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Generalized Ponce's inequality

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Abstract

We provide a generalization of a remarkable inequality by A. C. Ponce whose consequences are essential in several fields, such as a characterization of Sobolev spaces or nonlocal modelization.

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1 Definitions and preliminaries

Let Ω be an open bounded set in $\mathbb{R}^N$. We define the family of kernels $(k_\delta)_{\delta>0}$ as a set of radial positive functions fulfilling the following properties:

(1)

$$\frac{1}{C_N}\int_{B(0,\delta)} k_\delta\bigl(|s|\bigr)\,ds = 1,$$

where

$$C_N = \frac{1}{\operatorname{meas}(S^{N-1})}\int_{S^{N-1}} |\sigma\cdot\mathbf{e}|^p\,d\mathcal{H}^{N-1}(\sigma),$$

$\mathcal{H}^{N-1}$ stands for the $(N-1)$-dimensional Hausdorff measure on the unit sphere S^{N-1}, $\mathbf{e}$ is any unit vector in $\mathbb{R}^N$, $p>1$, and $B(x,\delta)$ is the ball with center x and radius δ.

(2) $\operatorname{supp} k_\delta \subset B(0,\delta)$.

We define the nonlocal operator $\mathcal{B}_h$ in $L^p(\Omega)\times L^p(\Omega)$ by

$$\mathcal{B}_h(u,u) = \int_\Omega\int_\Omega H\bigl(x',x\bigr)\frac{k_\delta(|x'-x|)}{|x'-x|^p}\bigl|u\bigl(x'\bigr)-u(x)\bigr|^p\,dx'\,dx,$$

where $H(x',x) = \frac{h(x')+h(x)}{2}$, $h\in\mathcal{H}$,

$$\mathcal{H} \doteq \bigl\{h:\Omega\to\mathbb{R}\mid h(x)\in[h_{\min},h_{\max}] \text{ a.e. } x\in\Omega, h=0 \text{ in } \mathbb{R}^N\setminus\Omega\bigr\},$$

and $0<h_{\min}<h_{\max}$ are given constants.

For $h = 1$, the following compactness result is well known (see, e.g., [4] and [9, proof of Theorem 1.2, p. 12]).

Theorem 1 *Let $(u_\delta)_\delta$ be a sequence uniformly bounded in $L^p(\Omega)$, and let C be a positive constant such that*

$$\int_\Omega \int_\Omega \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \le C \tag{1.1}$$

for any δ. Then from $(u_\delta)_\delta$ we can extract a subsequence, still denoted by $(u_\delta)_\delta$, and we can find $u \in W^{1,p}(\Omega)$ such that $u_\delta \to u$ strongly in $L^p(\Omega)$ as $\delta \to 0$ and

$$\lim_{\delta\to 0} \int_\Omega \int_\Omega \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \ge \int_\Omega \left|\nabla u(x)\right|^p dx. \tag{1.2}$$

Even though several authors are involved in the proof, we refer to estimate (1.2) as Ponce's inequality.

1.1 The objective

Our goal is to prove the following extension of (1.2):

$$\lim_{\delta\to 0} \int_\Omega \int_\Omega H(x',x) \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \ge \int_\Omega h(x) \left|\nabla u(x)\right|^p dx, \tag{1.3}$$

where Ω is an open bounded set, $H(x',x) = \frac{h(x')+h(x)}{2}$, and $h \in \mathcal{H}$.

As we will see, inequality (1.3) is equivalent to (1.2) for measurable sets, that is,

$$\lim_{\delta\to 0} \int_E \int_E \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \ge \int_E \left|\nabla u(x)\right|^p dx \tag{1.4}$$

for all measurable sets E in Ω.

1.2 Motivation and organization of the paper

The context in which we locate the present paper is the study of the nonlocal p-Laplacian problem. Before proceeding, we make precise some notation. We define the spaces

$$L^p_0(\Omega_\delta) = \left\{u \in L^p(\Omega_\delta) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\right\}$$

and

$$X = \left\{u \in L^p_0(\Omega_\delta) : \mathcal{B}(u,u) < \infty\right\},$$

where

$$\Omega_\delta = \Omega \cup \left(\bigcup_{x\in\partial\Omega} B(x,\delta)\right),$$

