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On a new extended half-discrete Hilbert's inequality involving partial sums

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Abstract

By applying the weight functions, the idea of introducing parameters, and Euler–Maclaurin summation formula, a new extended half-discrete Hilbert's inequality with the homogeneous kernel and the beta, gamma function is given. The equivalent statements of the best possible constant factor related to a few parameters are considered. As applications, a corollary about the case of the non-homogeneous kernel and some particular cases are obtained.

MSC: 26D15

Keywords: Weight function; Half-discrete Hilbert's inequality; Parameter; Euler–Maclaurin summation formula; Gamma function; Beta function

1 Introduction

If $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then we have the following discrete Hilbert's inequality with the best possible constant factor π (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2\right)^{1/2}. \tag{1}$$

Assuming that $0 < \int_0^{\infty} f^2(x)\,dx < \infty$ and $0 < \int_0^{\infty} g^2(y)\,dy < \infty$, we still have the following Hilbert's integral inequality (cf. [1], Theorem 316):

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\,dx\,dy < \pi\left(\int_0^{\infty} f^2(x)\,dx\int_0^{\infty} g^2(y)\,dy\right)^{1/2}, \tag{2}$$

where the constant factor π is the best possible. Inequalities (1) and (2) play an important role in the analysis and its applications (cf. [2–13]).

We still have the following half-discrete Hilbert-type inequality (cf. [1], Theorem 351): If $K(x)$ $(x > 0)$ is a decreasing function, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi(s) = \int_0^{\infty} K(x)x^{s-1}\,dx < \infty$, $f(x) \ge 0$, $0 < \int_0^{\infty} f^p(x)\,dx < \infty$, then

$$\sum_{n=1}^{\infty} n^{p-2}\left(\int_0^{\infty} K(nx)f(x)\,dx\right)^p < \phi^p\left(\frac{1}{q}\right)\int_0^{\infty} f^p(x)\,dx. \tag{3}$$

In recent years, some new extensions of (3) have been provided by [14–19].

In 2006, by using the Euler–Maclaurin summation formula, Krnic et al. [20] gave an extension of (1) with the kernel $\frac{1}{(m+n)^\lambda}$ $(0 < \lambda \le 4)$; and in 2019, according to the results of [20], Adiyasuren et al. [21] considered an extension of (1) involving the partial sums.

In 2016–2017, by applying the weight functions, Hong [22, 23] considered some equivalent statements of the extensions of (1) and (2) with a few parameters. Some similar interested works were provided by [24–26].

In this paper, according to the way of [21, 22], by the use of the weight functions, the idea of introducing parameters and the Euler–Maclaurin summation formula, a new extended half-discrete Hilbert's inequality with the homogeneous kernel $\frac{1}{(x+n)^\lambda}$ $(0 < \lambda \le 26)$ and the beta, gamma function is given. The equivalent statements of the best possible constant factor related to a few parameters are considered. As applications, a corollary about the case of non-homogeneous kernel and some particular cases are also obtained.

2 Some lemmas

In what follows, we assume that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in (-2, 26]$, $\lambda_2 \in (-1, 1]$, $\lambda_1, \lambda_2 \in (-1, \lambda + 1)$, $f(x) \ge 0$, $f \in L^1(R_+)$ $(R_+ = (0,\infty))$, $a_n \ge 0$ $(n \in \mathbb{N} = \{1, 2, \ldots\})$, $\{a_n\}_{n=1}^\infty \in l^1$,

$$F(x) := \int_0^x f(t)\,dt \quad (x \ge 0), \qquad A_n := \sum_{k=1}^n a_k \quad (n \in \mathbf{N})$$

such that

$$0 < \int_0^\infty x^{p[1-(\frac{\lambda+1-\lambda_2}{p}+\frac{\lambda_1+1}{q})]-1} F^p(x)\,dx < \infty \quad \text{and} \quad 0 < \sum_{n=1}^\infty n^{q[1-(\frac{\lambda_2+1}{p}+\frac{\lambda+1-\lambda_1}{q})]-1} A_n^q < \infty.$$

By the definition of the gamma function, for $\lambda, x > 0$, $n \in \mathbf{N}$, the following equality holds:

$$\frac{1}{(x+n)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-(x+n)t}\,dt. \tag{4}$$

Lemma 1 *For $t > 0$, we have*

$$\sum_{n=1}^\infty e^{-tn} a_n \le t \sum_{n=1}^\infty e^{-tn} A_n, \tag{5}$$

$$\int_0^\infty e^{-tx} f(x)\,dx = t \int_0^\infty e^{-tx} F(x)\,dx. \tag{6}$$

Proof Since $\{a_n\}_{n=1}^\infty \in l^1$, we find $\lim_{n\to\infty} A_n = \sum_{i=1}^\infty a_i \in [0,\infty)$. Using Abel's summation by parts formula and the inequality $1 - e^{-t} \le t$, we have (cf. [21])

$$\begin{aligned} \sum_{n=1}^\infty e^{-tn} a_n &= \lim_{n\to\infty} e^{-t(n+1)} A_n + \sum_{n=1}^\infty \big[e^{-tn} - e^{-t(n+1)}\big] A_n \\ &= \big(1 - e^{-t}\big) \sum_{n=1}^\infty e^{-tn} A_n \le t \sum_{n=1}^\infty e^{-tn} A_n, \end{aligned}$$

namely, inequality (5) follows. For $f \in L^1(R_+)$, $F(0) = 0$, $F(\infty) \in [0, \infty)$, we find

$$\int_0^\infty e^{-tx} f(x)\,dx = \int_0^\infty e^{-tx}\,dF(x) = e^{-tx}F(x)|_0^\infty - \int_0^\infty F(x)\,de^{-tx} = t\int_0^\infty e^{-tx}F(x)\,dx,$$

and then expression (6) follows. □

Lemma 2 *For* $1 < s \le 28$, $\sigma \in (0, 2] \cap (0, s)$, *define the following weight function*:

$$\varpi(\sigma, x) := x^{s-\sigma}\sum_{n=1}^\infty \frac{n^{\sigma-1}}{(x+n)^s} \quad (x \in \mathrm{R}_+). \tag{7}$$

