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Refinements of some integral inequalities for unified integral operators

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Abstract

In this paper we are presenting the refinements of integral inequalities established for convex functions. Consequently, we get refinements of several fractional integral inequalities for different kinds of fractional integral operators.

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1 Introduction

Integral operators are very useful in the theory of differential equations and boundary value problems. They are applied to formulate and solve mathematical models of real world problems; nowadays fractional integral operators are frequently studied to extend and generalize classical subjects. Fractional integral operators have converted the classical notions into modern concepts. In the recent past, fractional integral operators were utilized extensively to study the classical inequalities, see [2, 3, 6, 8, 9, 13, 14, 16–18, 20, 22] and the references therein.

The aim of this paper is to give several integral inequalities for strongly convex functions, resulting in refinements of the integral inequalities presented in [16], also [8, 9, 12]. For this purpose, we will need the following integral operators:

Definition 1 ([15]) Let $\tau_1 : [a,b] \to \mathbb{R}$ be an integrable function. Also let τ_2 be an increasing and positive function on $(a,b]$, having a continuous derivative τ_2' on (a,b). The left- and right-sided fractional integrals of a function τ_1 with respect to another function τ_2 on $[a,b]$ of order μ where $\Re(\mu) > 0$ are defined by:

$$ {}^{\mu}_{\tau_2}I_{a^+}\tau_1(x) = \frac{1}{\Gamma(\mu)} \int_a^x \bigl(\tau_2(x) - \tau_2(t)\bigr)^{\mu-1} \tau_2'(t)\tau_1(t)\,dt, \quad x > a, \tag{1.1} $$

and

$$ {}^{\mu}_{\tau_2}I_{b^-}\tau_1(x) = \frac{1}{\Gamma(\mu)} \int_x^b \bigl(\tau_2(t) - \tau_2(x)\bigr)^{\mu-1} \tau_2'(t)\tau_1(t)\,dt, \quad x < b, \tag{1.2} $$

where $\Gamma(\cdot)$ is the gamma function.

A k-analogue of the above definition is defined as follows:

Definition 2 ([1]) Let $\tau_1 : [a,b] \to \mathbb{R}$ be an integrable function. Also let τ_2 be an increasing and positive function on $(a,b]$, having a continuous derivative τ_2' on (a,b). The left- and right-sided fractional integrals of a function τ_1 with respect to another function τ_2 on $[a,b]$ of order $\mu; \Re(\mu), k > 0$ are defined by:

$$ {}^{\mu}_{\tau_2}I^{k}_{a^+}\tau_1(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x \big(\tau_2(x) - \tau_2(t)\big)^{\frac{\mu}{k}-1} \tau_2'(t)\tau_1(t)\,dt, \quad x > a, \tag{1.3} $$

$$ {}^{\mu}_{\tau_2}I^{k}_{b^-}\tau_1(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b \big(\tau_2(t) - \tau_2(x)\big)^{\frac{\mu}{k}-1} \tau_2'(t)\tau_1(t)\,dt, \quad x < b, \tag{1.4} $$

where $\Gamma_k(\cdot)$ is defined by [5]

$$ \Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t^k}{k}}\,dt, \quad \Re(x) > 0. \tag{1.5} $$

The integral operators (1.3) and (1.4) produce several fractional integral operators, see [16, Remark 1]. A well-known Mittag-Leffler function is defined by [19]

$$ E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1.6} $$

where $\alpha, z \in \mathbb{C}$ and $\Re(\alpha) > 0$.

This function has been extended and generalized in several different ways. In the following, we give a definition of an extended Mittag-Lefffler function with its corresponding fractional integral operator.

Definition 3 ([2]) Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Let $\tau_1 \in L_1[a,b]$ and $x \in [a,b]$. Then the generalized fractional integral operators $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}\tau_1$ and $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}\tau_1$ are defined by:

$$ \big(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}\tau_1\big)(x;p) = \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\big(\omega(x-t)^{\mu};p\big)\tau_1(t)\,dt, \tag{1.7} $$

$$ \big(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}\tau_1\big)(x;p) = \int_x^b (t-x)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\big(\omega(t-x)^{\mu};p\big)\tau_1(t)\,dt, \tag{1.8} $$

where

$$ E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}} \tag{1.9} $$

is the extended Mittag-Leffler function.

Recently, a unified integral operator was defined, which unifies several fractional integrals in a compact formula as follows:

Definition 4 ([11]) Let $\tau_1, \tau_2 : [a,b] \longrightarrow \mathbb{R}$, $0 < a < b$, be functions such that τ_1 is positive and $\tau_1 \in L_1[a,b]$ and τ_2 is differentiable and strictly increasing. Also let $\frac{\phi}{x}$ be an increasing function on $[a,\infty)$ and $\alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$, $p, \mu, \delta \geq 0$, and $0 < k \leq \delta + \mu$. Then for $x \in [a,b]$, the left and right integral operators are defined by

$$\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\tau_1\right)(x,\omega;p) = \int_a^x K_x^y\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\right)\tau_1(y)\,d\left(\tau_2(y)\right), \tag{1.10}$$

$$\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}\tau_1\right)(x,\omega;p) = \int_x^b K_y^x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\right)\tau_1(y)\,d\left(\tau_2(y)\right), \tag{1.11}$$

where $K_x^y(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi) = \frac{\phi(\tau_2(x)-\tau_2(y))}{\tau_2(x)-\tau_2(y)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\tau_2(x)-\tau_2(y))^{\mu};p)$.