$\mathcal{B} = \mathcal{B}_1$, and $\mathcal{B}_h$ is the operator defined in $X \times X$ by

$$\mathcal{B}_h(u,v) = \int_{\Omega_\delta} \int_{\Omega_\delta} H(x',x) \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u(x') - u(x)\right|^{p-2} \left(u(x') - u(x)\right)\left(v(x') - v(x)\right) dx'\,dx.$$

We define now the following nonlocal variational problem: given $f \in L^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$ and $p > 1$, find $u \in X$ such that

$$\mathcal{B}_h(u,w) = (f,w)_{L^{p'}(\Omega)\times L^p(\Omega)} \quad \text{in } X. \tag{1.5}$$

Note that (1.5) is equivalent to

$$\begin{aligned}\int_{\Omega_\delta}\int_{\Omega_\delta} H(x',x)k_\delta(|x'-x|)\frac{|u(x')-u(x)|^{p-2}(u(x')-u(x))(w(x')-w(x))}{|x'-x|^p}\,dx'\,dx \\ = \int_{\Omega_\delta} fw\,dx\end{aligned} \tag{1.6}$$

for all $w \in X$. Since the existence and uniqueness of solution for this problem is a well-known fact, for h fixed and any δ, there exists a solution u_δ. The aim is to check whether the sequence of solutions $(u_\delta)_\delta$ converges to the solution of the corresponding local p-Laplacian equation. This convergence (or G-convergence) clearly entails the study of the minimization principle

$$\min_{w\in X}\left\{\frac{1}{p}\mathcal{B}_h(w,w) - \int_\Omega f(x)w(x)\,dx\right\},$$

and, consequently, this task inevitably leads us to the study of the problem posed above; [1–3, 5] are some references where this type of convergence is analyzed.

The paper is organized by means of three sections containing different proofs of (1.3) and (1.4).

2 First proof

Our essential tool in to generalize (1.3) is a convenient Vitali covering of the set Ω (see [11, Chap. 4, Sect. 3, p. 109.] for details or [6, Chap. 2, Sect. 2, p. 26] for an elegant proof in the case of Lebesgue-measurable sets). Recall that the family $\{V_i\}_{i\in I}$ is a Vitali covering for $\Omega \subset \mathbb{R}^N$ if with any $x \in \Omega$ we can associate a number $\alpha > 0$, a sequence of V_i, and a sequence of balls $B(x,\epsilon_i)$ such that $V_i \subset B(x,\epsilon_i)$ and $|V_i| \geq \alpha|B(x,\epsilon_i)|$, where $\epsilon_i \to 0$ as $i \to \infty$.

Theorem 2 (Vitali covering theorem) *Let $\mathcal{A} = \{V_i\}_{k\in K}$ be a Vitali covering of closed subsets of $\mathbb{R}^N$ for Ω. There is a sequence of $(i_j)_j \in K$ such that $|\Omega \setminus \bigcup_j V_{i_j}| = 0$ and the sets $(V_{i_j})_j$ are pairwise disjoint.*

A particular and useful version of this chief result is the following:

Proposition 1 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, let K be a compact set included in Ω, and let ξ be a nonnegative function in $L^1(\Omega\times\Omega)$. Then there is a sequence of pairwise disjoint closed balls $(\overline{B}_i) \subset \Omega$ such that $|K \setminus \bigcup_{i=1}^\infty \overline{B}_i| = 0$ and*

$$\iint_{K\times K}\xi(x',x)\,dx'\,dx \geq \sum_{i=1}^\infty \iint_{\overline{B}_i\times\overline{B}_i}\xi(x',x)\,dx'\,dx.$$

Proof Since K is a compact inside Ω and Ω is open, we have $d \doteq \operatorname{dist}(K, \mathbb{R}^N \setminus \Omega) > 0$. In particular, any closed ball $\overline{B}_i = \overline{B(x,r)} \subset \Omega$ for any $r < d$. Moreover, the family $\mathcal{F} = \{\overline{B(x,s)} : x \in K, s < r/2\}$ is a Vitali covering of K, because every point of K is contained in an arbitrarily small ball belonging to $\mathcal{F}$. Consequently, there are disjoint balls $\overline{B}_i$ such that $|K \setminus \bigcup_{i=1}^{\infty} \overline{B}_i| = 0$. This covering also serves to approximate $K \times K$ because $|(K \times K) \setminus (\bigcup_{i,j=1}^{\infty} (\overline{B}_i \times \overline{B}_j))| = 0$, and therefore

$$\iint_{K\times K} \xi(x', x)\, dx'\, dx = \sum_{i,j=1}^{\infty} \iint_{\overline{B}_i \times \overline{B}_j} \xi(x', x)\, dx'\, dx \geq \sum_{i=1}^{\infty} \iint_{\overline{B}_i \times \overline{B}_i} \xi(x', x)\, dx'\, dx. \qquad \square$$

In a first step, we assume that h is continuous a.e. in Ω. We adapt [7, Lemma 7.9, p. 129] to prove our key result.