We have the following inequality:

$$\varpi(\sigma, x) < B(\sigma, s-\sigma) \quad (x \in \mathrm{R}_+). \tag{8}$$

Proof We set function $g(t) := \frac{t^{\sigma-1}}{(x+t)^s}$ $(t > 0)$. Using the Euler–Maclaurin summation formula (cf. [20]), for $\rho(t) := t - [t] - \frac{1}{2}$, we have

$$\sum_{n=1}^\infty g(n) = \int_1^\infty g(t)\,dt + \frac{1}{2}g(1) + \int_1^\infty \rho(t)g'(t)\,dt = \int_0^\infty g(t)\,dt - h(\sigma, s),$$

$$h(\sigma, s) := \int_0^1 g(t)\,dt - \frac{1}{2}g(1) - \int_1^\infty \rho(t)g'(t)\,dt.$$

We obtain $-\frac{1}{2}g(1) = \frac{-1}{2(x+1)^s}$. Integrating by parts, it follows that

$$\begin{aligned}\int_0^1 g(t)\,dt &= \int_0^1 \frac{t^{\sigma-1}}{(x+t)^s}\,dt = \frac{1}{\sigma}\int_0^1 \frac{dt^\sigma}{(x+t)^s} = \frac{1}{\sigma}\frac{t^\sigma}{(x+t)^s}\bigg|_0^1 + \frac{s}{\sigma}\int_0^1 \frac{t^\sigma\,dt}{(x+t)^{s+1}} \\ &= \frac{1}{\sigma}\frac{1}{(x+1)^s} + \frac{s}{\sigma(\sigma+1)}\int_0^1 \frac{dt^{\sigma+1}}{(x+t)^{s+1}} \\ &> \frac{1}{\sigma}\frac{1}{(x+1)^s} + \frac{s}{\sigma(\sigma+1)}\left[\frac{t^{\sigma+1}}{(x+t)^{s+1}}\right]_0^1 + \frac{s(s+1)}{\sigma(\sigma+1)}\int_0^1 \frac{t^{\sigma+1}}{(x+1)^{s+2}}\,dt \\ &= \frac{1}{\sigma}\frac{1}{(x+1)^s} + \frac{s}{\sigma(\sigma+1)}\frac{1}{(x+1)^{s+1}} + \frac{s(s+1)}{\sigma(\sigma+1)(\sigma+2)}\frac{1}{(x+1)^{s+2}}.\end{aligned}$$

Since we find

$$\begin{aligned}-g'(t) &= -\frac{(\sigma-1)t^{\sigma-2}}{(x+t)^s} + \frac{st^{\sigma-1}}{(x+t)^{s+1}} = \frac{(1-\sigma)t^{\sigma-2}}{(x+t)^s} + \frac{st^{\sigma-2}}{(x+t)^s} - \frac{sxt^{\sigma-2}}{(x+t)^{s+1}} \\ &= \frac{(s+1-\sigma)t^{\sigma-2}}{(x+t)^s} - \frac{sxt^{\sigma-2}}{(x+t)^{s+1}},\end{aligned}$$

and for $0 < \sigma \le 2$, $1 < s \le 28$,

$$(-1)^i \frac{d^i}{dt^i}\left[\frac{t^{\sigma-2}}{(x+t)^s}\right] > 0, \qquad (-1)^i \frac{d^i}{dt^i}\left[\frac{t^{\sigma-2}}{(x+t)^{s+1}}\right] > 0 \quad (i = 0, 1, 2, 3),$$

still by the Euler–Maclaurin summation formula (cf. [20]), for $s+1-\sigma>0$, we have

$$\begin{aligned}
&(s+1-\sigma)\int_1^\infty \rho(t)\frac{t^{\sigma-2}}{(x+t)^s}\,dt > -\frac{s+1-\sigma}{12(x+1)^s},\\
&-xs\int_1^\infty \rho(t)\frac{t^{\sigma-2}}{(x+t)^{s+1}}\,dt\\
&\quad > \frac{xs}{12(x+1)^{s+1}} - \frac{xs}{720}\left[\frac{t^{\sigma-2}}{(x+t)^{s+1}}\right]''_{t=1}\\
&\quad > \frac{(x+1)s-s}{12(x+1)^{s+1}} - \frac{(x+1)s}{720}\left[\frac{(s+1)(s+2)}{(x+1)^{s+3}} + \frac{2(s+1)(2-\sigma)}{(x+1)^{s+2}} + \frac{(2-\sigma)(3-\sigma)}{(x+1)^{s+1}}\right]\\
&\quad = \frac{s}{12(x+1)^s} - \frac{s}{12(x+1)^{s+1}}\\
&\qquad - \frac{s}{720}\left[\frac{(s+1)(s+2)}{(x+1)^{s+2}} + \frac{2(s+1)(2-\sigma)}{(x+1)^{s+1}} + \frac{(2-\sigma)(3-\sigma)}{(x+1)^s}\right].
\end{aligned}$$

Hence, we have $h(\sigma,s) > \frac{h_1(\sigma,s)}{(x+1)^s} + \frac{sh_2(\sigma,s)}{(x+1)^{s+1}} + \frac{s(s+1)h_3(\sigma,s)}{(x+1)^{s+2}}$, where

$$\begin{aligned}
h_1(\sigma,s) &:= \frac{1}{\sigma} - \frac{1}{2} - \frac{1-\sigma}{12} - \frac{s(2-\sigma)(3-\sigma)}{720},\\
h_2(\sigma,s) &:= \frac{1}{\sigma(\sigma+1)} - \frac{1}{12} - \frac{(s+1)(2-\sigma)}{720},
\end{aligned}$$

and $h_3(\sigma,s) := \frac{1}{\sigma(\sigma+1)(\sigma+2)} - \frac{s+2}{720}$.

