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# On a more accurate Hilbert-type inequality in the whole plane with the general homogeneous kernel

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**Abstract**

By the use of the weight coefficients, the idea of introduced parameters and the technique of real analysis, a more accurate Hilbert-type inequality in the whole plane with the general homogeneous kernel is given, which is an extension of the more accurate Hardy–Hilbert's inequality. An equivalent form is obtained. The equivalent statements of the best possible constant factor related to several parameters, the operator expressions and a few particular cases are considered.

**MSC:** 26D15

**Keywords:** Weight coefficient; More accurate Hilbert-type inequality; Equivalent form; Equivalent statement; Parameter; Operator expression

## 1 Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have the following more accurate Hardy–Hilbert's inequality with the best possible constant $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 323):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-1} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

For $p = q = 2$, inequality (1) reduces to the more accurate Hilbert's inequality. Since $\frac{1}{m+n} < \frac{1}{m+n-1}$, we still have the following Hardy–Hilbert's inequality (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{2}$$

Assuming that $f(x), g(y) \geq 0, 0 < \int_0^{\infty} f^p(x)\,dx < \infty$ and $0 < \int_0^{\infty} g^q(y)\,dy < \infty$, we have the following integral analogue of (2), namely Hardy–Hilbert's integral inequality:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\,dx\,dy$$

$$< \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)\,dy\right)^{\frac{1}{q}}, \tag{3}$$

with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 316).

By introducing an independent parameter $\lambda > 0$, Yang [2, 3] gave an extension of (2) (for $p = q = 2$) with the kernel $\frac{1}{(x+y)^\lambda}$ and the best possible constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ ($B(u,v) := \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\,dt$ $(u, v > 0)$ is the beta function) in 1998. Inequalities (1), (2) and (3) play an important role in analysis and its applications (cf. [4–15]).

The following half-discrete Hilbert-type inequality was provided in 1934 (cf. [1], Theorem 351): If $K(x)$ $(x > 0)$ is decreasing, $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \phi(s) = \int_0^\infty K(x)x^{s-1}\,dx < \infty$, $a_n \geq 0, 0 < \sum_{n=1}^\infty a_n^p < \infty$, then

$$\int_0^\infty x^{p-2}\left(\sum_{n=1}^\infty K(nx)a_n\right)^p dx < \phi^p\left(\frac{1}{q}\right)\sum_{n=1}^\infty a_n^p. \tag{4}$$

Some new extensions and applications of (4) were obtained in recent years [16–21]. In 2016, by the use of the technique of real analysis, Hong et al. [22] provided some equivalent statements of the extensions of (1) with the best possible constant factor related to several parameters. Other results about the extensions of (1)–(4) were given by [23–37].

In this paper, following the approach of [22], by means of the weight coefficients, the idea of introduced parameters and the technique of real analysis, a more accurate Hilbert-type inequality in the whole plane is given as follows: for $r > 1, \frac{1}{r} + \frac{1}{s} = 1$,

$$\begin{aligned}&\sum_{|n|=1}^\infty\sum_{|m|=1}^\infty \frac{a_m b_n}{|m-\frac{1}{2}| + |n-\frac{1}{2}|}\\ &\quad < \frac{2\pi}{\sin(\pi/r)}\left(\sum_{|m|=1}^\infty \left|m-\frac{1}{2}\right|^{\frac{p}{s}-1} a_m^p\right)^{\frac{1}{p}}\left(\sum_{|n|=1}^\infty \left|n-\frac{1}{2}\right|^{\frac{q}{r}-1} b_n^q\right)^{\frac{1}{q}},\end{aligned} \tag{5}$$

which is an extension of (1). The general form of (4), as well as an equivalent form, is obtained. The equivalent statements of the best possible constant factor related to several parameters, the operator expressions and a few particular cases are considered.

## 2 Some lemmas

In what follows, we suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, -\frac{1}{2} \leq \xi, \eta \leq \frac{1}{2}, -1 < \alpha, \beta < 1, \lambda, \lambda_1, \lambda_2 \in \mathrm{R} = (-\infty, \infty), d := \lambda - \lambda_1 - \lambda_2$, $k_\lambda(x, y)$ $(\geq 0)$ is a homogeneous function of degree $-\lambda$, satisfying

$$k_\lambda(ux, uy) = u^{-\lambda}k_\lambda(x, y) \quad (u, x, y > 0),$$

$k_\lambda(x, y)x^{\lambda_1-1}$ (resp. $k_\lambda(x, y)y^{\lambda_2-1}$) is strictly decreasing and strictly convex with respect to $x > 0$ (resp. $y > 0$), such that $(-1)^i \frac{\partial^i}{\partial x^i}(k_\lambda(x, y)x^{\lambda_1-1}) > 0, (-1)^i \frac{\partial^i}{\partial y^i}(k_\lambda(x, y)y^{\lambda_2-1}) > 0$ $(x, y > 0; i = 1, 2)$, and

$$k_\lambda(\gamma) := \int_0^\infty k_\lambda(1, u)u^{\gamma-1}\,du \in \mathrm{R}_+ = (0, \infty) \quad (\gamma = \lambda_2, \lambda - \lambda_1).$$

We still assume that $a_m, b_n \geq 0$ ($|m|, |n| \in \mathrm{N} = \{1, 2, \ldots\}$), satisfy

$$0 < \sum_{|m|=1}^{\infty} \big[|m - \xi| + \alpha(m - \xi)\big]^{p(1-\lambda_1)-d-1} a_m^p < \infty,$$

$$0 < \sum_{|n|=1}^{\infty} \big[|n - \eta| + \beta(n - \eta)\big]^{q(1-\lambda_2)-d-1} b_n^q < \infty,$$

where, $\sum_{|j|=1}^{\infty} = \cdots = \sum_{j=-1}^{-\infty} + \cdots + \sum_{j=1}^{\infty} \cdots$ ($j = m, n$).