Proposition 2 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set such that $|\partial\Omega| = 0$, and let f be a positive a.e. continuous function on Ω. Let $r_k : \Omega \setminus N \to \mathbb{R}^+$ be a sequence of functions, where N is the set of discontinuity points of f. There exist a set of points $\{a_{ki}\}_i \subset \Omega \setminus N$ and positive numbers $\{\epsilon_{ki}\}_i$ such that for each k, $\epsilon_{ki} \leq r_k(a_{ki})$,*

$$\begin{aligned} &\{a_{ki} + \epsilon_{ki}\overline{\Omega}\} \quad \textit{are pairwise disjoint}, \\ &\overline{\Omega} = \bigcup_i \{a_{ki} + \epsilon_{ki}\overline{\Omega}\} \cup N_k, \quad \textit{where } |N_k| = 0, \end{aligned}$$

and

$$\int_\Omega f(x)\xi(x)\, dx = \sum_i f(a_{ki}) \int_{a_{ki}+\epsilon_{ki}\Omega} \xi(x)\, dx + o(1) \quad \textit{as } k \to +\infty \tag{2.1}$$

for all $\xi \in L^1(\Omega)$.

Proof Let $C = \Omega \setminus N$ be the set of points of continuity of f. We define the families

$$\mathcal{F}_k = \left\{ a + \epsilon\overline{\Omega} \subset \Omega : a \in C, \epsilon \leq r_k(a), \left|f(x) - f(a)\right| \leq \frac{1}{k} \text{ for any } x \in a + \epsilon\Omega \right\}.$$

For each fixed $k > 0$, the family $\mathcal{F}_k$ covers C (and Ω) in the sense of Vitali. Thus, Theorem 2 allows us to choose a numerable sequence of disjoints sets $\{a_{kj} + \epsilon_{kj}\overline{\Omega}\}_j \in \mathcal{F}_k$ such that $|\overline{\Omega} \setminus \bigcup_j \{a_{kj} + \epsilon_{kj}\overline{\Omega}\}| = 0$. Since f is continuous at a_{kj}, the sequence ϵ_{kj} can be chosen so that

$$\left|f(x) - f(a_{kj})\right| \leq \frac{1}{k} \quad \text{for any } x \in a_{kj} + \epsilon_{kj}\Omega \text{ and any } j.$$

Consequently,

$$\begin{aligned} &\left| \int_\Omega \xi(x) f(x)\, dx - \sum_j f(a_{kj}) \int_{a_{kj}+\epsilon_{kj}\Omega} \xi(x)\, dx \right| \\ &\quad = \left| \sum_j \int_{a_{kj}+\epsilon_{kj}\Omega} \big(f(x) - f(a_{kj})\big) \xi(x)\, dx \right| \end{aligned}$$

$$\leq \sum_j \int_{a_{kj}+\epsilon_{kj}\Omega} \left|\left(f(x)-f(a_{kj})\right)\right|\left|\xi(x)\right| dx$$

$$\leq \frac{1}{k}\sum_j \int_{a_{kj}+\epsilon_{kj}\Omega} \left|\xi(x)\right| dx$$

$$= \frac{1}{k}\|\xi\|_{L^1(\Omega)}. \qquad \square$$

2.1 Application

We apply the previous analysis to the integral

$$I = \int_\Omega \int_\Omega H\left(x',x\right)\xi_\delta\left(x',x\right) dx'\, dx,$$

where

$$\xi_\delta\left(x',x\right) = \frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta\left(x'\right)-u_\delta(x)\right|. \tag{2.2}$$