For suitable settings of functions ϕ, τ_2 and certain values of parameters included in Mittag-Leffler function (1.9), very interesting consequences are obtained which are described in [16, Remarks 6 & 7].

The objective of this paper is to obtain some inequalities for unified integral operators via strongly convex functions.

Definition 5 A function $\tau_1 : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is said to be convex if

$$\tau_1\left(tx + (1-t)y\right) \leq t\tau_1(x) + (1-t)\tau_1(y) \tag{1.12}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The following well-known Hadamard inequality holds for convex functions:

Definition 6 ([7]) Let $\tau_1 : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$ and $a, b \in I$ where $a < b$. Then

$$\tau_1\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \tau_1(t)\,dt \leq \frac{\tau_1(a)+\tau_1(b)}{2}$$

holds.

Definition 7 ([21]) Let I be a nonempty convex subset of the normed space $\mathbf{X}$. A real valued function τ_1 is said to be strongly convex with modulus $G > 0$ on I if for each $a, b \in I$ and $t \in [0,1]$,

$$\tau_1\left(tx + (1-t)y\right) \leq t\tau_1(x) + (1-t)\tau_1(y) - Gt(1-t)\|b-a\|^2. \tag{1.13}$$

In the following, we give some results which are directly linked with the main findings of this paper. The following bounds of unified integral operators for convex functions are established in [16]:

Theorem 1 *Let $\tau_1 : [a,b] \longrightarrow \mathbb{R}$ be a positive integrable convex function with $m \in (0,1]$. Let $\tau_2 : [a,b] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function, also let $\frac{\phi}{x}$ be an increasing function on $[a,b]$. If $\alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$, $p, \mu, \delta \geq 0$, and $0 < k \leq$*

$\delta+\mu$, then for $x\in(a,b)$ we have

$$\begin{aligned}&\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\tau_1\right)(x,\omega;p)+\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}\tau_1\right)(x,\omega;p)\\&\quad\le K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\right)\left(\left(\tau_2(x)-\tau_2(a)\right)\left(\tau_1(x)+\tau_1(a)\right)\right)\\&\qquad+K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},\tau_2;\phi\right)\left(\left(\tau_2(b)-\tau_2(x)\right)\left(\tau_1(b)+\tau_1(x)\right)\right).\end{aligned}\tag{1.14}$$

The following Hadamard inequality for unified fractional integrals is proved in [16]:

Theorem 2 *Under the assumptions of Theorem 1, in addition if $\tau_1(x)=\tau_1(a+b-x)$, then we have*

$$\begin{aligned}&\tau_1\left(\frac{a+b}{2}\right)\left(\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a,\omega;p)+\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\right)(b,\omega;p)\right)\\&\quad\le\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}\tau_1\right)(a,\omega;p)+\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\tau_1\right)(b,\omega;p)\\&\quad\le 2K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\right)\left)\left(\tau_2(b)-\tau_2(a)\right)\left(\tau_1(b)+\tau_1(a)\right).\end{aligned}\tag{1.15}$$

The following modulus inequality is obtained for unified integrals in [16].

Theorem 3 *Let $\tau_1:[a,b]\longrightarrow\mathbb{R}$ be a differentiable function. Let $|\tau_1'|$ be convex with $m\in(0,1]$ and $\tau_2:[a,b]\longrightarrow\mathbb{R}$ be differentiable and strictly increasing function, also let $\frac{\phi}{x}$ be an increasing function on $[a,b]$. If $\alpha,l,\gamma,c\in\mathbb{C}$, $\Re(\alpha),\Re(l)>0$, $\Re(c)>\Re(\gamma)>0$, $p,\mu,\delta\ge 0$, and $0<k\le\delta+\mu$, then for $x\in(a,b)$ we have*

$$\begin{aligned}&\left|\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\tau_1*\tau_2\right)(x,\omega;p)+\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}\tau_1*\tau_2\right)(x,\omega;p)\right|\\&\quad\le K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\right)\left(\left(\tau_2(x)-\tau_2(a)\right)\left(\left|\tau_1'(x)\right|+\left|\tau_1'(a)\right|\right)\right)\\&\qquad+K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\right)\left(\left(\tau_2(b)-\tau_2(x)\right)\left(\left|\tau_1'(b)\right|+\left|\tau_1'(x)\right|\right),\end{aligned}\tag{1.16}$$

where

$$\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\tau_1*\tau_2\right)(x,\omega;p):=\int_a^x K^t_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\right)\tau_1'(t)\,d\left(\tau_2(t)\right),$$

$$\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}\tau_1*\tau_2\right)(x,\omega;p):=\int_x^b K^x_t\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\right)\tau_1'(t)\,d\left(\tau_2(t)\right).$$