For $s\in(1,28]$, $\frac{s}{720}<\frac{1}{24}$, $\sigma\in(0,2]$, it follows that

$$h_1(\sigma,s) > \frac{1}{\sigma} - \frac{1}{2} - \frac{1-\sigma}{12} - \frac{(2-\sigma)(3-\sigma)}{24} = \frac{24-20\sigma+7\sigma^2-\sigma^3}{24\sigma} > 0.$$

In fact, setting $g(\sigma) := 24-20\sigma+7\sigma^2-\sigma^3 (\sigma\in(0,2])$, we obtain

$$g'(\sigma) = -20+14\sigma^2-3\sigma^2 = -3\left(\sigma-\frac{7}{3}\right)^2 - \frac{11}{3} < 0,$$

and then $g(\sigma)\geq g(2)=4>0$ $(\sigma\in(0,2])$.

We still find that $h_2(\sigma,s) > \frac{1}{6} - \frac{1}{12} - \frac{30}{360} = 0$ and $h_3(\sigma,s) \geq \frac{1}{24} - \frac{30}{720} = 0$. Hence, we have $h(\sigma,s)>0$, and then

$$\sum_{n=1}^\infty g(n) < \int_0^\infty g(t)\,dt = \int_0^\infty \frac{t^{\sigma-1}}{(x+t)^s}\,dt = x^{\sigma-s}\int_0^\infty \frac{u^{\sigma-1}}{(1+u)^s}\,du = x^{\sigma-s}B(\sigma,s-\sigma),$$

namely, (8) follows. □

Lemma 3 *Suppose that* $s\in(1,28]$, $\mu,\sigma\in(1,s)$, $\sigma\in(0,2]$,

$$0 < \int_0^\infty x^{p[1-(\frac{s-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x)\,dx < \infty \quad \textit{and} \quad 0 < \sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{s-\mu}{q})]-1} a_n^q < \infty.$$

We have the following inequality:

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^s}\,dx < B^{\frac{1}{p}}(\sigma, s-\sigma)B^{\frac{1}{q}}(\mu, s-\mu) \times \left\{\int_0^\infty x^{p[1-(\frac{s-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{s-\mu}{q})]-1} a_n^q\right\}^{\frac{1}{q}}. \quad (9)$$

Proof For $n \in \mathbf{N}$, setting $x = nu$, we obtain the following weight function:

$$\omega(\mu, n) := n^{s-\mu} \int_0^\infty \frac{x^{\mu-1}\,dx}{(x+n)^s} = \int_0^\infty \frac{u^{\mu-1}\,du}{(u+1)^s} = B(\mu, s-\mu). \quad (10)$$

By Hölder's inequality (cf. [27]), we obtain

$$\begin{aligned} &\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^s}\,dx \\ &= \int_0^\infty \sum_{n=1}^\infty \frac{1}{(x+n)^s}\left[\frac{n^{(\sigma-1)/p}}{x^{(\mu-1)/q}} f(x)\right]\left[\frac{x^{(\mu-1)/q}}{n^{(\sigma-1)/p}} a_n\right] dx \\ &\le \left\{\int_0^\infty \left[\sum_{n=1}^\infty \frac{1}{(x+n)^s} \frac{n^{\sigma-1}}{x^{(\mu-1)(p-1)}}\right] f^p(x)\,dx\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\sum_{n=1}^\infty \left[\int_0^\infty \frac{1}{(x+n)^s} \frac{x^{\mu-1}}{n^{(\sigma-1)(q-1)}}\,dx\right] a_n^q\right\}^{\frac{1}{q}} \\ &= \left\{\int_0^\infty \varpi(\sigma, x) x^{p[1-(\frac{s-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty \omega(\mu, n) n^{q[1-(\frac{\sigma}{p}+\frac{s-\mu}{q})]-1} a_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

Then, by (8) and (10), we have (9). □

Remark 1 For $s = \lambda + 2$, $\lambda \in (-1, 26]$, $\lambda_1 = \mu - 1 \in (0, \lambda+1)$, $\lambda_2 = \sigma - 1 \in (0, 1] \cap (0, \lambda+1)$, we can reduce (9) as follows:

$$\int_0^\infty \sum_{n=1}^\infty \frac{F(x)A_n}{(x+n)^{\lambda+2}}\,dx < B^{\frac{1}{p}}(\lambda_2+1, \lambda+1-\lambda_2)B^{\frac{1}{q}}(\lambda_1+1, \lambda+1-\lambda_1) \times \left\{\int_0^\infty x^{p[1-(\frac{\lambda+1-\lambda_2}{p}+\frac{\lambda_1+1}{q})]-1} F^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty n^{q[1-(\frac{\lambda_2+1}{p}+\frac{\lambda+1-\lambda_1}{q})]-1} A_n^q\right\}^{\frac{1}{q}}. \quad (11)$$