We consider $\Omega \times \Omega$ instead of Ω, and now $f(x',x)$ is the symmetric function $H(x',x) = \frac{h(x')+h(x)}{2}$ with $h \in \mathcal{H}$. We assume that h is continuous, and we take $\bigcup_{i,j}(a_{ki}+\epsilon_{ki}\Omega) \times (a_{kj}+\epsilon_{kj}\Omega)$, the union of a family of pairwise of disjoint sets covering $\Omega \times \Omega$. Then, according to the previous discussion, we trivially deduce

$$\begin{aligned}
I &= \sum_{i,j} H(a_{ki},a_{kj}) \int_{a_{ki}+\epsilon_{ki}\Omega}\int_{a_{kj}+\epsilon_{kj}\Omega} \frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta\left(x'\right)-u_\delta(x)\right|^p dx'\, dx + o(1) \\
&\geq \sum_i H(a_{ki},a_{ki}) \int_{a_{ki}+\epsilon_{ki}\Omega}\int_{a_{ki}+\epsilon_{ki}\Omega} \frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta\left(x'\right)-u_\delta(x)\right|^p dx'\, dx + o(1) \\
&= \sum_i h(a_{ki}) \int_{a_{ki}+\epsilon_{ki}\Omega}\int_{a_{ki}+\epsilon_{ki}\Omega} \frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta\left(x'\right)-u_\delta(x)\right|^p dx'\, dx + o(1).
\end{aligned}$$

We pass to the limit as $\delta \to 0$ in I: we use (1.1), Fatou's lemma and (1.2) for open sets to derive

$$\begin{aligned}
\liminf_{\delta\to 0} I &\geq \liminf_{\delta\to 0} \sum_i h(a_{ki}) \int_{a_{ki}+\epsilon_{ki}\Omega}\int_{a_{ki}+\epsilon_{ki}\Omega} \frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta\left(x'\right)-u_\delta(x)\right|^p dx'\, dx + o(1) \\
&\geq \sum_i h(a_{ki})\left(\liminf_{\delta\to 0} \int_{a_{ki}+\epsilon_{ki}\Omega}\int_{a_{ki}+\epsilon_{ki}\Omega} \frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta\left(x'\right)-u_\delta(x)\right|^p dx'\, dx\right) \\
&\quad + o(1) \\
&\geq \sum_i h(a_{ki})\left(\int_{a_{ki}+\epsilon_{ki}\Omega} \left|\nabla u(x)\right|^p dx\right) + o(1).
\end{aligned}$$

If we take limits as $k \to +\infty$, then this estimate gives

$$\liminf_{\delta\to 0} I \geq \lim_{k\to+\infty} \sum_i h(a_{ki}) \int_{a_{ki}+\epsilon_{ki}\Omega} \left|\nabla u(x)\right|^p dx.$$

By using again Proposition 2 the last inequality clearly provides inequality (1.3).

Remark 1 The analysis and conclusion we have just arrived at remain valid if we consider any open set $O \subset \Omega$ such that $|\partial O| = 0$. We can go a step further: we have

$$\liminf_{\delta\to 0} \int_O \int_O F(x',x) \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \geq \int_O F(x,x)\left|\nabla u(x)\right|^p dx \tag{2.3}$$

for any symmetric nonnegative continuous function $F \in L^\infty(O \times O)$.

2.2 Extension to the case of measurable functions

Let now h be just measurable; without loss of generality, $\operatorname{supp} H \subset \Omega \times \Omega$ and $H = 0$ otherwise. By Luzin's theorem (see [10, Theorem 2.24, p. 62]), given arbitrary $\epsilon > 0$, there exists a continuous function $G \in C_c(\Omega \times \Omega)$ such that $\sup G(x,y) \leq \sup H(x,y)$ and $G(x,y) = H(x,y)$ for any $(x,y) \in (\Omega \times \Omega) \setminus \mathcal{E}$, where $\mathcal{E}$ is a measurable set such that $|\mathcal{E}| < \epsilon^2$. Since H is symmetric, we can assume that $(\Omega \times \Omega) \setminus \mathcal{E} = (\Omega \setminus E) \times (\Omega \setminus E)$, where $E \subset \Omega$ is a measurable set such that $|E| < \epsilon$.