**Lemma 1** *For any $\gamma > 0$, we have the following inequalities*:

$$\begin{aligned}
&\frac{(1 + |\xi|)^{-\chi}}{\gamma}\big[(1 - \alpha)^{-\gamma-1} + (1 + \alpha)^{-\gamma-1}\big] \\
&\quad < \sum_{|m|=1}^{\infty} \big[|m - \xi| + \alpha(m - \xi)\big]^{-\gamma-1} \\
&\quad < \frac{1}{\gamma}\big[(1 - \alpha)^{-\gamma-1} + (1 + \alpha)^{-\gamma-1}\big]\big[\gamma\big(1 - |\xi|\big)^{-\gamma-1} + 1\big].
\end{aligned} \tag{6}$$

*Proof* Since $(-1)^i \frac{d^i}{dt^i}\frac{1}{(t-|\xi|)^{\gamma+1}} > 0$ ($t > \frac{3}{2}; i = 1, 2$), for $\frac{3}{2} \geq 1 + |\xi|$, by Hermite–Hadamard's inequality (cf. [38]) and using the decreasing property of series, we find

$$\begin{aligned}
&\sum_{|m|=1}^{\infty} \big[|m - \xi| + \alpha(m - \xi)\big]^{-\gamma-1} \\
&\quad = \sum_{m=-1}^{-\infty} \big[(1 - \alpha)(\xi - m)\big]^{-\gamma-1} + \sum_{m=1}^{\infty} \big[(1 + \alpha)(m - \xi)\big]^{-\gamma-1} \\
&\quad = \sum_{m=1}^{\infty} \big[(1 - \alpha)(m + \xi)\big]^{-\gamma-1} + \sum_{m=1}^{\infty} \big[(1 + \alpha)(m - \xi)\big]^{-\gamma-1} \\
&\quad \leq \big[(1 - \alpha)^{-\gamma-1} + (1 + \alpha)^{-\gamma-1}\big]\left[\big(1 - |\xi|\big)^{-\gamma-1} + \sum_{m=2}^{\infty}\big(m - |\xi|\big)^{-\gamma-1}\right] \\
&\quad < \big[(1 - \alpha)^{-\gamma-1} + (1 + \alpha)^{-\gamma-1}\big]\left[\big(1 - |\xi|\big)^{-\gamma-1} + \int_{\frac{3}{2}}^{\infty}\big(x - |\xi|\big)^{-\gamma-1}\,dx\right] \\
&\quad \leq \big[(1 - \alpha)^{-\gamma-1} + (1 + \alpha)^{-\gamma-1}\big]\left[\big(1 - |\xi|\big)^{-\gamma-1} + \int_{1+|\xi|}^{\infty}\big(x - |\xi|\big)^{-\gamma-1}\,dx\right] \\
&\quad = \frac{1}{\gamma}\big[(1 - \alpha)^{-\gamma-1} + (1 + \alpha)^{-\gamma-1}\big]\big[\gamma\big(1 - |\xi|\big)^{-\gamma-1} + 1\big],
\end{aligned}$$

$$\begin{aligned}
&\sum_{|m|=1}^{\infty} \big[|m - \xi| + \alpha(m - \xi)\big]^{-\gamma-1} \\
&\quad \geq \big[(1 - \alpha)^{-\gamma-1} + (1 + \alpha)^{-\gamma-1}\big]\sum_{m=1}^{\infty}\big(m + |\xi|\big)^{-\gamma-1} \\
&\quad > \big[(1 - \alpha)^{-\gamma-1} + (1 + \alpha)^{-\gamma-1}\big]\int_{1}^{\infty}\big(x + |\xi|\big)^{-\gamma-1}\,dx
\end{aligned}$$

$$= \frac{(1+|\xi|)^{-\gamma}}{\gamma}\big[(1-\alpha)^{-\gamma-1} + (1+\alpha)^{-\gamma-1}\big],$$

and then we have (6).

The lemma is proved. □

**Definition 1** We set

$$k_{\xi,\eta}(m,n) := k_\lambda\big(|m-\xi| + \alpha(m-\xi), |n-\eta| + \beta(n-\eta)\big) \quad \big(|m|,|n| \in \mathrm{N}\big),$$

and define the following weight coefficients:

$$\omega(\lambda_2,m) := \big[|m-\xi| + \alpha(m-\xi)\big]^{\lambda-\lambda_2} \sum_{|n|=1}^{\infty} k_{\xi,\eta}(m,n)\big[|n-\eta| + \beta(n-\eta)\big]^{\lambda_2-1} \quad \big(|m| \in \mathrm{N}\big), \tag{7}$$

$$\varpi(\lambda_1,n) := \big[|n-\eta| + \beta(n-\eta)\big]^{\lambda-\lambda_1} \sum_{|m|=1}^{\infty} k_{\xi,\eta}(m,n)\big[|m-\xi| + \alpha(m-\xi)\big]^{\lambda_1-1} \quad \big(|n| \in \mathrm{N}\big). \tag{8}$$

**Lemma 2** *The following inequalities are valid*:

$$\omega(\lambda_2,m) < \frac{2}{1-\beta^2}k_\lambda(\lambda_2) \quad \big(|m| \in \mathrm{N}\big), \tag{9}$$

$$\varpi(\lambda_1,n) < \frac{2}{1-\alpha^2}k_\lambda(\lambda-\lambda_1) \quad \big(|n| \in \mathrm{N}\big). \tag{10}$$

*Proof* For fixed $|m| \in \mathrm{N}$, we set

$$k^{(1)}(m,y) := k_\lambda\big(|m-\xi| + \alpha(m-\xi), (1-\beta)(\eta-y)\big), \quad y < \eta,$$
$$k^{(2)}(m,y) := k_\lambda\big(|m-\xi| + \alpha(m-\xi), (1+\beta)(y-\eta)\big), \quad y > \eta,$$

where from for $y > -\eta, k^{(1)}(m,-y) = k_\lambda(|m-\xi| + \alpha(m-\xi), (1-\beta)(y+\eta))$. We find

$$\begin{aligned}\omega(\lambda_2,m) &= \big[|m-\xi| + \alpha(m-\xi)\big]^{\lambda-\lambda_2} \\ &\quad \times \left\{\sum_{n=-1}^{-\infty} k^{(1)}(m,n)\big[(1-\beta)(\eta-n)\big]^{\lambda_2-1} + \sum_{n=1}^{\infty} k^{(2)}(m,n)\big[(1+\beta)(n-\eta)\big]^{\lambda_2-1}\right\} \\ &= \big[|m-\xi| + \alpha(m-\xi)\big]^{\lambda-\lambda_2} \\ &\quad \times \left[(1-\beta)^{\lambda_2-1}\sum_{n=1}^{\infty} k^{(1)}(m,-n)(n+\eta)^{\lambda_2-1} \right. \\ &\quad \left. + (1+\beta)^{\lambda_2-1}\sum_{n=1}^{\infty} k^{(2)}(m,n)(n-\eta)^{\lambda_2-1}\right].\end{aligned}$$