In the upcoming section, we establish the bounds of a unified integral operator using strongly convexity. An Hadamard inequality is obtained for these integral operators via strongly convex function. A modulus inequality is obtained for differentiable functions by utilizing strongly convexity of $|\tau_1'|$ for unified integral operators. Furthermore, refinements of results given in [8, 16] are identified. In the whole paper we will use

$$I(a,b,\tau_2)=:\frac{1}{b-a}\int_a^b\tau_2(t)\,dt.$$

2 Main results

Bounds of unified integral operators (1.10) and (1.11) are studied in the following result:

Theorem 4 *Let $\tau_1 : [a,b] \longrightarrow \mathbb{R}$ be a positive, integrable and strongly convex function with $m \in (0,1]$. Let $\tau_2 : [a,b] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function, also let $\frac{\phi}{x}$ be an increasing function on $[a,b]$. If $\alpha, \beta, l, \gamma, c \in \mathbb{C}$, $\Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$, $p, \mu, \delta \geq 0$, and $0 < k \leq \delta + \mu$, then for $x \in (a,b)$ we have*

$$\begin{aligned}
&\big({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\tau_1\big)(x,\omega;p) + \big({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}\tau_1\big)(x,\omega;p) \\
&\quad \leq K^a_x\big(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\big)\big(\big(\tau_2(x)-\tau_2(a)\big)\big(\tau_1(x)+\tau_1(a)\big) \\
&\qquad - G(x-a)\big(2I(a,x,I_d\tau_2)-(a+x)I(a,x,\tau_2)\big)\big) \\
&\qquad + K^x_b\big(E^{\gamma,\delta,k,c}_{\mu,\beta,l},\tau_2;\phi\big)\big(\big(\tau_2(b)-\tau_2(x)\big)\big(\tau_1(b)+\tau_1(x)\big) \\
&\qquad - G(b-x)\big(2I(x,b,I_d\tau_2)-(x+b)I(x,b,\tau_2)\big)\big),
\end{aligned} \tag{2.1}$$

where I_d is the identity function.

Proof Let $t \in (a,x)$. Then the following inequality holds:

$$K^t_x\big(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\big)\tau_2'(t) \leq K^a_x\big(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\big)\tau_2'(t), \quad x \in (a,b). \tag{2.2}$$

Since τ_1 is a strongly convex function, for τ_1 the following inequality holds true:

$$\tau_1(t) \leq \left(\frac{x-t}{x-a}\right)\tau_1(a) + \left(\frac{t-a}{x-a}\right)\tau_1(x) - G(x-t)(t-a). \tag{2.3}$$

Multiplying (2.2) with (2.3) and integrating over $[a,x]$, one can obtain

$$\begin{aligned}
&\int_a^x K^t_x\big(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\big)\tau_1(t)\,d\big(\tau_2(t)\big) \\
&\quad \leq K^a_x\big(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\big)\left(\tau_1(a)\int_a^x\left(\frac{x-t}{x-a}\right)d\big(\tau_2(t)\big) + \tau_1(x)\int_a^x\left(\frac{t-a}{x-a}\right)d\big(\tau_2(t)\big)\right. \\
&\qquad \left. - G\int_a^x (x-t)(t-a)\,d\big(\tau_2(t)\big)\right).
\end{aligned}$$

By using (1.10) of Definition 4 and integrating by parts, the following inequality is obtained:

$$\begin{aligned}
&\big({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\tau_1\big)(x,\omega;p) \\
&\quad \leq K^a_x\big(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\big)\big(\big(\tau_2(x)-\tau_2(a)\big) \\
&\qquad \times \big(\tau_1(x)+\tau_1(a)\big)\big) - G(x-a)\big(2I(a,x,I_d\tau_2)-(a+x)I(a,x,\tau_2)\big)\big).
\end{aligned} \tag{2.4}$$

On the other hand, the following inequality holds true:

$$K^t_x\big(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\big)\tau_2'(t) \leq K^x_b\big(E^{\gamma,\delta,k,c}_{\mu,\beta,l},\tau_2;\phi\big)\tau_2'(t), \quad x \in (a,b). \tag{2.5}$$

Using strongly convexity of τ_1, we have

$$\tau_1(t) \leq \left(\frac{t-x}{b-x}\right)\tau_1(b) + \left(\frac{b-t}{b-x}\right)\tau_1(x) - G(t-x)(b-t). \tag{2.6}$$

Adopting the same procedure as we did for (2.2) and (2.3), the following inequality from (2.5) and (2.6) can be obtained:

$$\begin{aligned}
&\big({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}\tau_1\big)(x,\omega;p) \qquad (2.7)\\
&\quad\le K^x_b\big(E^{\gamma,\delta,k,c}_{\mu,\beta,l},\tau_2;\phi\big)\\
&\qquad\times\Big(\big(\tau_2(b)-\tau_2(x)\big)\big(\tau_1(b)+\tau_1(x)\big)-G(b-x)\big(2I(x,b,I_d\tau_2)-(x+b)I(x,b,\tau_2)\big)\Big).
\end{aligned}$$

By adding (2.4) and (2.7), (2.1) is obtained. □

Remark 1 (i) If we take $G = 0$ in (2.1), then (1.14) is obtained. In other words, (2.1) provides a refinement of (1.14).

