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Convergence rate of Euler–Maruyama scheme for SDDEs of neutral type

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Abstract

In this paper, we are concerned with the convergence rate of Euler–Maruyama (EM) scheme for stochastic differential delay equations (SDDEs) of *neutral type*, where the neutral, drift, and diffusion terms are allowed to be of polynomial growth. More precisely, for SDDEs of neutral type driven by Brownian motions, we reveal that the convergence rate of the corresponding EM scheme is one-half; Whereas for SDDEs of neutral type driven by pure jump processes, we show that the best convergence rate of the associated EM scheme is slower than one-half. As a result, the convergence rate of general SDDEs of neutral type, which is dominated by pure jump process, is slower than one-half.

MSC: 65C30; 60H10

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1 Introduction

There is numerous literature concerned with convergence rate of numerical schemes for stochastic differential equations (SDEs). It is well-known that the convergence rate of EM scheme for SDEs under global Lipschitz and linear growth condition is one-half (see, e.g., [12]). Under different conditions, the convergence rate of EM scheme for SDEs varies. For example, under the Khasminskii-type condition, Mao [11] revealed that the convergence rate of the truncated EM method is close to one-half; under the Hölder condition, the convergence rate of EM scheme for SDEs has been studied by many scholars (see, e.g., [7, 16, 17]); Sabanis [19] recovered the classical rate of convergence (i.e., one-half) for the tamed EM schemes, where, for the SDE involved, the drift coefficient satisfies a one-sided Lipschitz condition and a polynomial Lipschitz condition, and the diffusion term is Lipschitzian. In [2], Bao et al. investigated the convergence rate of EM scheme for SDEs with Hölder–Dini continuous drifts.

There is also some literature on the convergence rate of numerical schemes for stochastic functional differential equations (SFDEs). For example, under a log-Lipschitz condition, Bao et al. [5] studied the convergence rate of EM approximation for a range of SFDEs driven by jump processes; Bao and Yuan [4] investigated the convergence rate of EM approach for a class of SDDEs, where the drift and diffusion coefficients are allowed to be of

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polynomial growth with respect to the delay variables; Gyöngy and Sabanis [8] discussed the rate of almost sure convergence of Euler approximations for SDDEs under monotonicity conditions. In [31], Zhang et al. established the convergence of a class of highly nonlinear stochastic differential delay equations without the linear growth condition replacing by Khasminskii-type condition.

Increasingly real-world systems are modeled by SFDEs of neutral type, as they represent systems which evolve in a random environment and whose evolution depends on the past states and derivatives of states of the systems through either memory or time delay. In the last decade, for SFDEs of neutral type, there are a large number of papers on, e.g., stochastic stability (see, e.g., [12, 13, 26]), on large fluctuations (see, e.g., [1]), on large deviation principle (see, e.g., [6]), on transportation inequality (see, e.g., [3]), to name a few.

Since most SFDEs of neutral types cannot be solved explicitly, the topic on numerical approximations for SFDEs of neutral type has also been investigated considerably. For instance, under a global Lipschitz condition, Wu and Mao [21] revealed that the convergence rate of the EM scheme constructed is close to one-half; under a log-Lipschitz condition, Jiang et al. [9] generalized [24] by Yuan and Mao to the neutral case; under the Khasminskii-type condition, following the line of Yuan and Glover [25], Milosevic [15] and Zhou [27] studied the convergence in probability of the associated EM scheme; while in [22], Yan et al. proved the strong convergence of the split-step theta method for SFDEs of neutral type with convergence rate of one-half. In [28], Zhou and Jin investigated the strong convergence of the implicit numerical approximations for SFDEs of neutral type with superlinearly growing coefficients. For preserving stochastic stability (of the exact solutions) of variable numerical schemes, we refer to, e.g., [10, 23, 29, 30] and the references therein.

We remark that most of the existing literature on the convergence rate of explicit EM scheme for SFDEs of neutral type has dealt with the Lipschitz-type condition, where, in particular, the neutral term is contractive. For example, Obradović et al. [18] discussed the convergence in probability of the explicit EM method for neutral stochastic systems with unbounded delay and Markovian switching under local Lipschitz conditions. To the best of our knowledge, the convergence rate of explicit EM scheme for SFDEs of neutral type with non-Lipschitz conditions (hence nonlinear) has seen few results. Consider the following SDDE of neutral type:

$$d\{X(t) - X^2(t - \tau)\} = \{aX(t) + bX^3(t - \tau)\} dt + cX^2(t - \tau) dB(t), \quad t \geq 0, \tag{1.1}$$

in which $a, b, c \in \mathbb{R}$, $\tau > 0$ are some constants, and $B(t)$ is a scalar Brownian motion. Observe that all the neutral, drift, and diffusion coefficients in (1.1) are highly nonlinear with respect to the delay variable so that the existing results on the convergence rate of EM schemes associated with SFDEs of neutral type cannot be applied to the example above. In this paper we intend to establish the theory on the convergence rate of EM scheme for a class of SDDEs of neutral type, where, in particular, the neutral term is of polynomial growth, so that it would cover more interesting models.