In view of the assumptions, $k^{(1)}(m,-y)(y+\eta)^{\lambda_2-1}$ (resp. $k^{(2)}(m,y)(y-\eta)^{\lambda_2-1}$) is strictly decreasing and strictly convex with respect to $y\in(-\eta,\infty)$ (resp. $y\in(\eta,\infty)$). By Hermite–Hadamard's inequality and using the decreasing property of series, for $\frac{1}{2}\geq\pm\eta$, we obtain

$$\begin{aligned}
&\omega(\lambda_2,m)\\
&\quad<\bigl[|m-\xi|+\alpha(m-\xi)\bigr]^{\lambda-\lambda_2}\biggl[(1-\beta)^{\lambda_2-1}\int_{\frac{1}{2}}^{\infty}k^{(1)}(m,-y)(y+\eta)^{\lambda_2-1}\,dy\\
&\qquad+(1+\beta)^{\lambda_2-1}\int_{\frac{1}{2}}^{\infty}k^{(2)}(m,y)(y-\eta)^{\lambda_2-1}\,dy\biggr]\\
&\quad\leq\bigl[|m-\xi|+\alpha(m-\xi)\bigr]^{\lambda-\lambda_2}\\
&\qquad\times\biggl[(1-\beta)^{\lambda_2-1}\int_{-\eta}^{\infty}k_\lambda\bigl(|m-\xi|+\alpha(m-\xi),(1-\beta)(y+\eta)\bigr)(y+\eta)^{\lambda_2-1}\,dy\\
&\qquad+(1+\beta)^{\lambda_2-1}\int_{\eta}^{\infty}k_\lambda\bigl(|m-\xi|+\alpha(m-\xi),(1+\beta)(y-\eta)\bigr)(y-\eta)^{\lambda_2-1}\,dy\biggr],
\end{aligned}\tag{11}$$

$$\begin{aligned}
&\omega(\lambda_2,m)\\
&\quad>\bigl[|m-\xi|+\alpha(m-\xi)\bigr]^{\lambda-\lambda_2}\\
&\qquad\times\biggl[(1-\beta)^{\lambda_2-1}\int_{1}^{\infty}k_\lambda\bigl(|m-\xi|+\alpha(m-\xi),(1-\beta)(y+\eta)\bigr)(y+\eta)^{\lambda_2-1}\,dy\\
&\qquad+(1+\beta)^{\lambda_2-1}\int_{1}^{\infty}k_\lambda\bigl(|m-\xi|+\alpha(m-\xi),(1+\beta)(y-\eta)\bigr)(y-\eta)^{\lambda_2-1}\,dy\biggr].
\end{aligned}\tag{12}$$

Setting $u=\frac{(1-\beta)(y+\eta)}{|m-\xi|+\alpha(m+\xi)}$ (resp. $u=\frac{(1+\beta)(y-\eta)}{|m-\xi|+\alpha(m+\xi)}$) in the first (resp. second) integral of (11), we obtain

$$\omega(\lambda_2,m)<\bigl[(1-\beta)^{-1}+(1+\beta)^{-1}\bigr]\int_0^{\infty}k_\lambda(1,u)u^{\lambda_2-1}\,du=\frac{2k_\lambda(\lambda_2)}{1-\beta^2}.$$

Hence, we have (9).

In the same way, setting $v=\frac{1}{u}$, we obtain

$$\varpi(\lambda_1,n)<\frac{2}{1-\alpha^2}\int_0^{\infty}k_\lambda(u,1)u^{\lambda_1-1}\,du=\frac{2}{1-\alpha^2}\int_0^{\infty}k_\lambda(1,v)v^{(\lambda-\lambda_1)-1}\,dv=\frac{2k_\lambda(\lambda-\lambda_1)}{1-\alpha^2},$$

and then (10) follows.

The lemma is proved. □

**Lemma 3** *If $\lambda_1+\lambda_2=\lambda$ (or $d=0$), then for any $\varepsilon>0$, we have*

$$\begin{aligned}
\tilde{H}&:=\sum_{|m|=1}^{\infty}\omega\biggl(\lambda_2-\frac{\varepsilon}{q},m\biggr)\bigl[|m-\xi|+\alpha(m-\xi)\bigr]^{-\varepsilon-1}\\
&>\frac{4}{\varepsilon2^{\varepsilon}(1-\beta^2)(1-\alpha^2)}\biggl(\int_0^2k_\lambda(1,u)u^{\lambda_2+\frac{\varepsilon}{p}-1}\,du+\int_2^{\infty}k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}\,du\biggr).
\end{aligned}\tag{13}$$

*Proof* By (7) (for $\lambda_1 + \lambda_2 = \lambda$) and (12), replacing $\lambda_2$ (resp. $\lambda_1$) by $\lambda_2 - \frac{\varepsilon}{q}$ (resp. $\lambda_1 + \frac{\varepsilon}{q}$), we have

$$\begin{aligned}
&\omega\left(\lambda_2 - \frac{\varepsilon}{q}, m\right) \\
&\quad > \left[|m-\xi| + \alpha(m-\xi)\right]^{\lambda_1 + \frac{\varepsilon}{q}} \\
&\quad\quad \times \left[(1-\beta)^{(\lambda_2 - \frac{\varepsilon}{q})-1} \int_1^\infty k_\lambda\big(|m-\xi| + \alpha(m-\xi), (1-\beta)(y+\eta)\big)(y+\eta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\,dy\right. \\
&\quad\quad \left. + (1+\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1} \int_1^\infty k_\lambda\big(|m-\xi| + \alpha(m-\xi), (1+\beta)(y-\eta)\big)(y-\eta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\,dy\right].
\end{aligned}$$

Then we find

$$\begin{aligned}
\tilde{H} &> \sum_{|m|=1}^\infty \left[|m-\xi| + \alpha(m-\xi)\right]^{(\lambda_1-\frac{\varepsilon}{p})-1} (1-\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1} \\
&\quad \times \int_1^\infty k_\lambda\big(|m-\xi| + \alpha(m-\xi), (1-\beta)(y+\eta)\big)(y+\eta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\,dy \\
&\quad + \sum_{|m|=1}^\infty \left[|m-\xi| + \alpha(m-\xi)\right]^{(\lambda_1-\frac{\varepsilon}{p})-1} (1+\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1} \\
&\quad \times \int_1^\infty k_\lambda\big(|m-\xi| + \alpha(m-\xi), (1+\beta)(y-\eta)\big)(y-\eta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\,dy \\
&= (1-\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1} \int_1^\infty \sum_{|m|=1}^\infty k_\lambda\big(|m-\xi| + \alpha(m-\xi), (1-\beta)(y+\eta)\big) \\
&\quad \times \left[|m-\xi| + \alpha(m-\xi)\right]^{(\lambda_1-\frac{\varepsilon}{p})-1} (y+\eta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\,dy \\
&\quad + (1+\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1} \int_1^\infty \sum_{|m|=1}^\infty k_\lambda\big(|m-\xi| + \alpha(m-\xi), (1+\beta)(y-\eta)\big) \\
&\quad \times \left[|m-\xi| + \alpha(m-\xi)\right]^{(\lambda_1-\frac{\varepsilon}{p})-1} (y-\eta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\,dy = \sum_{i=1}^4 H_i,
\end{aligned}$$