(ii) If we take $\phi(t) = \frac{\Gamma(\alpha)t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $G = 0$, $\tau_2(x) = x$, and $p = \omega = 0$ in (1.14), then [9, Theorem 1] is obtained. For $G \neq 0$, we get its refinement.

(iii) If we take $\alpha = \beta$ in the result of (ii), then [9, Corollary 1] is obtained. For $G \neq 0$, we get its refinement.

(iv) If we take $\phi(t) = \frac{\Gamma(\alpha)t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $G = 0$, and $p = \omega = 0$ in (1.14), then [12, Theorem 6] is obtained. For $G \neq 0$, we get its refinement.

(v) If we take $\alpha = \beta$ in the result of (iv), then [12, Corollary 7] is obtained. For $G \neq 0$, we get its refinement.

(vi) If we take $\phi(t) = \Gamma(\alpha)t^{\alpha}$, $G = 0$, and $\tau_2(t) = t$ in (1.14), then [4, Corollary 1] is obtained. For $G \neq 0$, we get its refinement.

Corollary 1 *If we take $\phi(t) = \Gamma(\alpha)t^{\alpha}$, $p = \omega = 0$ in (2.1), then the following inequality is obtained for fractional integral operators defined in* [15]:

$$\begin{aligned}
&\Gamma(\alpha)I^{\alpha,\tau_2}_{a^+}\tau_1(x)+\Gamma(\beta)I^{\beta,\tau_2}_{b^-}\tau_1(x) \qquad (2.8)\\
&\quad\le\big(\tau_2(x)-\tau_2(a)\big)^{\alpha-1}\Big(\big(\tau_1(x)\tau_2(x)-\tau_1(a)\tau_2(a)\big)-\big(\tau_1(x)-\tau_1(a)\big)I(a,x,\tau_2)\Big)\\
&\qquad+\big(\tau_2(b)-\tau_2(x)\big)^{\beta-1}\Big(\big(\tau_1(b)\tau_2(b)-\tau_1(x)\tau_2(x)\big)-\big(\tau_1(b)-\tau_1(x)\big)I(x,b,g)\Big)\\
&\qquad-G(\big(\tau_2(b)-\tau_2(x)\big)^{\beta-1}(b-x)(2I(x,b,I_d\tau_2)-(x+b)I(x,b,\tau_2)\\
&\qquad+\big(\tau_2(x)-\tau_2(a)\big)^{\alpha-1}(x-a)\big(2I(a,x,I_d\tau_2)-(x+a)I(a,x,\tau_2)\big).
\end{aligned}$$

Remark 2 (i) If we take $G = 0$ and $\alpha = \beta$ in (2.8), then [13, Theorem 1] is obtained. For $G \neq 0$, we get its refinement.

(ii) If $\alpha = \beta$ in the result of (i), then [13, Corollary 1] is obtained. For $G \neq 0$, we get its refinement.

(iii) If we take $\tau_2(x) = x$ and $G = 0$ in (2.8), then [8, Theorem 1] is obtained. For $G \neq 0$, we get its refinement.

(iv) If we take $\alpha = \beta$ in the result of (iii), then [8, Corollary 1] is obtained. For $G \neq 0$, we get its refinement.

(v) If we take $\alpha = \beta = 1$ and $x = a$ or $x = b$ in the result of (iv), then [8, Corollary 2] is obtained. For $G \neq 0$, we get its refinement.

(vi) If we take $\alpha = \beta = 1$ and $x = \frac{a+b}{2}$ in the result of (iv), then [8, Corollary 3] is obtained. For $G \neq 0$, we get its refinement.

To prove the the next result we need the following lemma.

Lemma 1 *Let $\tau_1 : [a,b] \to \mathbb{R}$ be a strongly convex function. If τ_1 is symmetric about $\frac{a+b}{2}$, then the following inequality holds true*:

$$\tau_1\left(\frac{a+b}{2}\right) \le \tau_1(x) - \frac{G}{4}(a+b-2x)^2, \tag{2.9}$$

for all $x \in [a,b]$.

Proof Since τ_1 is strongly convex, we have

$$\begin{aligned}\tau_1\left(\frac{a+b}{2}\right) &\le \frac{1}{2}\left[\tau_1\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a\right) + \tau_1\left(\frac{x-a}{b-a}a + \frac{b-x}{b-a}b\right)\right] - \frac{G}{4}(a+b-2x)^2 \\ &= \frac{1}{2}\big(\tau_1(x) + \tau_1(a+b-x)\big) - \frac{G}{4}(a+b-2x)^2.\end{aligned}$$

As τ_1 is symmetric about $\frac{a+b}{2}$, we have $\tau_1(x) = \tau_1(a+b-x)$ and (2.9) holds. □

Remark 3 Lemma 1 is a refinement of [8, Lemma 1].