At this stage, we consider any compact set $K \subset \Omega \setminus E \subset \Omega$. Since Ω is open, we can use Proposition 1: there is a number $r > 0$ such that the family $\mathcal{F} = \{\overline{B(x,s)} : x \in K, s < r/2\}$ is a Vitali covering of K, and therefore there exists a sequence of pairwise disjoint closed balls $(\overline{B}_i)_{i=1}^\infty \subset \mathcal{F}$ such that $|K \setminus \bigcup_{i=1}^\infty \overline{B}_i|$, $\overline{B}_i \subset \Omega$, and

$$\begin{aligned}
&\int_\Omega \int_\Omega H(x',x) \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \\
&\quad\geq \int_{\Omega\setminus E} \int_{\Omega\setminus E} H(x',x) \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \\
&\quad\geq \iint_{(\Omega\setminus E)\times(\Omega\setminus E)} G(x',x) \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \\
&\quad\geq \iint_{K\times K} G(x',x) \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \\
&\quad\geq \sum_i \iint_{\overline{B}_i\times\overline{B}_i} G(x',x) \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx.
\end{aligned}$$

We take the limits as $\delta \to 0$ to get

$$\begin{aligned}
&\liminf_{\delta\to 0} \int_\Omega \int_\Omega H(x',x) \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \\
&\quad\geq \liminf_{\delta\to 0} \sum_i \iint_{\overline{B}_i\times\overline{B}_i} G(x',x) \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \\
&\quad\geq \sum_i \int_{B_i} G(x,x)\left|\nabla u(x)\right|^p dx \\
&\quad= \int_K G(x,x)\left|\nabla u(x)\right|^p dx,
\end{aligned}$$

where the second inequality is true because of (2.3) and Fatou's lemma. Then, since K is any compact set in $\Omega \setminus E$, we obtain

$$\liminf_{\delta\to 0} \int_\Omega \int_\Omega H(x',x) \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx$$

$$\begin{aligned}
&\geq \int_{\Omega\setminus E} G(x,x)\left|\nabla u(x)\right|^p dx \\
&= \int_{\Omega\setminus E} H(x,x)\left|\nabla u(x)\right|^p dx \\
&= \int_{\Omega} h(x)\left|\nabla u(x)\right|^p dx - \int_E h(x)\left|\nabla u(x)\right|^p dx.
\end{aligned}$$

By letting $\epsilon \downarrow 0$ and using $|E| \leq \epsilon$, we obtain (1.3), that is,

$$\liminf_{\delta\to 0} \int_\Omega \int_\Omega H(x',x)\frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \geq \int_\Omega H(x,x)\left|\nabla u(x)\right|^p dx. \quad (2.4)$$

Finally, to circumvent the assumption $|\partial\Omega| = 0$, the procedure we follow is identical to that just employed. Take any compact set K included in Ω. Since Ω is assumed to be open, thanks to Proposition 1, K can be exhaustively covered by the union of a numerable sequence of pairwise disjoint closed balls $\overline{B}_i \in \mathcal{F} = \{\overline{B(x,s)} : x \in K, s < r/2\} \subset \Omega$, $i = 1, 2, \ldots$. Then we realize that

$$\begin{aligned}
&\int_\Omega \int_\Omega H(x',x)\frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \\
&\quad\geq \sum_i \int_{B_i}\int_{B_i} H(x',x)\frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx. \quad (2.5)
\end{aligned}$$

By taking into account that $|\partial B_i| = 0$ we can apply (2.3) and Fatou's lemma in (2.5) to obtain

$$\liminf_{\delta\to 0} \int_\Omega \int_\Omega H(x',x)\frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \geq \int_K H(x,x)\left|\nabla u(x)\right|^p dx.$$

Since $K \subset \Omega$ is arbitrary, we arrive at (2.4) for any open set Ω,

$$\begin{aligned}
&\liminf_{\delta\to 0} \int_\Omega \int_\Omega H(x',x)\frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \\
&\quad\geq \int_\Omega H(x,x)\left|\nabla u(x)\right|^p dx. \quad (2.6)
\end{aligned}$$

2.3 Corollary

We apply (2.4) to the case $F(x',x) = I_{G\times G}(x',x)$, where G is any measurable set included in Ω: on the one hand, (2.6) guarantees

$$\begin{aligned}
&\liminf_{\delta\to 0} \int_\Omega \int_\Omega F(x',x)\frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \\
&\quad\geq \int_\Omega F(x,x)\left|\nabla u(x)\right|^p dx \\
&\quad= \int_G I_G(x)\left|\nabla u(x)\right|^p dx = \int_G \left|\nabla u(x)\right|^p dx,
\end{aligned}$$

and, on the other hand, it is obvious that

$$\int_\Omega \int_\Omega F(x',x)\frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx$$

$$= \int_G \int_G \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx.$$

Consequently, (1.4) is proved for any measurable set $G \subset \Omega$.

