|^2 + 2\alpha_{m_k} \langle x_0 - x^*, x_{m_k+1} - x^* \rangle, \\ \varphi(x_{m_k+1}, x^*) &\leq (1 - \alpha_{m_k}) \varphi(x_{m_k}, x^*) + \alpha_{m_k} \langle Jx_0 - Jx^*, x_{m_k+1} - x^* \rangle \\ &\leq (1 - \alpha_{m_k}) \varphi(x_{m_k+1}, x^*) + \alpha_{m_k} \langle Jx_0 - Jx^*, x_{m_k+1} - x^* \rangle. \end{aligned}$$

Since $\alpha_n > 0$, this implies that $\forall m_k \geq N_1$, we have

$$\varphi(x_{m_k}, x^*) \leq \varphi(x_{m_k+1}, x^*) \leq \langle Jx_0 - Jx^*, x_{m_k+1} - x^* \rangle.$$

And then

$$\limsup_{k \rightarrow \infty} \varphi(x_{m_k}, x^*) \leq \limsup_{k \rightarrow \infty} \langle Jx_0 - Jx^*, x_{m_k+1} - x^* \rangle \leq 0,$$

we obtain $\limsup_{k \rightarrow \infty} \varphi(x_{m_k}, x^*) = 0$, which means $\lim_{k \rightarrow \infty} \|x_{m_k} - x^*\|^2 = 0$. Since $\|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|$, we have $\lim_{k \rightarrow \infty} \|x_k - x^*\| = 0$. Therefore $x_k \rightarrow x^*$. This concludes the proof. \square

4 Numerical experiments

In this section, we present two numerical experiments relative to the variational inequalities.

Example 4.1 We compare the proposed algorithm with the Algorithm 3.5 in [23]. For Algorithm A and Algorithm 3.5 in [23], we take $\alpha_n = \frac{1}{100(n+2)}$. To terminate the algorithms, we use the condition $\|y_n - x_n\| \leq \varepsilon$ and $\varepsilon = 10^{-3}$ for all the algorithms.

Let $H = L^2([0, 2\pi])$ with norm $\|x\| = (\int_0^{2\pi} |x(t)|^2 dt)^{\frac{1}{2}}$ and inner product $\langle x, y \rangle = \int_0^{2\pi} x(t)y(t) dt$, $x, y \in H$. The operator $F : H \rightarrow H$ is defined by $Fx(t) = \max(0, x(t))$, $t \in [0, 2\pi]$ for all $x \in H$. It can be easily verified that F is Lipschitz-continuous and monotone. The feasible set is $C = \{x \in H : \int_0^{2\pi} (t^2 + 1)x(t) dt \leq 1\}$. Observe that $0 \in S$ and so $S \neq \emptyset$. We take $\lambda_0 = 0.7$ and $\mu = 0.9$ for Algorithm A. For Algorithm 3.5 in [23], we take $\lambda = 0.7$. The numerical results are showed in Table 1.

Example 4.2 The example is classical. The feasible set is $C = R^m$ and $F(x) = Ax$, where A is a square $m \times m$ matrix given by the condition

$$a_{i,j} = \begin{cases} -1, & \text{if } j = m + 1 - i \text{ and } j > i, \\ 1, & \text{if } j = m + 1 - i \text{ and } j < i, \\ 0, & \text{otherwise.} \end{cases}$$

Table 1 Comparison between the Algorithm A and Algorithm 3.5 in [23]

x_0	Algorithm A		Algorithm 3.5 in [23]	
	iter.	time	iter.	time
$\frac{1}{4}t^2e^{-3t}$	6	0.84	6	0.73
$\frac{1}{120}(1 - t^2)$	8	6.55	8	6.21
$\frac{1}{150}\sin(t)$	9	18.28	9	16.33

Table 2 Comparison between Algorithm A and Algorithm 3.5 in [17]

m	Algorithm A ($\lambda_0 = 0.7$)		Algorithm 3.5 in [17] ($\lambda = 0.7$)		Algorithm A ($\lambda_0 = 0.9$)		Algorithm 3.5 in [17] ($\lambda = 0.9$)	
	iter.	time	iter.	time	iter.	time	iter.	time
500	58	0.057	58	0.054	109	0.074	109	0.066
1000	258	1.03	258	1.01	109	0.78	109	0.78
2000	366	8.70	366	8.57	118	7.15	118	7.12
4000	517	64.8	517	64.5	469	63.7	469	63.2

This is a classical example of a problem where the usual gradient method does not converge. For even m , the zero vector is the solution of the Example 4.1. We take $\lambda = 0.7$ ($\lambda = 0.9$) and $\mu = 0.9$ for Algorithm A. For Algorithm 3.5 in [17], we take $\lambda = 0.7$ and $L = 1$. For all tests, we take $x_0 = (1, \dots, 1)$. The numerical results are showed in Table 2.

Tables 1 and 2 illustrate that algorithms may behave similarly to when we have knowledge of the Lipschitz constant.

5 Conclusions

In this paper, we consider a strong convergence result for monotone variational inequalities problem with Lipschitz continuous and monotone mapping in uniformly convex Banach spaces. Our algorithm is based on the subgradient extragradient methods with a new step size, the convergence of algorithm is established without the knowledge of the Lipschitz constant of the mapping. Our results extend the results of Yang and Liu in [8] from Hilbert spaces to uniformly convex Banach spaces which are also uniformly smooth and show strong convergence. Finally, a numerical experiment demonstrates the validity and advantage of the proposed method.

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The author worked on the results, and he read and approved the final manuscript.

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