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On Cauchy–Schwarz inequality for N-tuple diamond-alpha integral

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Abstract

In this paper, we present some new Cauchy–Schwarz inequalities for *N*-tuple diamond-alpha integral on time scales. The obtained results improve and generalize some Cauchy–Schwarz type inequalities given by many authors.

MSC: 26D15

Keywords: Cauchy–Schwarz inequality; Diamond-alpha integral; Time scales

1 Introduction

To unify discrete and continuous analysis and generalize the discrete and continuous theories to cases "in between", Stefan Hilger in [1] initiated the notions of a time scale and a delta derivative of a function defined on the time scale. The author then presented the calculus on time scales. If the time scale is an interval, the calculus is reduced to the classical calculus; if the time scale is discrete, the calculus is reduced to the calculus of finite differences. Since then, research in the area of the theory of time scales introduced by Stefan Hilger has exceeded by far a thousand publications, and numerous applications to all branches of science, such as operations research, engineering, economics, physics, finance, statistics, and biology, have been proposed. For more details on time scales theory, the interested readers may consult [2–12] and the references therein.

As we all know, inequality plays a very important and basic role in all mathematic areas, and it is also an indispensable and basic tool in engineering technology (see [13–27]). The classic Cauchy–Schwarz inequality is an important cornerstone in some branches of mathematic areas. It is also a bridge to help solve problems into depth. In [28], Agarwal et al. first gave the following Cauchy–Schwarz inequality for Δ-integral on time scale.

Theorem A *Let $\mathbb{T}$ be a time scale, $t_1, t_2 \in \mathbb{T}$ with $t_1 < t_2$, and let $s, t \in C_{rd}([t_1, t_2], \mathbb{R})$. Then*

$$\int_{t_1}^{t_2} \left| s(x)t(x) \right| \Delta x \le \left(\int_{t_1}^{t_2} s^2(x)\Delta x \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} t^2(x)\Delta x \right)^{\frac{1}{2}}. \tag{1}$$

Later, Wong et al. [29] presented the extension of inequality (1).

Theorem B *Let $\mathbb{T}$ be a time scale, $t_1, t_2 \in \mathbb{T}$ and $t_1 < t_2$, and let $s, t, \lambda \in C_{rd}([t_1, t_2], \mathbb{R})$. Then*

$$\int_{t_1}^{t_2} |\lambda(x)| |s(x)t(x)| \Delta x \leq \left(\int_{t_1}^{t_2} |\lambda(x)| s^2(x) \Delta x \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |\lambda(x)| t^2(x) \Delta x \right)^{\frac{1}{2}}.$$

In 2008, Özkan et al. [30] introduced the time scale versions of (1) for the ∇-integral and $\diamond_\alpha$-integral, respectively.

Theorem C *Let $\mathbb{T}$ be a time scale, $t_1, t_2 \in \mathbb{T}$ and $t_1 < t_2$, and let $s, t, \lambda \in C_{ld}([t_1, t_2], \mathbb{R})$. Then*

$$\int_{t_1}^{t_2} |\lambda(x)| |s(x)t(x)| \nabla x \leq \left(\int_{t_1}^{t_2} |\lambda(x)| s^2(x) \nabla x \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |\lambda(x)| t^2(x) \nabla x \right)^{\frac{1}{2}}. \tag{2}$$

Theorem D *Let $\mathbb{T}$ be a time scale, $t_1, t_2 \in \mathbb{T}$ with $t_1 < t_2$, and let $s, t, \lambda : [t_1, t_2] \to \mathbb{R}$ be $\diamond_\alpha$-integrable functions. Then*

$$\int_{t_1}^{t_2} |\lambda(x)| |s(x)t(x)| \diamond_\alpha x \leq \left(\int_{t_1}^{t_2} |\lambda(x)| s^2(x) \diamond_\alpha x \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |\lambda(x)| t^2(x) \diamond_\alpha x \right)^{\frac{1}{2}}. \tag{3}$$

Remark 1.1 Taking $\alpha = 0$ in Theorem D, inequality (3) is reduced to inequality (2). Taking $\alpha = 1$ in Theorem D, inequality (3) is reduced to inequality (1).