3 A second proof

We firstly prove (1.4) and then (1.3). By having a look at the work done in the previous section we will be able to provide a straightforward proof of (1.4). Indeed, if E is a measurable set included in Ω, then we can find a compact set $K \subset E$ such that $|E \setminus K|$ is arbitrarily small. Proposition 1 ensures the existence of a numerable sequence of pairwise disjoint balls $\overline{B}_i \in \mathcal{F}$ such that $|K \setminus \bigcup_{i=1}^\infty \overline{B}_i| = 0$, $\overline{B}_i \subset \Omega$ for any i and

$$\begin{aligned}
&\int_E \int_E \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \\
&\quad\geq \int_K \int_K \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \\
&\quad\geq \sum_i \int_{B_i} \int_{B_i} \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx.
\end{aligned}$$

We apply (1.2) for open sets and Fatou's lemma in the last chain of inequalities to derive

$$\liminf_{\delta\to 0} \int_E \int_E \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \geq \sum_i \int_{B_i} \left|\nabla u(x)\right|^p dx = \int_K \left|\nabla u(x)\right|^p dx.$$

Since $K \subset E$ is arbitrary, we arrive at (1.4), that is,

$$\liminf_{\delta\to 0} \int_E \int_E \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|u_\delta(x') - u_\delta(x)\right|^p dx'\,dx \geq \int_E \left|\nabla u(x)\right|^p dx. \tag{3.1}$$

3.1 Corollary

We prove (1.3). Let h be a given simple function defined in Ω. Then h can be written as $h(x) = \sum_{i=1}^m h_i I_{B_i}(x)$, where $\{B_i\}$ is a finite covering of disjoint measurable subsets of Ω, and $(h_i)_i$ is a set of numbers such that $h_{\min} \leq h_i \leq h_{\max}$. Consequently, we can easily check that

$$\begin{aligned}
I &\doteq \int_\Omega \int_\Omega H(x',x) k_\delta\left(|x'-x|\right) \frac{|u_\delta(x') - u_\delta(x)|^p}{|x'-x|^p}\,dx'\,dx \\
&\geq \sum_{i=1}^m h_i \int_{B_i} \int_{B_i} k_\delta\left(|x'-x|\right) \frac{|u_\delta(x') - u_\delta(x)|^p}{|x'-x|^p}\,dx'\,dx.
\end{aligned}$$

Using inequality (1.4) for measurable sets that we have just proved, we straightforwardly infer

$$\liminf_{\delta\to 0} I \geq \sum_{i=1}^m h_i \int_{B_i} \left|\nabla u(x)\right|^p dx = \int_\Omega h(x) \left|\nabla u(x)\right|^p dx.$$

Let h be a measurable function. By recalling that any measurable function h can be pointwise approximated by an increasing sequence $(s_n)_n$ of simple functions we can write

$$\liminf_{\delta\to 0} \int_\Omega \int_\Omega H(x',x) k_\delta\left(|x'-x|\right) \frac{|u_\delta(x') - u_\delta(x)|^p}{|x'-x|^p}\,dx'\,dx$$

$$
\begin{aligned}
&= \liminf_{\delta\to 0} \int_\Omega h(x) \int_\Omega k_\delta\big(|x'-x|\big) \frac{|u_\delta(x')-u_\delta(x)|^p}{|x'-x|^p}\,dx'\,dx \\
&\geq \liminf_{\delta\to 0} \int_\Omega s_n(x) \int_{\Omega_\delta} k_\delta\big(|x'-x|\big) \frac{|u_\delta(x')-u_\delta(x)|^p}{|x'-x|^p}\,dx'\,dx \\
&\geq \int_\Omega s_n(x)\big|\nabla u(x)\big|^p\,dx.
\end{aligned}
$$

It suffices to take the limits in n and apply the monotone convergence theorem to establish (1.3).

4 A third proof

The idea is reproducing the arguments from [9]. In a first step, we assume that $h:\overline{\Omega}\to[h_{\min},h_{\max}]$ is a continuous function. Moreover, without loss of generality, we suppose that h is a continuous function in the set $\Omega_s=\Omega\cup\{\bigcup_{p\in\partial\Omega}B(p,s)\}$, where s is a fixed positive number.