where we denote

$$\begin{aligned}
H_1 &:= (1-\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1}(1-\alpha)^{(\lambda_1-\frac{\varepsilon}{p})-1} \\
&\quad \times \int_1^\infty \sum_{m=1}^\infty k_\lambda\big((1-\alpha)(m+\xi), (1-\beta)(y+\eta)\big)(m+\xi)^{(\lambda_1-\frac{\varepsilon}{p})-1}(y+\eta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\,dy, \\
H_2 &:= (1-\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1}(1+\alpha)^{(\lambda_1-\frac{\varepsilon}{p})-1} \\
&\quad \times \int_1^\infty \sum_{m=1}^\infty k_\lambda\big((1+\alpha)(m-\xi), (1-\beta)(y+\eta)\big)(m-\xi)^{(\lambda_1-\frac{\varepsilon}{p})-1}(y+\eta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\,dy, \\
H_3 &:= (1+\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1}(1-\alpha)^{(\lambda_1-\frac{\varepsilon}{p})-1} \\
&\quad \times \int_1^\infty \sum_{m=1}^\infty k_\lambda\big((1-\alpha)(m+\xi), (1+\beta)(y-\eta)\big)(m+\xi)^{(\lambda_1-\frac{\varepsilon}{p})-1}(y-\eta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\,dy,
\end{aligned}$$

$$H_4 := (1+\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1}(1+\alpha)^{(\lambda_1-\frac{\varepsilon}{p})-1} \times \int_1^\infty \sum_{m=1}^\infty k_\lambda\big((1+\alpha)(m-\xi),(1+\beta)(y-\eta)\big)(m-\xi)^{(\lambda_1-\frac{\varepsilon}{p})-1}(y-\eta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\,dy.$$

In the following, we estimate $H_1$. Still using the decreasing property of series, for fixed $x > -\xi$, $\frac{2}{1-\eta} \geq 1 (\eta = \alpha, \beta)$, setting $u = \frac{(1-\beta)(y+\eta)}{(1-\alpha)(x+\xi)}$, we obtain

$$\begin{aligned}
H_1 &> (1-\beta)^{\lambda_2-\frac{\varepsilon}{q}-1}(1-\alpha)^{\lambda_1-\frac{\varepsilon}{p}-1}\int_{\frac{2}{1-\alpha}}^\infty \bigg[\int_{\frac{2}{1-\beta}}^\infty k_\lambda\big((1-\alpha)(x+\xi),(1-\beta)(y+\eta)\big) \\
&\quad \times (x+\xi)^{\lambda_1-\frac{\varepsilon}{p}-1}(y+\eta)^{\lambda_2-\frac{\varepsilon}{q}-1}\,dy\bigg]\,dx \\
&= \frac{(1-\alpha)^{-\varepsilon-1}}{1-\beta}\int_{\frac{2}{1-\alpha}}^\infty (x+\xi)^{-\varepsilon-1}\int_{\frac{2}{(1-\alpha)(x+\xi)}}^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}\,du\,dx \\
&\overset{v=(1-\alpha)(x+\xi)}{=} \frac{1}{(1-\beta)(1-\alpha)}\int_2^\infty v^{-\varepsilon-1}\int_{\frac{2}{v}}^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}\,du\,dv \\
&= \frac{1}{(1-\beta)(1-\alpha)}\bigg[\int_2^\infty v^{-\varepsilon-1}\int_{\frac{2}{v}}^2 k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}\,du\,dv \\
&\quad + \int_2^\infty v^{-\varepsilon-1}\int_2^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}\,du\,dv\bigg] \\
&= \frac{1}{(1-\beta)(1-\alpha)}\bigg[\int_0^2\bigg(\int_{\frac{2}{u}}^\infty v^{-\varepsilon-1}\,dv\bigg)k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}\,du \\
&\quad + \frac{1}{\varepsilon 2^\varepsilon}\int_2^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}\,du\bigg] \\
&= \tilde{H}_1 := \frac{1}{\varepsilon 2^\varepsilon(1-\beta)(1-\alpha)}\bigg(\int_0^2 k_\lambda(1,u)u^{\lambda_2+\frac{\varepsilon}{p}-1}\,du + \int_2^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}\,du\bigg).
\end{aligned}$$

In the same way, we can find that

$$\begin{aligned}
H_2 > \tilde{H}_2 &:= \frac{1}{\varepsilon 2^\varepsilon(1-\beta)(1+\alpha)}\bigg(\int_0^2 k_\lambda(1,u)u^{\lambda_2+\frac{\varepsilon}{p}-1}\,du + \int_2^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}\,du\bigg), \\
H_3 > \tilde{H}_3 &:= \frac{1}{\varepsilon 2^\varepsilon(1+\beta)(1-\alpha)}\bigg(\int_0^2 k_\lambda(1,u)u^{\lambda_2+\frac{\varepsilon}{p}-1}\,du + \int_2^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}\,du\bigg), \\
H_3 > \tilde{H}_3 &:= \frac{1}{\varepsilon 2^\varepsilon(1+\beta)(1+\alpha)}\bigg(\int_0^2 k_\lambda(1,u)u^{\lambda_2+\frac{\varepsilon}{p}-1}\,du + \int_2^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}\,du\bigg).
\end{aligned}$$

In view of the above results, we have

$$\tilde{H} > \sum_{i=1}^4 \tilde{H}_i = \frac{4}{\varepsilon 2^\varepsilon(1-\beta^2)(1-\alpha^2)}\bigg(\int_0^2 k_\lambda(1,u)u^{\lambda_2+\frac{\varepsilon}{p}-1}\,du + \int_2^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}\,du\bigg),$$

and then (13) follows.