In 2018, Tian [3] gave the triple diamond-alpha integral and proved that the Cauchy–Schwarz inequality holds for the triple diamond-alpha integral on the time scale.

Theorem E *Let $\lambda(x_1, x_2, x_3), s(x_1, x_2, x_3), t(x_1, x_2, x_3) : [a_i, b_i]^3_{\mathbb{T}} \to \mathbb{R}$ be $\diamond_\alpha$-integrable functions with $\lambda(x_1, x_2, x_3) \geq 0$. Then*

$$\begin{aligned} &\left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \lambda(x_1, x_2, x_3) s(x_1, x_2, x_3) t(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^2 \\ &\quad \leq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \lambda(x_1, x_2, x_3) s^2(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right) \\ &\qquad \times \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \lambda(x_1, x_2, x_3) t^2(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right). \end{aligned}$$

In 2019, Tian et al. [4] introduced the notion of n-tuple diamond-alpha integral for a function of n variables and established the Cauchy–Schwarz inequality for n-tuple diamond-alpha integral as follows.

Theorem F *Let $\lambda(\mathbf{x}), s(\mathbf{x}), t(\mathbf{x}) : [a_i, b_i]^n_{\mathbb{T}} \to \mathbb{R}$ be $\diamond_\alpha$-integrable functions with $\lambda(\mathbf{x}) \geq 0$. Then*

$$\begin{aligned} &\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) s(\mathbf{x}) t(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n \right)^2 \\ &\quad \leq \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) s^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n \right) \\ &\qquad \times \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) t^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n \right), \end{aligned} \tag{4}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Moreover, Yeh et al. in [31] presented some interesting complements of the Cauchy–Schwarz inequality via delta integral. Motivated by the above results, in the paper, by using methods similar to that in [31], we shall give some new variants, generalizations, and refinements of the Cauchy–Schwarz inequality for n-tuple diamond-alpha integral on time scales.

2 Main results

Throughout the paper, we use $\mathbb{T}$ to denote a time scale, denote $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $[a_i, b_i]_{\mathbb{T}}^n = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, where $x_i, a_i, b_i \in \mathbb{T}$ with $a_i < b_i$, $i = 1, 2, \dots, n$.

Proposition 2.1 ([4]) *Let $s(\mathbf{x})$, $t(\mathbf{x})$ be $\diamond_\alpha$-integrable functions on $[a_i, b_i]_{\mathbb{T}}^n$ $(i = 1, 2, \dots, n)$.*

(P1) *If $s(\mathbf{x}) \ge 0$ for $\mathbf{x} \in [a_i, b_i]_{\mathbb{T}}^n$, then*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} s(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n \ge 0.$$

(P2) *If $s(\mathbf{x}) \le t(\mathbf{x})$ for $\mathbf{x} \in [a_i, b_i]_{\mathbb{T}}^n$, then*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} s(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n \le \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} t(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n.$$

(P3) *If $f(\mathbf{x}) \ge 0$ for $\mathbf{x} \in [a_i, b_i]_{\mathbb{T}}^n$, then $s(\mathbf{x}) = 0$ if and only if*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} s(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n = 0.$$

Lemma 2.2 ((AG inequality) [32]) *Let $r_i > 0$ $(i = 1, 2, \dots, n)$, and let $\theta_1, \theta_2, \dots, \theta_n \in (1, +\infty)$ such that $\sum_{i=1}^n \frac{1}{\theta_i} = 1$. Then*

$$\prod_{i=1}^n r_i \le \sum_{i=1}^n \frac{r_i^{\theta_i}}{\theta_i}. \tag{5}$$

Remark 2.3 The Cauchy–Schwarz inequality for n-tuple diamond-alpha integral has the following variants.