The upcoming result gives the Hadamard inequality.

Theorem 5 *Under the assumptions of Theorem 4, in addition if $\tau_1(x) = \tau_1(a+b-x)$, then we have*

$$\begin{aligned}&\tau_1\left(\frac{a+b}{2}\right)\big({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\big)(a,\omega;p) + \frac{G}{4}\big({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}(a+b-2x)^2\big)(a,\omega;p) \\ &\quad + \tau_1\left(\frac{a+b}{2}\right)\big({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}1\big)(b,\omega;p) + \frac{G}{4}\big({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}(a+b-2x)^2\big)(b,\omega;p) \\ &\quad \le \big({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}\tau_1\big)(a,\omega;p) + \big({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}\tau_1\big)(b,\omega;p) \\ &\quad \le \big(K^a_b\big(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\big) + K^a_b\big(E^{\gamma,\delta,k,c}_{\mu,\beta,l},\tau_2;\phi\big)\big) \\ &\qquad \times \big(\big(\tau_2(b)-\tau_2(a)\big)\big(\tau_1(b)+\tau_1(a)\big) - (b-a)G\big(2I(a,b,I_d\tau_2) - (a+b)I(a,b,g)\big)\big).\end{aligned} \tag{2.10}$$

Proof The following inequality holds true:

$$K^a_x\big(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\big)\tau_2'(x) \le K^a_b\big(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi\big)\tau_2'(x), x \in (a,b). \tag{2.11}$$

Using strongly convexity of τ_1 for $x \in (a,b)$, we have

$$\tau_1(x) \le \left(\frac{x-a}{b-a}\right)\tau_1(b) + \left(\frac{b-x}{b-a}\right)\tau_1(a) - G(x-a)(b-x). \tag{2.12}$$

Multiplying (2.11) and (2.12) and integrating the resulting inequality over $[a,b]$, one can obtain

$$\begin{aligned}&\int_a^b K_x^a\big(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\tau_1(x)\,d\big(\tau_2(x)\big)\\&\quad\le K_b^a\big(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\bigg(\tau_1(a)\int_a^b\bigg(\frac{b-x}{b-a}\bigg)\,d\big(\tau_2(x)\big)\\&\qquad+\tau_1(b)\int_a^b\bigg(\frac{x-a}{b-a}\bigg)\,d\big(\tau_2(x)\big)-G\int_a^b(b-x)(x-a)\,d\big(\tau_2(x)\big)\bigg).\end{aligned}$$

By using Definition 4 and integrating by parts, the following inequality is obtained:

$$\begin{aligned}&\big({}_{\tau_2}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}\tau_1\big)(a,\omega;p)\\&\quad\le K_b^a\big(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\\&\qquad\times\big(\big(\tau_2(b)-\tau_2(a)\big)\big(\tau_1(b)+\tau_1(a)\big)-G(b-a)\big(2I(a,b,I_d\tau_2)-(a+b)I(a,b,g)\big)\big).\end{aligned}\tag{2.13}$$

On the other hand, the following inequality holds true:

$$K_b^x\big(E_{\mu,\beta,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\tau_2'(x)\le K_b^a\big(E_{\mu,\beta,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\tau_2'(x),x\in(a,b).\tag{2.14}$$

Adopting the same pattern of simplification as we did for (2.11) and (2.12), the following inequality can be observed from (2.12) and (2.14):

$$\begin{aligned}&\big({}_{\tau_2}F_{\mu,\beta,l,a^+}^{\phi,\gamma,\delta,k,c}\tau_1\big)(b,\omega;p)\\&\quad\le K_b^a\big(E_{\mu,\beta,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\\&\qquad\times\big(\big(\tau_2(b)-\tau_2(a)\big)\big(\tau_1(b)+\tau_1(a)\big)-G(b-a)\big(2I(a,b,I_d\tau_2)-(a+b)I(a,b,g)\big)\big).\end{aligned}\tag{2.15}$$

By adding (2.13) and (2.15), the following inequality can be obtained:

$$\begin{aligned}&\big({}_{\tau_2}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}\tau_1\big)(a,\omega;p)+\big({}_{\tau_2}F_{\mu,\beta,l,a^+}^{\phi,\gamma,\delta,k,c}\tau_1\big)(b,\omega;p)\\&\quad\le\big(K_b^a\big(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)+K_b^a\big(E_{\mu,\beta,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\big)\\&\qquad\times\big(\big(\tau_2(b)-\tau_2(a)\big)\big(\tau_1(b)+\tau_1(a)\big)-G(b-a)\big(2I(a,b,I_d\tau_2)-(a+b)I(a,b,g)\big)\big).\end{aligned}\tag{2.16}$$