3 Main results

Theorem 1 *If* $\lambda \in (0, 26]$, $\lambda_1, \lambda_2 \in (0, \lambda + 1)$, $\lambda_2 \in (0, 1]$, *then we have the following inequality*:

$$\begin{aligned} I &:= \int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda}\,dx \\ &< \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} B^{\frac{1}{p}}(\lambda_2+1, \lambda+1-\lambda_2) B^{\frac{1}{q}}(\lambda_1+1, \lambda+1-\lambda_1) \\ &\quad \times \left\{\int_0^\infty x^{p[1-(\frac{\lambda+1-\lambda_2}{p}+\frac{\lambda_1+1}{q})]-1} F^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty n^{q[1-(\frac{\lambda_2+1}{p}+\frac{\lambda+1-\lambda_1}{q})]-1} A_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{12}$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, *we also have*

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda}\,dx < \lambda_1\lambda_2 B(\lambda_1, \lambda_2)\left(\int_0^\infty x^{-p\lambda_1-1}F^p(x)\,dx\right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{-q\lambda_2-1}A_n^q\right)^{\frac{1}{q}}, \tag{13}$$

where the constant factor $\lambda_1\lambda_2 B(\lambda_1, \lambda_2)$ *is the best possible.*

Proof Using (4), (5), and (6), we find

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \sum_{n=1}^\infty a_n f(x)\left(\int_0^\infty t^{\lambda-1}e^{-(x+n)t}\,dt\right)dx \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1}\left(\int_0^\infty e^{-xt}f(x)\,dx\right)\left(\sum_{n=1}^\infty e^{-nt}a_n\right)dt \\ &\le \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+1}\left(\int_0^\infty e^{-xt}F(x)\,dx\right)\left(\sum_{n=1}^\infty e^{-nt}A_n\right)dt \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \sum_{n=1}^\infty F(x)A_n\left(\int_0^\infty t^{\lambda+1}e^{-(x+n)t}\,dt\right)dx \\ &= \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} \int_0^\infty \sum_{n=1}^\infty \frac{F(x)A_n}{(x+n)^{\lambda+2}}\,dx. \end{aligned} \tag{14}$$

In view of (11), we have (12).

In the case of $\lambda_1 + \lambda_2 = \lambda$, we find

$$\begin{aligned} &\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} B^{\frac{1}{p}}(\lambda_2+1, \lambda+1-\lambda_2) B^{\frac{1}{q}}(\lambda_1+1, \lambda+1-\lambda_1) \\ &\quad = \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} B^{\frac{1}{p}}(\lambda_2+1, \lambda_1+1) B^{\frac{1}{q}}(\lambda_1+1, \lambda_2+1) = \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} B(\lambda_1+1, \lambda_2+1) \\ &\quad = \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} \frac{\Gamma(\lambda_1+1)\Gamma(\lambda_2+1)}{\Gamma(\lambda+2)} = \lambda_1\lambda_2 \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda)} = \lambda_1\lambda_2 B(\lambda_1, \lambda_2), \end{aligned}$$

and then (13) follows.

For any $0 < \varepsilon < \min\{p\lambda_1, q\lambda_2\}$, we set

$$\tilde{f}(t) := \begin{cases} 0, & 0 < t < 1, \\ t^{\lambda_1 - \frac{\varepsilon}{p} - 1}, & t \geq 1 \end{cases}, \qquad \tilde{a}_k := k^{\lambda_2 - \frac{\varepsilon}{q} - 1} \quad (k \in \mathbf{N}).$$

We obtain from $\lambda_1, \lambda_2 \in (0, \lambda + 1)$, $\lambda_2 \in (0, 1]$, and $0 < \varepsilon < \min\{p\lambda_1, q\lambda_2\}$ that $\tilde{F}(x) = 0$ $(0 < x < 1)$,

$$\tilde{F}(x) = \int_0^x \tilde{f}(t)\, dt = \int_1^x t^{\lambda_1 - \frac{\varepsilon}{p} - 1}\, dt \leq \frac{x^{\lambda_1 - \frac{\varepsilon}{p}}}{\lambda_1 - \frac{\varepsilon}{p}} \quad (x \geq 1),$$

$$\tilde{A}_n := \sum_{k=1}^{n} \tilde{a}_k = \sum_{k=1}^{n} k^{\lambda_2 - \frac{\varepsilon}{q} - 1} < \int_0^n t^{\lambda_2 - \frac{\varepsilon}{q} - 1}\, dt = \frac{n^{\lambda_2 - \frac{\varepsilon}{q}}}{\lambda_2 - \frac{\varepsilon}{q}} \quad (n \in \mathbf{N}).$$

If there exists a positive constant M $(M \leq \lambda_1 \lambda_2 B(\lambda_1, \lambda_2))$ such that (13) is valid when replacing $\lambda_1 \lambda_2 B(\lambda_1, \lambda_2)$ by M, then in particular, by substitution of $f(x) = \tilde{f}(x)$ and $a_n = \tilde{a}_n$, we have

$$\tilde{I} := \int_0^\infty \sum_{n=1}^\infty \frac{\tilde{f}(x)\tilde{a}_n}{(x+n)^\lambda}\, dx < M \left(\int_0^\infty x^{-p\lambda_1 - 1} \tilde{F}^p(x)\, dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{-q\lambda_2 - 1} \tilde{A}_n^q \right)^{\frac{1}{q}}.$$