(V1) Let $t(\mathbf{x}) > 0$, $p, q \in \mathbb{N}^+$, and let $s(\mathbf{x})$ and $t(\mathbf{x})$ be replaced by $s^p(\mathbf{x})/\sqrt{t^q(\mathbf{x})}$ and $\sqrt{t^q(\mathbf{x})}$ in (4), respectively. Then

$$\begin{aligned}\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) s^p(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right)^2 &\le \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \frac{h(\mathbf{x}) s^{2p}(\mathbf{x})}{t^q(\mathbf{x})} \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\ &\quad \times \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) t^q(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right).\end{aligned}$$

(V2) Let $s(\mathbf{x})$ and $t(\mathbf{x})$ be replaced by $(s^{2p}(\mathbf{x})+t^{2q}(\mathbf{x}))^{\frac{1}{2}}$ and $s^p(\mathbf{x})t^q(\mathbf{x})/(s^{2p}(\mathbf{x})+t^{2q}(\mathbf{x}))^{\frac{1}{2}}$ in (4), respectively, where $p,q\in\mathbb{N}^+$. Then we get

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^p(\mathbf{x})t^q(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\&\quad\le\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\big(s^{2p}(\mathbf{x})+t^{2q}(\mathbf{x})\big)\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\&\qquad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{\lambda(\mathbf{x})s^{2p}(\mathbf{x})t^{2q}(\mathbf{x})}{s^{2p}(\mathbf{x})+t^{2q}(\mathbf{x})}\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right).\end{aligned}$$

(V3) Let $s(\mathbf{x})$ and $t(\mathbf{x})$ be replaced by $\sqrt{t^q(\mathbf{x})/s^p(\mathbf{x})}$ and $\sqrt{s^p(\mathbf{x})t^q(\mathbf{x})}$ in (4), respectively, where $s^p(\mathbf{x})t^q(\mathbf{x})\ge 0$, $s(\mathbf{x})\ne 0$, and $p,q\in\mathbb{N}^+$. Then we get

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^q(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\&\quad\le\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{\lambda(\mathbf{x})t^q(\mathbf{x})}{s^p(\mathbf{x})}\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\&\qquad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^p(\mathbf{x})t^q(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right).\end{aligned}$$

Theorem 2.4 *Let $\lambda(\mathbf{x}),s(\mathbf{x}),t(\mathbf{x}):[a_i,b_i]^n_{\mathbb{T}}\to\mathbb{R}$ be $\diamond_\alpha$-integrable functions with $\lambda(\mathbf{x})\ge 0$, and let there be constants $m,M,h,H\in\mathbb{R}$ such that*

$$\big(Ms(\mathbf{x})-ht(\mathbf{x})\big)\big(Ht(\mathbf{x})-ms(\mathbf{x})\big)\ge 0. \tag{6}$$

Then

$$\begin{aligned}&mM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\quad+hH\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\le(mh+MH)\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\le|mh+MH|\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^{1/2}\\&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^{1/2}.\end{aligned}\tag{7}$$

Specially, if $Mm>0$, $Hh>0$, then

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\&\le\frac{(hm+HM)^2}{4hmHM}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2.\end{aligned}\tag{8}$$

Proof It is easy to find from (6) that

$$\lambda(\mathbf{x})\big(Ms(\mathbf{x}) - ht(\mathbf{x})\big)\big(Ht(\mathbf{x}) - ms(\mathbf{x})\big) \geq 0.$$

Then

$$\begin{aligned}&\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} MH\lambda(\mathbf{x})t(\mathbf{x})s(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\quad - \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} hH\lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\quad - mM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\quad + mh\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})t(\mathbf{x})s(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n \geq 0.\end{aligned} \tag{9}$$

From (9) and Cauchy–Schwarz inequality (4), we find that (7) holds.