The lemma is proved. □

**Lemma 4** *The following inequality holds*:

$$\begin{aligned} H &:= \sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} k_{\xi,\eta}(m,n)a_m b_n \\ &< \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left\{\sum_{|m|=1}^{\infty}\bigl[|m-\xi|+\alpha(m-\xi)\bigr]^{p(1-\lambda_1)-d-1}a_m^p\right\}^{\frac{1}{p}} \\ &\quad\times\left\{\sum_{|n|=1}^{\infty}|n-\eta|+\beta(n-\eta)]^{q(1-\lambda_2)-d-1}b_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{14}$$

*Proof* By Hölder's inequality with weight (cf. [38]), (7) and (8), we obtain

$$\begin{aligned} H &= \sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} k_{\xi,\eta}(m,n)\left\{\frac{[|n-\eta|+\beta(n-\eta)]^{(\lambda_2-1)/p}}{[|m-\xi|+\alpha(m-\xi)]^{(\lambda_1-1)/q}}a_m\right\} \\ &\quad\times\left\{\frac{[|m-\xi|+\alpha(m-\xi)]^{(\lambda_1-1)/q}}{[|n-\eta|+\beta(n-\eta)]^{(\lambda_2-1)/p}}b_n\right\} \\ &\le\left\{\sum_{|m|=1}^{\infty}\sum_{|n|=1}^{\infty} k_{\xi,\eta}(m,n)\frac{[|n-\eta|+\beta(n-\eta)]^{\lambda_2-1}}{[|m-\xi|+\alpha(m-\xi)]^{(\lambda_1-1)(p-1)}}a_m^p\right\}^{\frac{1}{p}} \\ &\quad\times\left\{\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} k_{\xi,\eta}(m,n)\frac{[|m-\xi|+\alpha(m-\xi)]^{\lambda_1-1}}{[|n-\eta|+\beta(n-\eta)]^{(\lambda_2-1)(q-1)}}b_n^q\right\}^{\frac{1}{q}} \\ &=\left\{\sum_{|m|=1}^{\infty}\omega(\lambda_2,m)\bigl[|m-\xi|+\alpha(m-\xi)\bigr]^{p(1-\lambda_1)-d-1}a_m^p\right\}^{\frac{1}{p}} \\ &\quad\times\left\{\sum_{|n|=1}^{\infty}\varpi(\lambda_1,n)(|n-\eta|+\beta(n-\eta)]^{q(1-\lambda_2)-d-1}b_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

Then by (9) and (10), we have (14).
The lemma is proved. □

*Remark* 1 (i) By (14), for $\lambda_1+\lambda_2=\lambda$ (or $d=0$), we find

$$0<\sum_{|m|=1}^{\infty}\bigl[|m-\xi|+\alpha(m-\xi)\bigr]^{p(1-\lambda_1)-1}a_m^p<\infty,$$

$$0<\sum_{|n|=1}^{\infty}\bigl[|n-\eta|+\beta(n-\eta)\bigr]^{q(1-\lambda_2)-1}b_n^q<\infty,$$

and the following more accurate Hilbert-type inequality in the whole plane:

$$\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} k_\lambda\bigl(|m-\xi|+\alpha(m-\xi),|n-\eta|+\beta(n-\eta)\bigr)a_m b_n$$

$$< \frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \left\{ \sum_{|m|=1}^{\infty} \bigl[|m-\xi| + \alpha(m-\xi)\bigr]^{p(1-\lambda_1)-1} a_m^p \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{|n|=1}^{\infty} \bigl[|n-\eta| + \beta(n-\eta)\bigr]^{q(1-\lambda_2)-1} b_n^q \right\}^{\frac{1}{q}}. \tag{15}$$

In particular, for $\alpha = \beta = \xi = \eta = 0, a_{-m} = a_m, b_{-n} = b_n$ $(m, n \in \mathrm{N})$ in (15), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda(m,n)a_m b_n < k_\lambda(\lambda_2)\left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}. \tag{16}$$

(ii) For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$ in (16), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_1(m,n)a_m b_n < k_1\left(\frac{1}{p}\right)\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}; \tag{17}$$

for $\lambda = 1, \lambda_1 = \frac{1}{p}, \lambda_2 = \frac{1}{q}$ in (16), we have the dual form of (17) as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_1(m,n)a_m b_n < k_1\left(\frac{1}{q}\right)\left(\sum_{m=1}^{\infty} m^{p-2} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{q-2} b_n^q\right)^{\frac{1}{q}}; \tag{18}$$

for $p = q = 2$, both (17) and (18) reduce to the following Hilbert-type inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_1(m,n)a_m b_n < k_1\left(\frac{1}{2}\right)\left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}}. \tag{19}$$

(iii) For $\alpha = \beta = 0, \xi = \eta = \frac{1}{2}, \lambda = 1, k_1(m,n) = \frac{1}{m+n}, \lambda_1 = \frac{1}{r}, \lambda_2 = \frac{1}{s}$ $(r > 1, \frac{1}{r} + \frac{1}{s} = 1)$, (15) reduces to (5). Hence, (14) and (15) are general extensions of (5).

**Lemma 5** *The constant factor* $\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *in* (15) *is the best possible.*

*Proof* For any $\varepsilon > 0$, we set

$$\tilde{a}_m := \bigl[|m-\xi| + \alpha(m-\xi)\bigr]^{(\lambda_1-\frac{\varepsilon}{p})-1}, \tilde{b}_n := \bigl[|n-\eta| + \beta(n-\eta)\bigr]^{(\lambda_2-\frac{\varepsilon}{q})-1} \quad \bigl(|m|, |n| \in \mathrm{N}\bigr).$$

If there exists a constant $M(\le \frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}})$, such that (15) is valid when replacing $\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ by $M$, then in particular, in view of $\lambda_1 + \lambda_2 = \lambda$, by Lemma 1, we have

$$\tilde{H} = \sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} k_\lambda\bigl(|m-\xi| + \alpha(m-\xi), |n-\eta| + \beta(n-\mu)\bigr)\tilde{a}_m\tilde{b}_n$$

$$< M\left\{\sum_{|m|=1}^{\infty}\bigl[|m-\xi| + \alpha(m-\xi)\bigr]^{p(1-\lambda_1)-1}\tilde{a}_m^p\right\}^{\frac{1}{p}} \{\sum_{|n|=1}^{\infty}\bigl[|n-\eta| + \beta(n-\eta)\bigr]^{q(1-\lambda_2)-1}\tilde{b}_n^q]^{\frac{1}{q}}.$$

$$= M\left\{\sum_{|m|=1}^{\infty}\bigl[|m-\xi| + \alpha(m-\xi)\bigr]^{-\varepsilon-1}\right\}^{\frac{1}{p}} \left\{\sum_{|n|=1}^{\infty}\bigl[|n-\eta| + \beta(n-\eta)\bigr]^{-\varepsilon-1}\right\}^{\frac{1}{q}}$$

$$< \frac{M}{\varepsilon}\Big\{\big[(1-\alpha)^{-\varepsilon-1}+(1+\alpha)^{-\varepsilon-1}\big]\big[\varepsilon\big(1-|\xi|\big)^{-\varepsilon-1}+1\big]\Big\}^{\frac{1}{p}} \times \Big\{\big[(1-\beta)^{-\varepsilon-1}+(1+\beta)^{-\varepsilon-1}\big]\big[\varepsilon\big(1-|\eta|\big)^{-\varepsilon-1}+1\big]\Big\}^{\frac{1}{q}}.$$