We find

$$\begin{aligned}
\tilde{J} &:= \left(\int_0^\infty x^{-p\lambda_1 - 1} \tilde{F}^p(x)\, dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{-q\lambda_2 - 1} \tilde{A}_n^q \right)^{\frac{1}{q}} \\
&< \frac{1}{(\lambda_1 - \frac{\varepsilon}{p})(\lambda_2 - \frac{\varepsilon}{q})} \left[\int_1^\infty x^{-p\lambda_1 - 1} \left(x^{\lambda_1 - \frac{\varepsilon}{p}}\right)^p dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{-q\lambda_2 - 1} \left(n^{\lambda_2 - \frac{\varepsilon}{q}}\right)^q \right]^{\frac{1}{q}} \\
&= \frac{1}{(\lambda_1 - \frac{\varepsilon}{p})(\lambda_2 - \frac{\varepsilon}{q})} \left(\int_1^\infty x^{-\varepsilon - 1}\, dx \right)^{\frac{1}{p}} \left(1 + \sum_{n=2}^\infty n^{-\varepsilon - 1} \right)^{\frac{1}{q}} \\
&< \frac{1}{(\lambda_1 - \frac{\varepsilon}{p})(\lambda_2 - \frac{\varepsilon}{q})} \left(\int_1^\infty x^{-\varepsilon - 1}\, dx \right)^{\frac{1}{p}} \left(1 + \int_1^\infty t^{-\varepsilon - 1}\, dt \right)^{\frac{1}{q}} = \frac{(\varepsilon + 1)^{1/q}}{\varepsilon(\lambda_1 - \frac{\varepsilon}{p})(\lambda_2 - \frac{\varepsilon}{q})}.
\end{aligned}$$

In view of Fubini's theorem (cf. [28]), it follows that

$$\begin{aligned}
\tilde{I} &= \int_1^\infty \sum_{n=1}^\infty \frac{n^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(x+n)^\lambda} x^{\lambda_1 - \frac{\varepsilon}{p} - 1}\, dx \geq \int_1^\infty \left(\int_1^\infty \frac{t^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(x+t)^\lambda}\, dt \right) x^{\lambda_1 - \frac{\varepsilon}{p} - 1}\, dx \\
&= \int_1^\infty x^{-\varepsilon - 1} \int_{1/x}^\infty \frac{u^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+u)^\lambda}\, du\, dx \\
&= \int_1^\infty x^{-\varepsilon - 1} \int_{1/x}^1 \frac{u^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+u)^\lambda}\, du\, dx + \int_1^\infty x^{-\varepsilon - 1} \int_1^\infty \frac{u^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+u)^\lambda}\, du\, dx
\end{aligned}$$

$$= \int_0^1 \left(\int_{1/u}^\infty x^{-\varepsilon-1}\, dx \right) \frac{u^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+u)^\lambda}\, du + \frac{1}{\varepsilon} \int_1^\infty \frac{u^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+u)^\lambda}\, du$$
$$= \frac{1}{\varepsilon} \left[\int_0^1 \frac{u^{\lambda_2 + \frac{\varepsilon}{p} - 1}}{(1+u)^\lambda}\, du + \int_1^\infty \frac{u^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+u)^\lambda}\, du \right].$$

So we obtain

$$\int_0^1 \frac{u^{\lambda_2 + \frac{\varepsilon}{p} - 1}}{(1+u)^\lambda}\, du + \int_1^\infty \frac{u^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+u)^\lambda}\, du \le \varepsilon \tilde{I} < \varepsilon M \tilde{J} < \frac{M(\varepsilon+1)^{1/q}}{(\lambda_1 - \frac{\varepsilon}{p})(\lambda_2 - \frac{\varepsilon}{q})}.$$

For $\varepsilon \to 0^+$ in the above inequality, in view of the continuity of the beta function, we find $B(\lambda_1, \lambda_2) \le \frac{M}{\lambda_1 \lambda_2}$, namely $\lambda_1 \lambda_2 B(\lambda_1, \lambda_2) \le M$. Hence $M = \lambda_1 \lambda_2 B(\lambda_1, \lambda_2)$ is the best possible constant factor of (13). □

Remark 2 We set $\hat{\lambda}_1 := \frac{\lambda+1-\lambda_2}{p} + \frac{\lambda_1+1}{q} - 1$, $\hat{\lambda}_2 := \frac{\lambda_2+1}{p} + \frac{\lambda+1-\lambda_1}{q} - 1$. It follows that

$$\hat{\lambda}_1 + \hat{\lambda}_2 = \frac{\lambda + 1 - \lambda_2}{p} + \frac{\lambda_1 + 1}{q} - 1 + \frac{\lambda_2 + 1}{p} + \frac{\lambda + 1 - \lambda_1}{q} - 1 = \lambda,$$

$0 < \hat{\lambda}_1, \hat{\lambda}_2 < \lambda + 1$, and then we reduce (12) as follows:

$$\begin{aligned} I &:= \int_0^\infty \sum_{n=1}^\infty \frac{f(x) a_n}{(x+n)^\lambda}\, dx \\ &< \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} B^{\frac{1}{p}}(\lambda_2 + 1, \lambda + 1 - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + 1, \lambda + 1 - \lambda_1) \\ &\quad \times \left(\int_0^\infty x^{-p\hat{\lambda}_1 - 1} F^p(x)\, dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{-q\hat{\lambda}_2 - 1} A_n^q \right)^{\frac{1}{q}}. \end{aligned} \tag{15}$$

Theorem 2 *Assuming that* $\lambda \in (0, 26]$, $\lambda_1, \lambda_2 \in (0, \lambda + 1)$, $\lambda_2 \in (0, 1]$, *if the constant factor*

$$\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} B^{\frac{1}{p}}(\lambda_2 + 1, \lambda + 1 - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + 1, \lambda + 1 - \lambda_1)$$

in (15) *is the best possible, then* $\lambda_1 + \lambda_2 = \lambda$.