Moreover, from $mM > 0$, $hH > 0$, and

$$\begin{aligned}&\Bigg[\bigg(hH\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\bigg)^{\frac{1}{2}}\\&\quad - \bigg(mM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\bigg)^{\frac{1}{2}}\Bigg]^2 \geq 0,\end{aligned} \tag{10}$$

we have

$$\begin{aligned}&hH\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\quad + mM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\quad \geq 2\bigg(Mm\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\bigg)^{1/2}\\&\qquad \times \bigg(Hh\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\bigg)^{1/2}.\end{aligned} \tag{11}$$

Therefore, by using (11) and (7), we find that (8) is valid. The proof of Theorem 2.10 is completed. □

Remark 2.5 Obviously, inequality (8) extends the result in [33].

Remark 2.6 Under the assumptions of Theorem 2.4, and letting $mM > 0$, $hH > 0$, $0 < q \leq p < 1$, and $p + q = 1$, from the AG inequality (5) and (7) we have

$$\begin{aligned}&\bigg(\frac{mM}{p}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\bigg)^p\\&\quad \cdot \bigg(\frac{hH}{q}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\bigg)^q\end{aligned}$$

$$\begin{aligned}&\leq mM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\quad+hH\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\leq(mh+MH)\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n,\end{aligned}$$

which implies that

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^q\\&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^p\\&\leq p^p(1-p)^{1-p}\frac{mh+MH}{(mM)^p(hH)^{1-p}}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n.\end{aligned}\tag{12}$$

Letting $p\to 1^-$ on the both sides of inequality (12), we find

$$\begin{aligned}&\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\leq\left(\frac{H}{m}+\frac{h}{M}\right)\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&=\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\frac{Ht(\mathbf{x})}{m}\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\quad+\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\frac{ht(\mathbf{x})}{M}\diamond_\alpha x_1\cdots\diamond_\alpha x_n.\end{aligned}$$

Letting $p\to 0^+$ on the both sides of inequality (12), we find

$$\begin{aligned}&\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\leq\left(\frac{m}{H}+\frac{M}{h}\right)\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n.\end{aligned}\tag{13}$$

Remark 2.7 Let $S(\mathbf{x}):[a_i,b_i]_{\mathbb{T}}^n\to(0,+\infty)$ be a $\diamond_\alpha$-integrable function. If $s(\mathbf{x})=S^{\frac{-1}{2}}(\mathbf{x})$, $t(\mathbf{x})=S^{\frac{1}{2}}(\mathbf{x})$, then inequality (7) reduces to

$$\begin{aligned}&hH\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})S(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n+mM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{\lambda(\mathbf{x})}{S(\mathbf{x})}\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\leq(mh+MH)\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n,\end{aligned}$$

which generalizes the result in [34].

Remark 2.8 Letting $\lambda(\mathbf{x}) = 1$ in (8), then inequality (8) reduces to

$$\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} s^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} t^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ \leq \frac{(hm+HM)^2}{4hmHM}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} s(\mathbf{x})t(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2,$$

which extends Pólya and Szegö's result [35].

Remark 2.9 Let $S(\mathbf{x}) : [a_i, b_i]_{\mathbb{T}}^n \to (0, +\infty)$ be a $\diamond_\alpha$-integrable function. If $s(\mathbf{x}) = S^{\frac{-1}{2}}(\mathbf{x})$, $t(\mathbf{x}) = S^{\frac{1}{2}}(\mathbf{x})$, then inequality (8) reduces to

$$\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \frac{\lambda(\mathbf{x})}{S(\mathbf{x})} \diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})S(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ \leq \frac{(hm+HM)^2}{4hmHM}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2,$$

which extends some results in [36, 37].

Remark 2.10 Letting $h = H = 1$ in (8), inequality (8) reduces to

$$\begin{aligned} &\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n \\ &\quad + mM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n \\ &\leq (m+M)\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})t(\mathbf{x})s(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n \\ &\leq |m+M|\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^{\frac{1}{2}} \\ &\quad \times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^{\frac{1}{2}}. \end{aligned} \tag{14}$$

Moreover, if $mM > 0$, then

$$\begin{aligned} &\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ &\quad \times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ &\leq \frac{(m+M)^2}{4mM}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x}) \diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2. \end{aligned} \tag{15}$$

The above inequalities (14) and (15) extend some results in [31].