Throughout the paper, the shorthand notation $a \lesssim b$ is used to express that there exists a positive constant c such that $a \leq cb$, where c is a generic constant whose value may change from line to line. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration

$(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and \mathcal{F}_0 contains all \mathbb{P} -null sets). For each integer $n \geq 1$, let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle, |\cdot|)$ be an n -dimensional Euclidean space. For $A \in \mathbb{R}^n \otimes \mathbb{R}^m$, the collection of all $n \times m$ matrices, $\|A\|$ stands for the Hilbert–Schmidt norm, i.e., $\|A\| = (\sum_{i=1}^m |Ae_i|^2)^{1/2}$, where $(e_i)_{i \geq 1}$ is an orthogonal basis of \mathbb{R}^m . For $\tau > 0$, which is referred to as delay or memory, $\mathcal{C} := C([-\tau, 0]; \mathbb{R}^n)$ means the space of all continuous functions $\phi : [-\tau, 0] \mapsto \mathbb{R}^n$ with the uniform norm $\|\phi\|_\infty := \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$. Let $(B(t))_{t \geq 0}$ be a standard m -dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

The result of this paper will be organized as follows: The convergence rate of two special cases of SDDEs of neutral type, one driven by Brownian motions and the other driven by Pure jump processes, will be discussed in Sects. 2 and 3, respectively. The convergence result of a general SDDEs of neutral type will be demonstrated in Sect. 4. Some numerical examples will be illustrated in Sect. 5. A conclusion will be presented in Sect. 6.

2 The Brownian motion case

To begin, we focus on an SDDE of neutral type on $(\mathbb{R}^n, \langle \cdot, \cdot \rangle, |\cdot|)$ in the form

$$d\{X(t) - G(X(t - \tau))\} = b(X(t), X(t - \tau)) dt + \sigma(X(t), X(t - \tau)) dB(t), \quad t > 0, \tag{2.1}$$

with the initial value $X(\theta) = \xi(\theta)$ for $\theta \in [-\tau, 0]$, where $G : \mathbb{R}^n \mapsto \mathbb{R}^n$, $b : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^{n \times m}$.

We assume that there exist constants $L > 0$ and $q \geq 1$ such that, for any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$,

$$(A1) \quad |G(y) - G(\bar{y})| \leq L(1 + |y|^q + |\bar{y}|^q)|y - \bar{y}|;$$

$$(A2) \quad |b(x, y) - b(\bar{x}, \bar{y})| + \|\sigma(x, y) - \sigma(\bar{x}, \bar{y})\| \leq L|x - \bar{x}| + L(1 + |y|^q + |\bar{y}|^q)|y - \bar{y}|, \text{ where } \|\cdot\| \text{ stands for the Hilbert–Schmidt norm};$$

$$(A3) \quad |\xi(t) - \xi(s)| \leq L|t - s| \text{ for any } s, t \in [-\tau, 0].$$

Remark 2.1 There are some examples such that (A1) and (A2) hold. For instance, if $G(y) = y^2$, $b(x, y) = \sigma(x, y) = ax + y^3$ for any $x, y \in \mathbb{R}$ and some $a \in \mathbb{R}$, then both (A1) and (A2) hold.

By following a similar argument to [12, Theorem 3.1, p. 210], (2.1) has a unique solution $\{X(t)\}$ under (A1) and (A2). In the sequel, we introduce the EM scheme associated with (2.1). Without loss of generality, we assume that $h = T/M = \tau/m \in (0, 1)$ for some integers $M, m > 1$. For every integer $k = -m, \dots, 0$, set $Y_h^{(k)} := \xi(kh)$, and for each integer $k = 1, \dots, M - 1$, we define

$$Y_h^{(k+1)} - G(Y_h^{(k+1-m)}) = Y_h^{(k)} - G(Y_h^{(k-m)}) + b(Y_h^{(k)}, Y_h^{(k-m)})h + \sigma(Y_h^{(k)}, Y_h^{(k-m)})\Delta B_h^{(k)}, \tag{2.2}$$

where $\Delta B_h^{(k)} := B((k + 1)h) - B(kh)$. For any $t \in [kh, (k + 1)h]$, set $\bar{Y}(t) := Y_h^{(k)}$. To avoid the complex calculation, we define the continuous-time EM approximation solution $Y(t)$ as follows: for any $\theta \in [-\tau, 0]$, $Y(\theta) = \xi(\theta)$, and

$$Y(t) = G(\bar{Y}(t - \tau)) + \xi(0) - G(\xi(-\tau)) + \int_0^t b(\bar{Y}(s), \bar{Y}(s - \tau)) ds + \int_0^t \sigma(\bar{Y}(s), \bar{Y}(s - \tau)) dB(s), \quad t \in [0, T]. \tag{2.3}$$

A straightforward calculation shows that the continuous-time EM approximate solution $Y(t)$ coincides with the discrete-time approximation solution $\bar{Y}(t)$ at the grid points $t = nh$.