Proof As regards to the assumptions, we find $0 < \hat{\lambda}_1, \hat{\lambda}_2 < \lambda + 1$. By (13), the unified best possible constant factor in (15) must be of the following form:

$$\hat{\lambda}_1 \hat{\lambda}_2 B(\hat{\lambda}_1, \hat{\lambda}_2) \left(= \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} B(\hat{\lambda}_1 + 1, \hat{\lambda}_2 + 1) \right),$$

namely, it follows that

$$B(\hat{\lambda}_1 + 1, \hat{\lambda}_2 + 1) = B^{\frac{1}{p}}(\lambda_2 + 1, \lambda + 1 - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + 1, \lambda + 1 - \lambda_1).$$

By Hölder's inequality (cf. [27]), we obtain

$$\begin{aligned}
B(\hat{\lambda}_1+1,\hat{\lambda}_2+1) &= \int_0^\infty \frac{u^{(\hat{\lambda}_1+1)-1}}{(1+u)^{\lambda+2}}\,du = \int_0^\infty \frac{u^{\hat{\lambda}_1}}{(1+u)^{\lambda+2}}\,du \\
&= \int_0^\infty \frac{1}{(1+u)^{\lambda+2}} u^{\frac{\lambda+1-\lambda_2}{p}+\frac{\lambda_1+1}{q}-1}\,du = \int_0^\infty \frac{1}{(1+u)^{\lambda+2}}\bigl(u^{\frac{\lambda-\lambda_2}{p}}\bigr)\bigl(u^{\frac{\lambda_1}{q}}\bigr)\,du \\
&\leq \left\{\int_0^\infty \frac{u^{\lambda-\lambda_2}}{(1+u)^{\lambda+2}}\,du\right\}^{\frac{1}{p}} \left\{\int_0^\infty \frac{u^{\lambda_1}}{(1+u)^{\lambda+2}}\,du\right\}^{\frac{1}{q}} \\
&= B^{\frac{1}{p}}(\lambda_2+1,\lambda+1-\lambda_2)B^{\frac{1}{q}}(\lambda_1+1,\lambda+1-\lambda_1). \qquad (16)
\end{aligned}$$

We observe that (16) keeps the form of equality if and only if there exist constants A and B such that they are not all zero and $Au^{\lambda-\lambda_2} = Bu^{\lambda_1}$ a.e. in R_+. Assuming that $A \neq 0$, it follows that $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ a.e. in R_+, namely $\lambda - \lambda_2 - \lambda_1 = 0$, and then $\lambda_1 + \lambda_2 = \lambda$. □

Theorem 3 *If* $\lambda \in (0, 26]$, $\lambda_1, \lambda_2 \in (0, \lambda + 1)$, $\lambda_2 \in (0, 1]$, *then the following statements are equivalent*:

(i) $B^{\frac{1}{p}}(\lambda_2+1,\lambda+1-\lambda_2)B^{\frac{1}{q}}(\lambda_1+1,\lambda+1-\lambda_1)$ *is independent of* p, q;

(ii) $B^{\frac{1}{p}}(\lambda_2+1,\lambda+1-\lambda_2)B^{\frac{1}{q}}(\lambda_1+1,\lambda+1-\lambda_1)$ *is expressible as a single integral*;

(iii) $\lambda_1 + \lambda_2 = \lambda$;

(iv) *The constant factor*

$$\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}B^{\frac{1}{p}}(\lambda_2+1,\lambda+1-\lambda_2)B^{\frac{1}{q}}(\lambda_1+1,\lambda+1-\lambda_1)$$

in (12) *is the best possible*.

Proof (i) ⇒ (ii). We find

$$\begin{aligned}
&B^{\frac{1}{p}}(\lambda_2+1,\lambda+1-\lambda_2)B^{\frac{1}{q}}(\lambda_1+1,\lambda+1-\lambda_1) \\
&\quad= \lim_{p\to\infty}\lim_{q\to1^+} B^{\frac{1}{p}}(\lambda_2+1,\lambda+1-\lambda_2)B^{\frac{1}{q}}(\lambda_1+1,\lambda+1-\lambda_1) \\
&\quad= B(\lambda_1+1,\lambda+1-\lambda_1) = \int_0^\infty \frac{u^{\lambda_1}}{(1+u)^{\lambda+2}}\,du,
\end{aligned}$$

which is a single integral. (ii) ⇒ (iii). Suppose that $B^{\frac{1}{p}}(\lambda_2+1,\lambda+1-\lambda_2)B^{\frac{1}{q}}(\lambda_1+1,\lambda+1-\lambda_1)$ is expressible as a single integral $\int_0^\infty \frac{1}{(1+u)^{\lambda+2}} u^{\frac{\lambda+1-\lambda_2}{p}+\frac{\lambda_1+1}{q}-1}\,du$. Then (16) keeps the form of equality. By the proof of Theorem 2, we have $\lambda_1 + \lambda_2 = \lambda$. (iii) ⇒ (i). If $\lambda_1 + \lambda_2 = \lambda$, then

$$B^{\frac{1}{p}}(\lambda_2+1,\lambda+1-\lambda_2)B^{\frac{1}{q}}(\lambda_1+1,\lambda+1-\lambda_1) = B(\lambda_1+1,\lambda_2+1),$$

which is a single integral.

(iii) ⇒ (iv). By Theorem 1, for $\lambda_1 + \lambda_2 = \lambda$, the constant factor

$$\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}B^{\frac{1}{p}}(\lambda_2+1,\lambda+1-\lambda_2)B^{\frac{1}{q}}(\lambda_1+1,\lambda+1-\lambda_1) = \lambda_1\lambda_2 B(\lambda_1,\lambda_2)$$

in (12) is the best possible. (iv) ⇒ (iii). By Theorem 2, we have $\lambda_1 + \lambda_2 = \lambda$.