In view of the above result and (13), we have

$$\begin{aligned}&\frac{4}{2^{\varepsilon}(1-\beta^2)(1-\alpha^2)}\left(\int_0^2 k_\lambda(1,u)u^{\lambda_2+\frac{\varepsilon}{p}-1}\,du+\int_2^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}\,du\right)\\ &\quad< \varepsilon\tilde{H}< M\Big\{\big[(1-\alpha)^{-\varepsilon-1}+(1+\alpha)^{-\varepsilon-1}\big]\big[\varepsilon\big(1-|\xi|\big)^{-\varepsilon-1}+1\big]\Big\}^{\frac{1}{p}}\\ &\qquad\times\Big\{\big[(1-\beta)^{-\varepsilon-1}+(1+\beta)^{-\varepsilon-1}\big]\big[\varepsilon\big(1-|\eta|\big)^{-\varepsilon-1}+1\big]\Big\}^{\frac{1}{q}}.\end{aligned}$$

For $\varepsilon\to 0^+$, by Fatou lemma (cf. [39]), we find

$$\begin{aligned}&\frac{4}{(1-\beta^2)(1-\alpha^2)}k_\lambda(\lambda_2)\\ &\quad\le \lim_{\varepsilon\to 0^+}\frac{4}{2^{\varepsilon}(1-\beta^2)(1-\alpha^2)}\left(\int_0^2 k_\lambda(1,u)u^{\lambda_2+\frac{\varepsilon}{p}-1}\,du+\int_2^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}\,du\right)\\ &\quad\le\frac{2M}{(1-\alpha^2)^{1/p}(1-\beta^2)^{1/q}},\end{aligned}$$

namely, $\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\le M$, which means that $M=\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ is the best possible constant factor of (15).

The lemma is proved. □

*Remark* 2 (i) In view of Lemma 5, the constant factors in (16)–(19) are also the best possible.

(ii) Setting $\hat{\lambda}_1:=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}=\lambda_1+\frac{d}{p}$, $\hat{\lambda}_2:=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\lambda_2+\frac{d}{q}$, we find

$$\hat{\lambda}_1+\hat{\lambda}_2=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\frac{\lambda}{p}+\frac{\lambda}{q}=\lambda,$$

and then by Hölder's inequality (cf. [38]), it follows that

$$\begin{aligned}0&<k_\lambda(\hat{\lambda}_2)=k_\lambda\left(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q}\right)\\ &=\int_0^\infty k_\lambda(1,u)u^{\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q}-1}\,du=\int_0^\infty k_\lambda(1,u)\big(u^{\frac{\lambda_2-1}{p}}\big)\big(u^{\frac{\lambda-\lambda_1-1}{q}}\big)\,du\\ &\le\left(\int_0^\infty k_\lambda(1,u)u^{\lambda_2-1}\,du\right)^{\frac{1}{p}}\left(\int_0^\infty k_\lambda(1,u)u^{(\lambda-\lambda_1)-1}\,du\right)^{\frac{1}{q}}\\ &=k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)<\infty.\end{aligned}\tag{20}$$

We can rewrite (14) as follows:

$$H<\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left\{\sum_{|m|=1}^\infty\big[|m-\xi|+\alpha(m-\xi)\big]^{p(1-\hat{\lambda}_1)-1}a_m^p\right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{|n|=1}^{\infty} \bigl[ |n-\eta| + \beta(n-\eta) \bigr]^{q(1-\hat{\lambda}_2)-1} b_n^q \right\}^{\frac{1}{q}}. \tag{21}$$

**Lemma 6** *If the constant factor* $\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *in* (14) (*or* (21)) *is the best possible, then we have* $\lambda_1 + \lambda_2 = \lambda$.

*Proof* If the constant factor $\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ in (14) (or (21)) is the best possible, then by (21) and (15) (for $\lambda_i = \hat{\lambda}_i$ $(i = 1, 2)$), we have the following inequality:

$$\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \le \frac{2k_\lambda(\hat{\lambda}_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} (\in \mathrm{R}_+),$$

namely, $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1) \le k_\lambda(\hat{\lambda}_2)$, which means that (20) is an equality.

We observe that (20) is an equality if and only if there exist constants $A$ and $B$, such that they are not both zero and (cf. [38])

$$Au^{\lambda_2-1} = Bu^{\lambda-\lambda_1-1} \quad \text{a.e. in } \mathrm{R}_+.$$

Assuming that $A \ne 0$, it follows that $u^{\lambda_2+\lambda_1-\lambda} = \frac{B}{A}$ a.e. in $\mathrm{R}_+$, and then $\lambda_2 + \lambda_1 - \lambda = 0$, namely, $\lambda_1 + \lambda_2 = \lambda$.

The lemma is proved. □

## 3 Main results

**Theorem 1** *Inequality* (14) *is equivalent to the following more accurate Hilbert-type inequality in the whole plane*:

$$L := \left\{ \sum_{|n|=1}^{\infty} \bigl[ |n-\eta| + \beta(n-\eta) \bigr]^{p(\lambda_2+d)-d-1} \left( \sum_{|m|=1}^{\infty} k_{\xi,\eta}(m,n) a_m \right)^p \right\}^{\frac{1}{p}}$$
$$< \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \left\{ \sum_{|m|=1}^{\infty} \bigl[ |m-\xi| + \alpha(m-\xi) \bigr]^{p(1-\lambda_1)-d-1} a_m^p \right\}^{\frac{1}{p}}. \tag{22}$$

*Proof* Suppose that (22) is valid. By Hölder's inequality (cf. [38]), we find

$$H = \sum_{|n|=1}^{\infty} \left\{ \bigl[ |n-\eta| + \beta(n-\eta) \bigr]^{\frac{-1}{p}+\lambda_2+\frac{d}{q}} \sum_{|m|=1}^{\infty} k_{\xi,\eta}(m,n) a_m \right\} \left\{ \bigl[ |n-\eta| + \beta(n-\eta) \bigr]^{\frac{1}{p}-\lambda_2-\frac{d}{q}} b_n \right\}$$
$$\le L \cdot \left\{ \sum_{|n|=1}^{\infty} \bigl[ |n-\xi| + \beta(n-\xi) \bigr]^{q(1-\lambda_2)-d-1} b_n^q \right\}^{\frac{1}{q}}. \tag{23}$$