Now, for the proof of (1.3), the key idea is extending the Stein inequality (see [8, Lemma 4, p. 245]) in the following sense: by using Jensen's inequality and performing a change of variables we deduce the inequality

$$
\begin{aligned}
&\int_\Omega\int_\Omega H_r(x',x)k_\delta\big(|x'-x|\big)\frac{|u_\delta(x')-u_\delta(x)|^p}{|x'-x|^p}\,dx'\,dx \\
&\quad\geq \int_{\Omega_{-r}}\int_{\Omega_{-r}} H(x',x)k_\delta\big(|x'-x|\big)\frac{|u_{r,\delta}(x')-u_{r,\delta}(x)|^p}{|x'-x|^p}\,dx'\,dx
\end{aligned}
$$

for any $\delta<r$, where $u_{r,\delta}=\eta_r*u_\delta$, $\eta_r(x)=\frac{1}{r^N}\eta(\frac{x}{r})$, $x\in\mathbb{R}^N$, η is a nonnegative radial function from $C_c^\infty(B(0,1))$ such that $\int\eta(x)\,dx=1$,

$$
H_r(x',x)=\frac{(\eta_r*h)(x')+(\eta_r*h)(x)}{2},
$$

and $\Omega_{-r}=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>r\}$. By the continuity of H in $\Omega_s\times\Omega_s$ we know that $H_r(x',x)\to H(x',x)$ uniformly on compact sets of $\Omega_s\times\Omega_s$, whereby, for any $\epsilon>0$, we can choose $r_0>0$ such that

$$
\left|\int_\Omega\int_\Omega\big(H(x',x)-H_r(x',x)\big)k_\delta\big(|x'-x|\big)\frac{|u_\delta(x')-u_\delta(x)|^p}{|x'-x|^p}\,dx'\,dx\right|\leq\epsilon C
$$

for any $r<r_0$ and uniformly in $\delta>0$. Then

$$
\begin{aligned}
&\lim_{\delta\to 0}\int_\Omega\int_\Omega H(x',x)k_\delta\big(|x'-x|\big)\frac{|u_\delta(x')-u_\delta(x)|^p}{|x'-x|^p}\,dx'\,dx \\
&\quad\geq\lim_{\delta\to 0}\int_{\Omega_{-r}}\int_{\Omega_{-r}} H(x',x)k_\delta\big(|x'-x|\big)\frac{|u_{r,\delta}(x')-u_{r,\delta}(x)|^p}{|x'-x|^p}\,dx'\,dx-\epsilon C
\end{aligned}
$$

for any $r<r_0$. At this point, we notice that Proposition 1 from [8, p. 242] can be modified by including the term $H(x',x)$ within the integrand; this is factually what Remark 1 establishes. Then passing to the limit as $\delta\to 0$ and using the convergence of $\rho_r*u_\delta\to\rho_r*u$ in $C^2(\overline{\Omega}_{-r})$, we get

$$
\lim_{\delta\to 0}\int_{\Omega_{-r}}\int_{\Omega_{-r}} H(x',x)k_\delta\big(|x'-x|\big)\frac{|u_{r,\delta}(x')-u_{r,\delta}(x)|^p}{|x'-x|^p}\,dx'\,dx
$$

$$\geq \int_{\Omega_{-r}} h(x)\left|\nabla(\rho_r * u)(x)\right|^p \,dx'\,dx.$$

Consequently, letting $r \to 0$ in this inequality and taking into account that $\nabla(\rho_r * u)$ strongly converges to ∇u in $L^p(\Omega)$, we derive

$$\lim_{\delta\to 0}\int_\Omega\int_\Omega H\bigl(x',x\bigr)k_\delta\bigl(\left|x'-x\right|\bigr)\frac{|u_\delta(x')-u_\delta(x)|^p}{|x'-x|^p}\,dx'\,dx \geq \int_\Omega h(x)\left|\nabla u(x)\right|^p \,dx'\,dx - \epsilon C.$$

Now, since ϵ is arbitrarily small, the statement is proved under the assumption that h is continuous in Ω_s.

If $h:\Omega\to[h_{\min},h_{\max}]$ is a measurable function, then we extend it by zero to Ω_s and then apply Luzin's theorem to this extended function. The remaining details follow along the lines of Sect. 2.2.

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