Hence, statements (i), (ii), (iii), and (iv) are equivalent. □

Remark 3 If $\mu + \sigma = s$,then inequality (9) reduces to

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^s}\,dx < B(\mu,\sigma)\left[\int_0^\infty x^{p(1-\mu)-1} f^p(x)\,dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \tag{17}$$

We confirm that the constant factor $B(\mu,\sigma)$ in (17) is the best possible. Otherwise, we would reach a contradiction by (14) that the constant factor in (13) is not the best possible.

4 A corollary and some particular cases

Replacing x by $\frac{1}{x}$ in (12), setting $g(x) = x^{\lambda-2} f(\frac{1}{x})$, we define

$$G_\lambda(x) := F(x) = \int_0^x f(t)\,dt = \int_{\frac{1}{x}}^\infty f\left(\frac{1}{u}\right)\frac{1}{u^2}\,du = \int_{\frac{1}{x}}^\infty t^{-\lambda} g(t)\,dt.$$

Then we obtain the following inequality with the non-homogeneous kernel:

$$\begin{aligned}&\int_0^\infty \sum_{n=1}^\infty \frac{g(x)a_n}{(1+xn)^\lambda}\,dx\\&\quad< \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} B^{\frac{1}{p}}(\lambda_2+1,\lambda+1-\lambda_2) B^{\frac{1}{q}}(\lambda_1+1,\lambda+1-\lambda_1)\\&\qquad\times\left\{\int_0^\infty x^{p[1-(\frac{\lambda+1-\lambda_2}{p}+\frac{\lambda_1+1}{q})]-1} G_\lambda^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^\infty n^{q[1-(\frac{\lambda_2+1}{p}+\frac{\lambda+1-\lambda_1}{q})]-1} A_n^q\right\}^{\frac{1}{q}}.\end{aligned} \tag{18}$$

It is obvious that inequality (18) is equivalent to (12).

In view of Theorem 3, we have the following.

Corollary 1 *Assuming that* $\lambda \in (0,26]$, $\lambda_1,\lambda_2 \in (0,\lambda+1)$, $\lambda_2 \in (0,1]$, *the constant factor*

$$\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} B^{\frac{1}{p}}(\lambda_2+1,\lambda+1-\lambda_2) B^{\frac{1}{q}}(\lambda_1+1,\lambda+1-\lambda_1)$$

in (18) *is the best possible if and only if* $\lambda_1 + \lambda_2 = \lambda$. *In the case of* $\lambda_1 + \lambda_2 = \lambda$, (18) *reduces to the following inequality with the best possible constant factor* $\lambda_1\lambda_2 B(\lambda_1,\lambda_2)$:

$$\begin{aligned}&\int_0^\infty \sum_{n=1}^\infty \frac{g(x)a_n}{(1+xn)^\lambda}\,dx\\&\quad< \lambda_1\lambda_2 B(\lambda_1,\lambda_2)\left(\int_0^\infty x^{-p\lambda_1-1} G_\lambda^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-q\lambda_2-1} A_n^q\right)^{\frac{1}{q}},\end{aligned} \tag{19}$$

which is equivalent to (13).

Remark 4 (i) In (13) and (19), for $0 < \lambda \le \min\{p, 26\}$, $\lambda_1 = \frac{\lambda}{q}$, $\lambda_2 = \frac{\lambda}{p}$ (≤ 1), we have the following equivalent inequalities:

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda}\,dx < \frac{\lambda^2}{pq}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left(\int_0^\infty x^{\lambda(1-p)-1}F^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{\lambda(1-q)-1}A_n^q\right)^{\frac{1}{q}}, \tag{20}$$

$$\int_0^\infty \sum_{n=1}^\infty \frac{g(x)a_n}{(1+xn)^\lambda}\,dx < \frac{\lambda^2}{pq}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left(\int_0^\infty x^{\lambda(1-p)-1}G_\lambda^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{\lambda(1-q)-1}A_n^q\right)^{\frac{1}{q}}; \tag{21}$$

if $0 < \lambda \le \min\{q, 26\}$, $\lambda_1 = \frac{\lambda}{p}$, $\lambda_2 = \frac{\lambda}{q}$ (≤ 1), then we have the following equivalent inequalities:

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda}\,dx < \frac{\lambda^2}{pq}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left(\int_0^\infty x^{-\lambda-1}F^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-\lambda-1}A_n^q\right)^{\frac{1}{q}}, \tag{22}$$

$$\int_0^\infty \sum_{n=1}^\infty \frac{g(x)a_n}{(1+xn)^\lambda}\,dx < \frac{\lambda^2}{pq}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left(\int_0^\infty x^{-\lambda-1}G_\lambda^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-\lambda-1}A_n^q\right)^{\frac{1}{q}}. \tag{23}$$

In particular, for $p = q = 2$, $0 < \lambda \le 2$, both inequalities (20) and (22) reduce to

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda}\,dx < \frac{\lambda^2}{4}B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\int_0^\infty x^{-\lambda-1}F^2(x)\,dx\sum_{n=1}^\infty n^{-\lambda-1}A_n^2\right)^{\frac{1}{2}}, \tag{24}$$

and both (21) and (23) reduce to the equivalent form of (24) as follows:

$$\int_0^\infty \sum_{n=1}^\infty \frac{g(x)a_n}{(1+xn)^\lambda}\,dx < \frac{\lambda^2}{4}B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\int_0^\infty x^{-\lambda-1}G_\lambda^2(x)\,dx\sum_{n=1}^\infty n^{-\lambda-1}A_n^2\right)^{\frac{1}{2}}. \tag{25}$$