Multiplying both sides of (2.9) by $K_b^x(E_{\mu,\beta,l}^{\gamma,\delta,k,c},\tau_2;\phi)d(\tau_2(x))$ and integrating over $[a,b]$, we have

$$\begin{aligned}&\tau_1\bigg(\frac{a+b}{2}\bigg)\int_a^b K_b^x\big(E_{\mu,\beta,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\,d\big(\tau_2(x)\big)\\&\quad\le\int_a^b K_b^x\big(E_{\mu,\beta,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\tau_1(x)\,d\big(\tau_2(x)\big)\\&\qquad-\frac{G}{4}\int_a^b K_b^x\big(E_{\mu,\beta,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)(a+b-2x)^2\,d\big(\tau_2(x)\big).\end{aligned}$$

From Definition 4, the following inequality is obtained:

$$\begin{aligned}&\tau_1\left(\frac{a+b}{2}\right)\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}1\right)(b,\omega;p)\\&\le\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}\tau_1\right)(b,\omega;p)\\&\quad-\frac{G}{4}\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}(a+b-2x)^2\right)(b,\omega;p).\end{aligned}\tag{2.17}$$

Similarly, multiplying both sides of (2.9) by $K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\tau_2;\phi)\,d(\tau_2(x))$ and integrating over $[a,b]$, we have

$$\begin{aligned}&\tau_1\left(\frac{a+b}{2}\right)\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a,\omega;p)\\&\le\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}\tau_1\right)(a,\omega;p)\\&\quad-\frac{G}{4}\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}(a+b-2x)^2\right)(a,\omega;p).\end{aligned}\tag{2.18}$$

By adding (2.17) and (2.18), following inequality is obtained:

$$\begin{aligned}&\tau_1\left(\frac{a+b}{2}\right)\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a,\omega;p)+\frac{G}{4}\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}(a+b-2x)^2\right)(a,\omega;p)\\&\quad+\tau_1\left(\frac{a+b}{2}\right)\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}1\right)(b,\omega;p)+\frac{G}{4}\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}(a+b-2x)^2\right)(b,\omega;p)\\&\le\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}\tau_1\right)(a,\omega;p)+\left({}_{\tau_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}\tau_1\right)(b,\omega;p).\end{aligned}\tag{2.19}$$

Using (2.16) and (2.19), inequality (2.10) can be established. □

Remark 4 (i) If we take $G=0$ and $\alpha=\beta$ in (2.10), then (1.15) is obtained, for $G\neq 0$, we get its refinement.

(ii) If we take $\phi(t)=\Gamma(\alpha)t^{\alpha+1}$, $p=\omega=0$ and $G=0$ in (2.10), then [13, Theorem 3] is obtained. For $G\neq 0$, we get its refinement.

(iii) If $\alpha=\beta$ in the result of (ii), then [13, Corollary 3] is obtained. For $G\neq 0$, we get its refinement.

(iv) If we take $\phi(t)=\Gamma(\alpha)t^{\frac{\alpha}{k}+1}$, $G=0$, $\tau_2(x)=x$ and $p=\omega=0$ in (2.10), then [9, Theorem 3] is obtained. For $G\neq 0$, we get its refinement.

(v) If we take $\alpha=\beta$ in the result of (iv), then [9, Corollary 6] is obtained. For $G\neq 0$, we get its refinement.

(vi) If we take $\phi(t)=\Gamma(\alpha)t^{\frac{\alpha}{k}+1}$, $G=0$ and $p=\omega=0$ in (2.10), then [12, Theorem 11] is obtained. For $G\neq 0$, we get its refinement.

(vii) If we take $\alpha=\beta$ in the result of (vi), then [12, Corollary 12] is obtained. For $G\neq 0$, we get its refinement.

(viii) If we take $\phi(t)=t^{\alpha+1}$, $\tau_2(t)=t$ and $G=0$ in (2.10), then [4, Corollary 3] is obtained. For $G\neq 0$, we get its refinement.

Corollary 2 *If we take $p = \omega = 0$ in (2.10), then the following Hadamard inequality is obtained for fractional integral operators defined in [10]:*

$$\begin{aligned}&\tau_1\left(\frac{a+b}{2}\right)\left(\frac{1}{\Gamma(\alpha)}F_{b^-}^{\phi,\tau_2}1\right)(a)+\frac{G}{4}\left(\frac{1}{\Gamma(\alpha)}F_{b^-}^{\phi,\tau_2}(a+b-2x)^2\right)(a)\\&\quad+\tau_1\left(\frac{a+b}{2}\right)\left(\frac{1}{\Gamma(\beta)}F_{a^+}^{\phi,\tau_2}1\right)(b)+\frac{G}{4}\left(\frac{1}{\Gamma(\beta)}F_{a^+}^{\phi,\tau_2}(a+b-2x)^2\right)(b)\\&\quad\le\left(\frac{1}{\Gamma(\alpha)}F_{b^-}^{\phi,\tau_2}\tau_1\right)(a)+\left(\frac{1}{\Gamma(\beta)}F_{a^+}^{\phi,\tau_2}\tau_1\right)(b)\\&\quad\le 2K_{\tau_2}(t,x;\phi)\\&\qquad\times\big(\big(\tau_2(b)-\tau_2(a)\big)\big(\tau_1(b)+\tau_1(a)\big)-G(b-x)\big(2I(x,b,I_d\tau_2)-(x+b)I(x,b,\tau_2)\big)\big).\end{aligned}\tag{2.20}$$