Theorem 2.11 *Let $\lambda(\mathbf{x}), s(\mathbf{x}), t(\mathbf{x}) : [a_i, b_i]_{\mathbb{T}}^n \to [0, +\infty)$ be $\diamond_\alpha$-integrable functions, let there be constants $h, H, m, M, p, q > 0$ such that $m \leq t(\mathbf{x}) \leq M$, $h \leq s(\mathbf{x}) \leq H$, $0 < q \leq p < 1$, and*

$p + q = 1$. *Then*

$$\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^p\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^q \\ \leq \frac{pHM+qhm}{(hH)^q(mM)^p}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right). \tag{16}$$

Proof As $(pMs(\mathbf{x}) - hqt(\mathbf{x}))(ms(\mathbf{x}) - Ht(\mathbf{x})) \leq 0$, we have

$$pmMs^2(\mathbf{x}) - (pHM + qhm)s(\mathbf{x})t(\mathbf{x}) + qHht^2(\mathbf{x}) \leq 0.$$

Then

$$pmMs^2(\mathbf{x}) + qHht^2(\mathbf{x}) \leq (pHM + qhm)s(\mathbf{x})t(\mathbf{x}). \tag{17}$$

By the AG inequality (5) and (17), we find

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\right)^p\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^q \\ &= \frac{1}{(hH)^q(mM)^p}\left(mM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^p \\ &\quad\times\left(hH\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^q \\ &\leq \frac{1}{(hH)^q(mM)^p}\left(pmM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right. \\ &\quad\left.+\,qhH\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ &\leq \frac{pHM+qhm}{(hH)^q(mM)^p}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right),\end{aligned}$$

which implies (16) holds. □

Theorem 2.12 *Let $\lambda(\mathbf{x}), s(\mathbf{x}), t(\mathbf{x}) : [a_i, b_i]^n_{\mathbb{T}} \to [0, +\infty)$ be $\diamond_\alpha$-integrable functions.*

(i) *If there are constants $h, H, m, M \in \mathbb{R}$ such that $[Ht(\mathbf{x}) - ms(\mathbf{y})][Ms(\mathbf{y}) - ht(\mathbf{x})] \geq 0$ for all $\mathbf{x}, \mathbf{y} \in [a_i, b_i]^n_{\mathbb{T}}$, then*

$$\begin{aligned}&(hm+HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ &\qquad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ &\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ &\qquad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\end{aligned}$$

$$+ mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \times \left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right). \tag{18}$$

(ii) *If* $hH > 0$, $mM > 0$ *such that* $[Ht(\mathbf{x}) - ms(\mathbf{y})][Ms(\mathbf{y}) - ht(\mathbf{x})] \geq 0$ *for all* $\mathbf{x}, \mathbf{y} \in [a_i, b_i]_{\mathbb{T}}^n$, *and if* $p, q > 0$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\begin{aligned}
&(hm + HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\
&\quad\times \left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\
&\geq \left[\frac{hH}{p}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\right. \\
&\quad\left.\times \left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\right]^p \\
&\quad\times \left[\frac{mM}{q}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\right. \\
&\quad\left.\times \left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\right]^q.
\end{aligned} \tag{19}$$

(iii) *If* $mM > 0$, $hH > 0$ *such that* $[Ht(\mathbf{x}) - ms(\mathbf{x})][Ms(\mathbf{x}) - ht(\mathbf{x})] \geq 0$, *then*

$$\begin{aligned}
&(hm + HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\
&\quad\times \left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\
&\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2 \\
&\quad+ mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2.
\end{aligned}$$