Then by (22), we obtain (14). On the other hand, assuming that (14) is valid, we set

$$b_n := \bigl[ |n-\eta| + \beta(n-\eta) \bigr]^{p(\lambda_2+d)-d-1} \left( \sum_{|m|=1}^{\infty} k_{\xi,\eta}(m,n) a_m \right)^{p-1}, \quad |n| \in \mathbf{N}.$$

Then we have

$$L^p = \sum_{|n|=1}^{\infty} \big[|n-\eta| + \beta(n-\eta)\big]^{q(1-\lambda_2)-d-1} b_n^q = H. \tag{24}$$

If $L = 0$, then (22) is naturally valid; if $L = \infty$, then it is impossible that (22) is valid, namely, $L < \infty$. Suppose that $0 < L < \infty$. By (14), it follows that

$$\begin{aligned}
&\sum_{|n|=1}^{\infty} \big[|n-\eta| + \beta(n-\eta)\big]^{q(1-\lambda_2)-d-1} b_n^q \\
&\quad = L^p = H < \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \left\{ \sum_{|m|=1}^{\infty} \big[|m-\xi| + \alpha(m-\xi)\big]^{p(1-\lambda_1)-d-1} a_m^p \right\}^{\frac{1}{p}} L^{p-1}, \\
&L = \left\{ \sum_{|n|=1}^{\infty} \big[|n-\eta| + \beta(n-\eta)\big]^{q(1-\lambda_2)-d-1} b_n^q \right\}^{\frac{1}{p}} \\
&\quad < \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \left\{ \sum_{|m|=1}^{\infty} \big[|m-\xi| + \alpha(m-\xi)\big]^{p(1-\lambda_1)-d-1} a_m^p \right\}^{\frac{1}{p}},
\end{aligned}$$

namely, (22) follows, which is equivalent to (14).

The theorem is proved. □

**Theorem 2** *The following statements are equivalent*:

(i) *Both* $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)$ *and* $k_\lambda(\frac{\lambda_2}{p} + \frac{\lambda-\lambda_1}{q})$ *are independent of* $p,q$;

(ii) $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1) \le k_\lambda(\frac{\lambda_2}{p} + \frac{\lambda-\lambda_1}{q})$;

(iii) $\lambda_1 + \lambda_2 = \lambda$;

(iv) $\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *is the best possible constant factor of* (14);

(v) $\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *is the best possible constant factor of* (22).

If the statement (iii) follows, namely, $\lambda_1 + \lambda_2 = \lambda$ (or $d = 0$), then we have the following inequality equivalent to (15) with the best possible constant factor $\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$:

$$\begin{aligned}
&\left\{ \sum_{|n|=1}^{\infty} \big[|n-\eta| + \beta(n-\eta)\big]^{p\lambda_2-1} \left[ \sum_{|m|=1}^{\infty} k_\lambda\big(|m-\xi| + \alpha(m-\xi), |n-\eta| + \beta(n-\eta)\big) a_m \right]^p \right\}^{\frac{1}{p}} \\
&\quad < \frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \left\{ \sum_{|m|=1}^{\infty} \big[|m-\xi| + \alpha(m-\xi)\big]^{p(1-\lambda_1)-1} a_m^p \right\}^{\frac{1}{p}}.
\end{aligned} \tag{25}$$

In particular, for $\alpha = \beta = \xi = \eta = 0, a_{-m} = a_m, b_{-n} = b_n$ $(m, n \in \mathrm{N})$ in (25), we have the following inequality equivalent to (16) with the best possible constant factor $k_\lambda(\lambda_2)$:

$$\left[ \sum_{n=1}^{\infty} n^{p\lambda_2-1} \left( \sum_{m=1}^{\infty} k_\lambda(m,n) a_m \right)^p \right]^{\frac{1}{p}} < k_\lambda(\lambda_2) \left[ \sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}}. \tag{26}$$

*Proof* (i) ⇒ (ii). Since $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)$ is independent of $p,q$, we find

$$k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)=\lim_{q\to\infty}\lim_{p\to 1^+}k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)=k_\lambda(\lambda_2).$$

Then by Fatou lemma (cf. [39]), we have the following inequality:

$$k_\lambda\left(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q}\right)=\lim_{q\to\infty}k_\lambda\left(\lambda_2+\frac{c}{q}\right)\geq k_\lambda(\lambda_2)=k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1).$$

(ii) ⇒ (iii). If $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)\leq k_\lambda(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})$, then (20) is an equality. By the proof of Lemma 6, it follows that $\lambda_1+\lambda_2=\lambda$.

(iii) ⇒ (i). If $\lambda_1+\lambda_2=\lambda$, then we have

$$k_\lambda\left(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q}\right)=k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)=k_\lambda(\lambda_2).$$

Both $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)$ and $k_\lambda(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})$ are independent of $p,q$.

Hence, we have (i) ⇔ (ii) ⇔ (iii).

(iii) ⇔ (iv). By Lemmas 5 and 6, we obtain the conclusions.

(iv) ⇔ (v). If the constant factor in (14) is the best possible, then so is the constant factor in (22). Otherwise, by (23), we would reach a contradiction that the constant factor in (14) is not the best possible. On the other hand, if the constant factor in (22) is the best possible, then so is the constant factor in (14). Otherwise, by (24), we would reach a contradiction that the constant factor in (22) is not the best possible.

Therefore, the statements (i)–(v) are equivalent.