Remark 5 (i) If we take $\phi(t)=\Gamma(\alpha)t^{\alpha+1}$, $p=\omega=0$, $G=0$, and $\tau_2(t)=t$ in (2.20), [8, Theorem 3] is obtained. For $G\neq 0$, we get its refinement.

(ii) If we take $\alpha=\beta$ in the result of (i), then [8, Corollary 6] is obtained. For $G\neq 0$, we get its refinement.

Theorem 6 *Let $\tau_1:[a,b]\longrightarrow\mathbb{R}$ be a differentiable function. Let $|\tau_1'|$ be strongly convex with $m\in(0,1]$ and $\tau_2:[a,b]\longrightarrow\mathbb{R}$ be differentiable and strictly increasing function, also let $\frac{\phi}{x}$ be an increasing function on $[a,b]$. If $\alpha,\beta,l,\gamma,c\in\mathbb{C}$, $\Re(\alpha),\Re(l)>0$, $\Re(c)>\Re(\gamma)>0$, $p,\mu,\delta\ge 0$, and $0<k\le\delta+\mu$, then for $x\in(a,b)$ we have*

$$\begin{aligned}&\left|\left({}_{\tau_2}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}\tau_1*\tau_2\right)(x,\omega;p)+\left({}_{\tau_2}F_{\mu,\beta,l,b^-}^{\phi,\gamma,\delta,k,c}\tau_1*\tau_2\right)(x,\omega;p)\right|\\&\quad\le K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\tau_2;\phi\right)\\&\qquad\times\left(\big(\tau_2(x)-\tau_2(a)\big)\left(\left|\tau_1'(x)\right|+\left|\tau_1'(a)\right|\right)-G(x-a)\big(2I(a,x,I_d\tau_2)-(a+x)I(a,x,\tau_2)\big)\right)\\&\qquad+K_b^x\left(E_{\mu,\beta,l}^{\gamma,\delta,k,c},\tau_2;\phi\right)\left(\big(\tau_2(b)-\tau_2(x)\big)\left(\left|\tau_1'(b)\right|+\left|\tau_1'(x)\right|\right)\right.\\&\qquad\left.-G(b-x)\big(2I(x,b,I_d\tau_2)-(x+b)I(x,b,\tau_2)\big)\right).\end{aligned}\tag{2.21}$$

Proof Using strongly convexity of $|\tau_1'|$, we have

$$\left|\tau_1'(t)\right|\le\left(\frac{x-t}{x-a}\right)\left|\tau_1'(a)\right|+\left(\frac{t-a}{x-a}\right)\left|\tau_1'(x)\right|-G(x-t)(t-a).\tag{2.22}$$

Inequality (2.22) can be written as follows:

$$\begin{aligned}&-\left(\left(\frac{x-t}{x-a}\right)\left|\tau_1'(a)\right|+\left(\frac{t-a}{x-a}\right)\left|\tau_1'(x)\right|-G(x-t)(t-a)\right)\\&\quad\le\tau_1'(t)\\&\quad\le\left(\left(\frac{x-t}{x-a}\right)\left|\tau_1'(a)\right|+\left(\frac{t-a}{x-a}\right)\left|\tau_1'(x)\right|-G(x-t)(t-a)\right).\end{aligned}\tag{2.23}$$

Let us consider the second inequality of (2.23), namely

$$\tau_1'(t)\le\left(\frac{x-t}{x-a}\right)\left|\tau_1'(a)\right|+\left(\frac{t-a}{x-a}\right)\left|\tau_1'(x)\right|-G(x-t)(t-a).\tag{2.24}$$

Multiplying (2.2) and (2.24) and integrating over $[a,x]$, we can obtain

$$\begin{aligned}
&\int_a^x K_x^t\big(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\tau_1'(t)\,d\big(\tau_2(t)\big) \\
&\quad\le K_x^a\big(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\bigg(\big|\tau_1'(a)\big|\int_a^x\left(\frac{x-t}{x-a}\right)d\big(\tau_2(t)\big) \\
&\qquad+\big|\tau_1'(x)\big|\int_a^x\left(\frac{t-a}{x-a}\right)d\big(\tau_2(t)\big)-G\int_a^x(x-t)(t-a)\,d\big(\tau_2(t)\big)\bigg).
\end{aligned}$$