(iv) *If* $mM > 0$, $hH > 0$ *such that* $[Ht(\mathbf{x}) - ms(\mathbf{y})][Ms(\mathbf{y}) - ht(\mathbf{x})] \geq 0$ *for all* $\mathbf{x}, \mathbf{y} \in [a_i, b_i]_{\mathbb{T}}^n$, *then*

$$\begin{aligned}
&(hm + HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\
&\quad\times \left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\
&\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2 \\
&\quad+ mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2.
\end{aligned}$$

Proof Case (i). From the assumption we find that

$$\lambda(\mathbf{x})\lambda(\mathbf{y})\big(Ht(\mathbf{x}) - ms(\mathbf{y})\big)\big(Ms(\mathbf{y}) - ht(\mathbf{x})\big) \geq 0,$$

which means that

$$\begin{aligned} &HM\lambda(\mathbf{x})\lambda(\mathbf{y})s(\mathbf{y})t(\mathbf{x}) + hm\lambda(\mathbf{x})\lambda(\mathbf{y})s(\mathbf{y})t(\mathbf{x}) \\ &\quad \geq hH\lambda(\mathbf{x})\lambda(\mathbf{y})t^2(\mathbf{x}) + mM\lambda(\mathbf{x})\lambda(\mathbf{y})s^2(\mathbf{x}). \end{aligned}$$

Therefore

$$\begin{aligned} &(hm + HM)\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})t(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\ &\quad \times \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{y})s(\mathbf{y}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\ &\quad \geq hH\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\ &\quad \times \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\ &\quad + mM\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\ &\quad \times \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right). \end{aligned}$$

Case (ii). From AG inequality (5) and Case (i), it is easy to find that (19) holds.

Case (iii). From Cauchy–Schwarz inequality (4) we find that

$$\begin{aligned} &\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right)\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\ &\quad \geq \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})s(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right)^2, \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right)\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\ &\quad \geq \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})t(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right)^2. \end{aligned} \tag{21}$$

Combining (7), (20), and (21), we have

$$\begin{aligned} &(hm + HM)\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\ &\quad \times \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \end{aligned}$$

$$
\begin{aligned}
&\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad + mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad \times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\quad + mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2.
\end{aligned}
$$

The proof of Case (iii) is completed.

Case (iv). Combining (18), (20), and (21), we have

$$
\begin{aligned}
&(hm+HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad \times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad \times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad + mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad \times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\quad + mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2,
\end{aligned}
$$

which implies that Case (iv) holds. □

Remark 2.13 From Case (i) of Theorem 2.12 we find that

$$
\begin{aligned}
&(hm+HM)^2\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\quad \times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\geq h^2H^2\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\quad \times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&+ m^2M^2\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&+ 2hmHM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
\ge{}& 4hmHM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2.
\end{aligned}
$$

Therefore, if $hmHM > 0$, we find

$$
\begin{aligned}
&(hm + HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
\ge{}& 2\sqrt{hmHM}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right).
\end{aligned}
$$

Similarly, from Case (iv) of Theorem 2.12 we have

$$
\begin{aligned}
&(hm + HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
\ge{}& 2\sqrt{hmHM}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right).
\end{aligned}
$$

By using methods similar to that in [31], we can prove the following theorem.

Theorem 2.14 *Let* $\lambda(\mathbf{x}), s(\mathbf{x}), t(\mathbf{x}) : [a_i, b_i]_{\mathbb{T}}^n \to [0, +\infty)$ *be* $\diamond_\alpha$*-integrable functions.*

Case (1).

$$\begin{aligned}
&\left[\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\right.\\
&\quad\left.+\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\right]\\
&\quad\times\left[\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\right.\\
&\quad\left.+\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\right]\\
&\geq\left[\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\right)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\right)\right.\\
&\quad\left.+\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\right)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\right)\right]^2.
\end{aligned}$$