(ii) In (13) and (19), for $\frac{1}{p} < \lambda \le 26$, $\lambda_1 = \lambda - \frac{1}{p}$, $\lambda_2 = \frac{1}{p}$ (< 1), we have the following equivalent inequalities:

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda}\,dx < \frac{p\lambda-1}{p^2}B\left(\frac{p\lambda-1}{p},\frac{1}{p}\right)\left(\int_0^\infty x^{-p\lambda}F^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-q}A_n^q\right)^{\frac{1}{q}}, \tag{26}$$

$$\int_0^\infty \sum_{n=1}^\infty \frac{g(x)a_n}{(1+xn)^\lambda}\,dx < \frac{p\lambda-1}{p^2}B\left(\frac{p\lambda-1}{p},\frac{1}{p}\right)\left(\int_0^\infty x^{-p\lambda}G_\lambda^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-q}A_n^q\right)^{\frac{1}{q}}; \tag{27}$$

if $\frac{1}{q} < \lambda \le 26$, $\lambda_1 = \lambda - \frac{1}{q}$, $\lambda_2 = \frac{1}{q}$ (< 1), then we have the following equivalent inequalities:

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda}\,dx < \frac{q\lambda-1}{q^2}B\left(\frac{q\lambda-1}{q},\frac{1}{q}\right)\left(\int_0^\infty x^{-2\lambda}F^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-2}A_n^q\right)^{\frac{1}{q}}, \tag{28}$$

$$\int_0^\infty \sum_{n=1}^\infty \frac{g(x)a_n}{(1+xn)^\lambda}\,dx < \frac{q\lambda-1}{q^2}B\left(\frac{q\lambda-1}{q},\frac{1}{q}\right)\left(\int_0^\infty x^{-2\lambda}G_\lambda^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-2}A_n^q\right)^{\frac{1}{q}}. \tag{29}$$

In particular, for $p = q = 2$, $\frac{1}{2} < \lambda \le 26$, both inequalities (26) and (28) reduce to

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda}\,dx < \frac{2\lambda-1}{4}B\left(\frac{2\lambda-1}{2},\frac{1}{2}\right)\left(\int_0^\infty x^{-2\lambda}F^2(x)\,dx\sum_{n=1}^\infty n^{-2}A_n^2\right)^{\frac{1}{2}}, \tag{30}$$

and both (27) and (29) reduce to the equivalent form of (30) as follows:

$$\int_0^\infty \sum_{n=1}^\infty \frac{g(x)a_n}{(1+xn)^\lambda}\,dx < \frac{2\lambda-1}{4}B\left(\frac{2\lambda-1}{2},\frac{1}{2}\right)\left(\int_0^\infty x^{-2\lambda}G_\lambda^2(x)\,dx\sum_{n=1}^\infty n^{-2}A_n^2\right)^{\frac{1}{2}}. \tag{31}$$

(iii) In (13) and (19), for $1 < \lambda \le 26$, $\lambda_1 = \lambda - 1$, $\lambda_2 = 1$, we have the following equivalent inequalities:

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda}\,dx < \left(\int_0^\infty x^{p(1-\lambda)-1}F^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-q-1}A_n^q\right)^{\frac{1}{q}}, \tag{32}$$

$$\int_0^\infty \sum_{n=1}^\infty \frac{g(x)a_n}{(1+xn)^\lambda}\,dx\left(\int_0^\infty x^{p(1-\lambda)-1}G_\lambda^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-q-1}A_n^q\right)^{\frac{1}{q}}; \tag{33}$$

if $1 < \lambda \le 2$, $\lambda_1 = 1$, $\lambda_2 = \lambda - 1$ (≤ 1), we have the following equivalent inequalities:

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda}\,dx < \left(\int_0^\infty x^{-p-1}F^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{q(1-\lambda)-1}A_n^q\right)^{\frac{1}{q}}, \tag{34}$$

$$\int_0^\infty \sum_{n=1}^\infty \frac{g(x)a_n}{(1+xn)^\lambda}\,dx\left(\int_0^\infty x^{-p-1}G_\lambda^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{q(1-\lambda)-1}A_n^q\right)^{\frac{1}{q}}. \tag{35}$$

In particular, for $\lambda = 2$, both (32) and (34) reduce to

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^2}\,dx < \left(\int_0^\infty x^{-p-1}F^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-q-1}A_n^q\right)^{\frac{1}{q}}, \tag{36}$$

both (33) and (35) reduce to the equivalent form of (36) as follows:

$$\int_0^\infty \sum_{n=1}^\infty \frac{g(x)a_n}{(1+xn)^2}\,dx\left(\int_0^\infty x^{-p-1}G_2^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-q-1}A_n^q\right)^{\frac{1}{q}}. \tag{37}$$

The constant factors in the above inequalities are the best possible.

5 Conclusions

In this paper, according to the way of [21, 22], by applying the weight functions, the idea of introduced parameters, and the Euler–Maclaurin summation formula, a new extended half-discrete Hilbert's inequality with the homogeneous kernel and the beta, gamma function is given in Theorem 1. The preliminaries are obtained in Theorem 2. The equivalent statements of the best possible constant factor related to some parameters are proved in Theorem 3. As applications, a corollary about the case of the non-homogeneous kernel and some particular cases are considered in Corollary 1 and Remark 4. The lemmas and theorems provide an extensive account of this type of inequalities.

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Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. XH and RL participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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