By using (1.10) of Definition 4 and integrating by parts, the following inequality is obtained:

$$\begin{aligned}
&\big({}_{\tau_2}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}\tau_1*\tau_2\big)(x,\omega;p) \\
&\quad\le K_x^a\big(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\big(\big(\tau_2(x)-\tau_2(a)\big) \\
&\qquad\times\big(\big|\tau_1'(x)\big|+\big|\tau_1'(a)\big|\big)-G(x-a)\big(2I(a,x,I_d\tau_2)-(a+x)I(a,x,\tau_2)\big)\big).
\end{aligned} \tag{2.25}$$

If we consider the left-hand side of inequality (2.23) and adopt the same argument as for the right-hand side inequality, then we get

$$\begin{aligned}
&\big({}_{\tau_2}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}(\tau_1*\tau_2)\big)(x,\omega;p) \\
&\quad\ge -K_x^a\big(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\big(\big(\tau_2(x)-\tau_2(a)\big) \\
&\qquad\times\big(\big|\tau_1'(x)\big|+\big|\tau_1'(a)\big|\big)-G(x-a)\big(2I(a,x,I_d\tau_2)-(a+x)I(a,x,\tau_2)\big)\big).
\end{aligned} \tag{2.26}$$

From (2.25) and (2.26), the following inequality is obtained:

$$\begin{aligned}
&\big|\big({}_{\tau_2}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}(\tau_1*\tau_2)\big)(x,\omega;p)\big| \\
&\quad\le K_x^a\big(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\big(\big(\tau_2(x)-\tau_2(a)\big) \\
&\qquad\times\big(\big|\tau_1'(x)\big|+\big|\tau_1'(a)\big|\big)-G(x-a)\big(2I(a,x,I_d\tau_2)-(a+x)I(a,x,\tau_2)\big)\big).
\end{aligned} \tag{2.27}$$

Now using the strongly convexity of $|\tau_1'|$, we have

$$\big|\tau_1'(t)\big|\le\left(\frac{t-x}{b-x}\right)\big|\tau_1'(b)\big|+\left(\frac{b-t}{b-x}\right)\big|\tau_1'(x)\big|-G(t-x)(b-t). \tag{2.28}$$

With the same procedure as that used for (2.2) and (2.22), one can obtain the following inequality from (2.5) and (2.28):

$$\begin{aligned}
&\big|\big({}_{\tau_2}F_{\mu,\beta,l,b^-}^{\phi,\gamma,\delta,k,c}(\tau_1*\tau_2)\big)(x,\omega;p)\big| \\
&\quad\le K_b^x\big(E_{\mu,\beta,l}^{\gamma,\delta,k,c},\tau_2;\phi\big)\big(\big(\tau_2(b)-\tau_2(x)\big) \\
&\qquad\times\big(\big|\tau_1'(b)\big|+\big|\tau_1'(x)\big|\big)-G(b-x)\big(2I(x,b,I_d\tau_2)-(x+b)I(x,b,\tau_2)\big)\big).
\end{aligned} \tag{2.29}$$

By adding (2.27) and (2.29), inequality (2.21) can be achieved. □

Remark 6 (i) If we take $G = 0$ and $\alpha = \beta$ in (2.21), then (1.16) is obtained. For $G \neq 0$, we get its refinement.

(ii) If we take $\phi(t) = \Gamma(\alpha)t^{\frac{\alpha}{k}+1}$, $G = 0$, $\tau_2(x) = x$ and $p = \omega = 0$ in (2.21), then [9, Theorem 2] is obtained. For $G \neq 0$, we get its refinement.

(iii) If we take $\alpha = \beta$ in the result of (ii), then [9, Corollary 4] is obtained. For $G \neq 0$, we get its refinement.

(iv) If we take $\alpha = \beta = k = 1$ and $x = \frac{a+b}{2}$, in the result of (iii), then [9, Corollary 5] is obtained. For $G \neq 0$, we get its refinement.

(v) If we take $\phi(t) = t^{\alpha+1}$, $\tau_2(x) = x$, $p = \omega = 0$ and $G = 0$ in (2.21), then [8, Theorem 2] is obtained. For $G \neq 0$, we get its refinement.

(vi) If we take $\alpha = \beta$ in the result of (v), then [8, Corollary 5] is obtained. For $G \neq 0$, we get its refinement.

(vii) If we take $\phi(t) = \Gamma(\alpha)t^{\frac{\alpha}{k}+1}$, $\tau_2(x) = x$, $p = \omega = 0$ and $G = 0$ in (2.21), then [12, Theorem 8] is obtained. For $G \neq 0$, we get its refinement.

(viii) If we take $\alpha = \beta$ in the result of (vii), then [12, Corollary 9] is obtained. For $G \neq 0$, we get its refinement.

(ix) If we take $\phi(t) = t^{\alpha}$, $\tau_2(x) = x$ and $G = 0$ in (2.21), then [4, Corollary 2] is obtained. For $G \neq 0$, we get its refinement.

3 Concluding remarks

This research provides inequalities for unified integral operators for strongly convex functions, refined form of convex functions. These inequalities provide refinements of the results proved in already published works. The special cases also provide results for fractional integral operators and their refinements.

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