Case (2). *If there are constants* $h, H, m, M \in \mathbb{R}$ *such that* $[Ht(\mathbf{x}) - ms(\mathbf{x})][Ms(\mathbf{x}) - ht(\mathbf{x})] \geq 0$, *then*

$$\begin{aligned}
&(hm+HM)^2\left[\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\right.\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad+\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad\left.\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\right]^2\\
&\geq 4hmHM\left[\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\right.\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad\left.+\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\right]\\
&\quad\times\left[\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\right.\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad\left.+\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\right]^2.
\end{aligned}$$

Case (3). *If there are constants* $h, H, m, M \in \mathbb{R}$ *such that* $[Ht(\mathbf{x}) - ms(\mathbf{y})][Ms(\mathbf{y}) - ht(\mathbf{x})] \geq 0$ *for all* $\mathbf{x}, \mathbf{y} \in [a_i, b_i]_{\mathbb{T}}^n$, *then*

$$
\begin{aligned}
1 \leq{}& \Biggl[\biggl(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\biggr)\biggl(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) s^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\biggr) \\
&+ \biggl(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) s(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\biggr)^2\Biggr] \\
&\times \Biggl[\biggl(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\biggr)\biggl(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) t^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\biggr) \\
&+ \biggl(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) t(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\biggr)^2\Biggr] \\
&\Big/ \Biggl[\biggl(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\biggr) \\
&\times \biggl(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) s(\mathbf{x}) t(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\biggr) \\
&+ \biggl(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) s(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\biggr) \\
&\times \biggl(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) t(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\biggr)\Biggr]^2 \\
\leq{}& \frac{(hm + HM)^2}{4hmHM}.
\end{aligned}
$$

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Authors' contributions
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References

1. Hilger, S.: Ein maßkettenkalkül mit anwendung auf zentrumsmannigfaltigkeiten. Ph.D. thesis, Universität Würzburg (1988)
2. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Basel (2001)
3. Tian, J.-F.: Triple diamond-alpha integral and Hölder-type inequalities. J. Inequal. Appl. **2018**, Article ID 111 (2018)