The theorem is proved. □

## 4 Operator expressions

We define functions

$$\phi(m):=\bigl[|m-\xi|+\alpha(m-\xi)\bigr]^{p(1-\lambda_1)-d-1},\qquad \psi(n):=\bigl[|n-\eta|+\beta(n-\eta)\bigr]^{q(1-\lambda_2)-d-1},$$

where

$$\psi^{1-p}(n)=\bigl[|n-\eta|+\beta(n-\eta)\bigr]^{p(\lambda_2+d)-d-1}\quad\bigl(|m|,|n|\in\mathrm{N}\bigr).$$

Define the following real normed spaces:

$$l_{p,\phi}:=\left\{a=\{a_m\}_{|m|=1}^{\infty};\|a\|_{p,\phi}:=\left(\sum_{|m|=1}^{\infty}\phi(m)|a_m|^p\right)^{\frac{1}{p}}<\infty\right\},$$

$$l_{q,\psi}:=\left\{b=\{b_n\}_{|n|=1}^{\infty};\|b\|_{q,\psi}:=\left(\sum_{|n|=1}^{\infty}\psi(n)|b_n|^q\right)^{\frac{1}{q}}<\infty\right\},$$

$$l_{p,\psi^{1-p}}:=\left\{c=\{c_n\}_{|n|=1}^{\infty};\|c\|_{p,\psi^{1-p}}:=\left(\sum_{|n|=1}^{\infty}\psi^{1-p}(n)|b_n|^p\right)^{\frac{1}{p}}<\infty\right\}.$$

Assuming that $a \in l_{p,\phi}$ and setting

$$c = \{c_n\}_{|n|=1}^{\infty}, c_n := \sum_{|m|=1}^{\infty} k_{\xi,\eta}(m,n)a_m, \quad |n| \in \mathrm{N},$$

we can rewrite (22) as follows:

$$\|c\|_{p,\psi^{1-p}} < \frac{2k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\|a\|_{p,\phi} < \infty,$$

namely, $c \in l_{p,\psi^{1-p}}$.

**Definition 2** Define a Hilbert-type operator $T : l_{p,\phi} \to l_{p,\psi^{1-p}}$ as follows: For any $a \in l_{p,\phi}$, there exists a unique representation $Ta = c \in l_{p,\psi^{1-p}}$, satisfying for any $|n| \in \mathrm{N}$, $Ta(n) = c_n$. Define the formal inner product of $Ta$ and $b \in l_{q,\psi}$, and the norm of $T$, as follows:

$$(Ta, b) := \sum_{|n|=1}^{\infty}\left(\sum_{|m|=1}^{\infty} k_{\xi,\eta}(m,n)a_m\right)b_n = H,$$

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\phi}} \frac{\|Ta\|_{p,\psi^{1-p}}}{\|a\|_{p,\phi}}.$$

By Theorems 1 and 2, we have

**Theorem 3** *If $a \in l_{p,\phi}, b \in l_{q,\psi}, \|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$, then we have the following equivalent inequalities*:

$$(Ta, b) < \frac{2k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\|a\|_{p,\phi}\|b\|_{q,\psi}, \tag{27}$$

$$\|Ta\|_{p,\psi^{1-p}} < \frac{2k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\|a\|_{p,\phi}. \tag{28}$$

*Moreover, $\lambda_1 + \lambda_2 = \lambda$ if and only if the constant factor*

$$\frac{2k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left(= \frac{2k_{\lambda}(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\right)$$

*in (27) and (28) is the best possible, namely,*

$$\|T\| = \frac{2k_{\lambda}(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}. \tag{29}$$

*Example* 1 For $\lambda > 0, 0 < \sigma \leq 1, \lambda_i \in (0,\lambda)\cap(0,1]$ $(i = 1,2)$, setting $k_{\lambda}(x,y) = \frac{1}{(x^{\sigma}+y^{\sigma})^{\lambda/\sigma}}$ $(x,y > 0)$ yields that

$$k_{\xi,\eta}(m,n) = \frac{1}{\{[|m-\xi| + \alpha(m-\xi)]^{\sigma} + [|n-\eta| + \beta(n-\eta)]^{\sigma}\}^{\lambda/\sigma}} \quad (|m|, |n| \in \mathrm{N}),$$

$k_\lambda(x,y)x^{\lambda_1-1}$ (resp. $k_\lambda(x,y)y^{\lambda_2-1}$) is strictly decreasing and strictly convex with respect to $x>0$ (resp. $y>0$), such that

$$k_\lambda(\gamma)=\int_0^\infty \frac{u^{\gamma-1}}{(1+u^\sigma)^{\lambda/\sigma}}\,du \overset{v=u^\sigma}{=} \frac{1}{\sigma}\int_0^\infty \frac{v^{(\gamma/\sigma)-1}}{(1+v)^{\lambda/\sigma}}\,dv$$
$$=\frac{1}{\sigma}B\left(\frac{\gamma}{\sigma}.\frac{\lambda-\gamma}{\sigma}\right)\in \mathrm{R}_+ \quad (\gamma=\lambda_2,\lambda-\lambda_1).$$

By Theorem 3, it follows that $\lambda_1+\lambda_2=\lambda$ if and only if

$$\|T\|=\frac{2}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\frac{1}{\sigma}B\left(\frac{\lambda_1}{\sigma}.\frac{\lambda_2}{\sigma}\right). \tag{30}$$

*Example* 2 For $0<\lambda\leq 1,\lambda_i\in(0,\lambda)\cap(0,1]$ $(i=1,2)$, setting $k_\lambda(x,y)=\frac{\ln(x/y)}{x^\lambda-y^\lambda}$ $(x,y>0)$ yields that

$$k_{\xi,\eta}(m,n)=\frac{\ln\frac{|m-\xi|+\alpha(m-\xi)}{|n-\eta|+\beta(n-\eta)}}{[|m-\xi|+\alpha(m-\xi)]^\lambda-[|n-\eta|+\beta(n-\eta)]^\lambda} \quad \big(|m|,|n|\in \mathrm{N}\big),$$

$k_\lambda(x,y)x^{\lambda_1-1}$ (resp. $k_\lambda(x,y)y^{\lambda_2-1}$) is strictly decreasing and strictly convex with respect to $x>0$ (resp. $y>0$), such that

$$k_\lambda(\gamma)=\int_0^\infty \frac{u^{\gamma-1}\ln u}{u^\lambda-1}\,du=\left[\frac{\pi}{\lambda\sin(\pi\gamma/\lambda)}\right]^2\in \mathrm{R}_+ \quad (\gamma=\lambda_2,\lambda-\lambda_1).$$

By Theorem 3, it follows that $\lambda_1+\lambda_2=\lambda$ if and only if

$$\|T\|=\frac{2}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\frac{\pi}{\lambda\sin(\pi\lambda_2/\lambda)}\right]^2. \tag{31}$$

## 5 Conclusions

In this paper, by means of the weight coefficients, the idea of introduced parameters and the technique of real analysis, a more accurate Hilbert-type inequality in the whole plane is obtained in Lemma 4, which is an extension of (1). An equivalent form is given in Theorem 1. The equivalent statements of the best possible constant factor related to several parameters are considered in Theorem 2. The operator expressions and some particular cases are provided in Theorem 3, Remark 1 and Examples 1–2. The lemmas and theorems provide an extensive account of this type of inequalities.

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**Availability of data and materials**
The data used to support the findings of this study are included within the article.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. XH participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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