4. Tian, J.-F., Zhu, Y.-R., Cheung, W.-S.: N-tuple diamond-alpha integral and inequalities on time scales. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**(3), 2189–2200 (2019)
5. Hilger, S.: Analysis on measure chains—A unified approach to continuous and discrete calculus. Results Math. **18**, 18–56 (1990)
6. Hilger, S.: Differential and difference calculus—Unified! Nonlinear Anal. **30**, 2683–2694 (1997)
7. Boukerrioua, K., Diabi, D., Meziri, I.: New explicit bounds on Gamidov type integral inequalities on time scales and applications. J. Math. Inequal. **12**(3), 807–825 (2018)
8. Saker, S.H., Mahmoud, R.R.: A connection between weighted Hardy's inequality and half-linear dynamic equations. Adv. Differ. Equ. **2019**, Article ID 129 (2019)
9. Rogers, J.W.J., Sheng, Q.: Notes on the diamond-α dynamic derivative on time scales. J. Math. Anal. Appl. **326**, 228–241 (2007)
10. Sheng, Q., Fadag, M., Henderson, J., Davis, J.M.: An exploration of combined dynamic derivatives on time scales and their applications. Nonlinear Anal. **7**, 395–413 (2006)
11. Agarwal, R., O'Regan, D., Saker, S.: Dynamic Inequalities on Time Scales. Springer, Berlin (2014)
12. Liu, H., Meng, F.: Nonlinear retarded integral inequalities on time scales and their applications. J. Math. Inequal. **12**(1), 219–234 (2018)
13. Watanabe, K.: A Cauchy–Bunyakovsky–Schwarz type inequality related to the Möbius addition. J. Math. Inequal. **12**(4), 989–996 (2018)
14. Yang, Z.-H., Tian, J.-F.: Sharp bounds for the ratio of two zeta functions. J. Comput. Appl. Math. **364**, Article ID 112359 (2020)
15. Benge, P.: A two-weight inequality for essentially well localized operators with general measures. J. Math. Anal. Appl. **479**(2), 1506–1518 (2019)
16. Xi, B.-Y., Qi, F.: Inequalities of Hermite–Hadamard type for extended s-convex functions and applications to means. J. Nonlinear Convex Anal. **16**(5), 873–890 (2015)
17. Wang, M.K., Zhang, W., Chu, Y.M.: Monotonicity, convexity and inequalities involving the generalized elliptic integrals. Acta Math. Sci. Ser. B Engl. Ed. **39**(5), 1440–1450 (2019)
18. Wang, M.K., Li, Y.M., Chu, Y.M.: Inequalities and infinite product formula for Ramanujan generalized modular equation function. Ramanujan J. **46**(1), 189–200 (2018)
19. Wang, M.K., Chu, Y.M., Qiu, S.L., Jiang, Y.P.: Convexity of the complete elliptic integrals of the first kind with respect to Hölder means. J. Math. Anal. Appl. **388**(2), 1141–1146 (2012)
20. Yang, Z.-H., Tian, J.-F.: A class of completely mixed monotonic functions involving the gamma function with applications. Proc. Am. Math. Soc. **146**(11), 4707–4721 (2018)
21. Osaka, H., Tsurumi, Y., Wada, S.: Generalized reverse Cauchy inequality and applications to operator means. J. Math. Inequal. **12**(4), 1029–1039 (2018)
22. Xi, B.-Y., Qi, F.: Hermite–Hadamard type inequalities for geometrically r-convex functions. Studia Sci. Math. Hung. **51**(4), 530–546 (2014)
23. Tian, J.: Note on common fixed point theorems in fuzzy metric spaces using the CLRg property. Fuzzy Sets Syst. **379**, 134–137 (2020)
24. Zhao, C.-J., Cheung, W.: On improvements of Kantorovich type inequalities. Mathematics **7**, Article ID 259 (2019)
25. Fayyaz, T., Irshad, N., Khan, A.R., Rahman, G., Roqia, G.: Generalized integral inequalities on time scales. J. Inequal. Appl. **2016**, Article ID 235 (2016)
26. Nwaeze, E.R.: Time scale versions of the Ostrowski–Grüss type inequality with a parameter function. J. Math. Inequal. **12**(2), 531–543 (2018)
27. Nwaeze, E.R., Kermausuor, S., Tameru, A.M.: New time scale generalizations of the Ostrowski–Grüss type inequality for k points. J. Inequal. Appl. **2017**, Article ID 245 (2017)
28. Agarwal, R., Bohner, M., Peterson, A.: Inequalities on time scales: a survey. Math. Inequal. Appl. **4**(4), 535–557 (2001)
29. Wong, F.H., Yeh, C.C., Yu, S.L., Hong, C.H.: Young's inequality and related results on time scales. Appl. Math. Lett. **18**, 983–988 (2005)
30. Özkan, U.M., Sarikaya, M.Z., Yildirim, H.: Extensions of certain integral inequalities on time scales. Appl. Math. Lett. **21**, 993–1000 (2008)
31. Yeh, C.-C., Hong, H.-L., Li, H.-J., Yu, S.-L.: Some complements of Cauchy's inequality on time scales. J. Inequal. Appl. **2006**, Article ID 97430 (2006)
32. Mitrinović, D.S.: Analytic Inequalities. Springer, New York (1970)
33. Greub, W., Rheinboldt, W.: On a generalization of an inequality of L. V. Kantorovich. Proc. Am. Math. Soc. **10**, 407–415 (1959)
34. Rennie, B.C.: On a class of inequalities. J. Aust. Math. Soc. **3**, 442–448 (1963)
35. Pólya, G., Szegö, G.: Aufgaben und Lebrsätze aus der Analysis. Spinger, Berlin (1925)
36. Kantorovich, L.V.: Functional analysis and applied mathematics. Usp. Mat. Nauk **3**(6), 89–185 (1945) (Russian)
37. Schweitzer, P.: An inequality concerning the arithmetic mean. Mat. Fiz. Lapok **23**, 257–261 (1914) (Hungarian)