2.1 p th moment bound

The lemma below provides estimates of the p th moment of the solution to (2.1) and the corresponding EM scheme, alongside with the p th moment of the displacement.

Lemma 2.1 *Under (A1) and (A2), for any $p \geq 2$ there exists a constant $C_T > 0$ such that*

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |X(t)|^p\right) \vee \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y(t)|^p\right) \leq C_T \tag{2.4}$$

and

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Gamma(t)|^p\right) \lesssim h^{p/2}, \tag{2.5}$$

where $\Gamma(t) := Y(t) - \bar{Y}(t)$.

Proof We focus only on the following estimate:

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |Y(t)|^p\right) \leq C_T \tag{2.6}$$

for some constant $C_T > 0$ since the uniform p th moment of $X(t)$ in a finite time interval can be done similarly. From (A1) and (A2), one has

$$|G(y)| \lesssim 1 + |y|^{1+q} \tag{2.7}$$

and

$$|b(x, y)| + \|\sigma(x, y)\| \lesssim 1 + |x| + |y|^{1+q} \tag{2.8}$$

for any $x, y \in \mathbb{R}^n$. By the Hölder inequality, the Burkholder–Davis–Gundy (BDG) inequality (see, e.g., [12, Theorem 7.3, p. 40]), we derive from (2.7) and (2.8) that

$$\begin{aligned} \mathbb{E}\left(\sup_{-\tau \leq s \leq t} |Y(s)|^p\right) &\lesssim 1 + \|\xi\|_\infty^{p(1+q)} + \mathbb{E}\left(\sup_{-\tau \leq s \leq t-\tau} |\bar{Y}(s)|^{p(1+q)}\right) \\ &\quad + \int_0^t \{\mathbb{E}|\bar{Y}(s)|^p + \mathbb{E}|\bar{Y}(s-\tau)|^{p(1+q)}\} ds \\ &\lesssim 1 + \|\xi\|_\infty^{p(1+q)} + \mathbb{E}\left(\sup_{-\tau \leq s \leq t-\tau} |Y(s)|^{p(1+q)}\right) \\ &\quad + \int_0^T \mathbb{E}\left(\sup_{-\tau \leq r \leq s} |Y(r)|^p\right) ds, \end{aligned}$$

where we have used $Y(kh) = \bar{Y}(kh)$ in the last display. This, together with Gronwall’s inequality, yields that

$$\mathbb{E}\left(\sup_{0 \leq s \leq t} |Y(s)|^p\right) \lesssim 1 + \|\xi\|_\infty^{p(1+q)} + \mathbb{E}\left(\sup_{0 \leq s \leq (t-\tau) \vee 0} |X(s)|^{p(1+q)}\right),$$

which further implies that

$$\mathbb{E} \left(\sup_{0 \leq t \leq \tau} |X(t)|^p \right) \lesssim 1 + \|\xi\|_\infty^{p(1+q)}$$

and

$$\mathbb{E} \left(\sup_{0 \leq t \leq 2\tau} |X(t)|^p \right) \lesssim 1 + \mathbb{E} \|\xi\|_\infty^{p(1+q)} + \left(\sup_{0 \leq t \leq \tau} |X(t)|^{p(1+q)} \right) \lesssim 1 + \|\xi\|_\infty^{p(1+q)^2},$$

where we use the fact that $p_1 = p(1 + q) > 2$ and

$$\mathbb{E} \left(\sup_{0 \leq t \leq 2\tau} |X(t)|^{p_1} \right) \lesssim 1 + \mathbb{E} \|\xi\|_\infty^{p_1(1+q)} \lesssim 1 + \mathbb{E} \|\xi\|_\infty^{p(1+q)^2}.$$

Thus (2.6) follows from an inductive argument.

Employing Hölder’s and BDG inequalities, we deduce from (2.3) and (2.8) that

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq T} |\Gamma(t)|^p \right) &\lesssim \sup_{0 \leq k \leq M-1} \left\{ \mathbb{E} \left(\sup_{kh \leq t \leq (k+1)h} \left| \int_{kh}^t b(\bar{Y}(s), \bar{Y}(s - \tau)) \, ds \right|^p \right) \right. \\ &\quad \left. + \mathbb{E} \left(\sup_{nh \leq t \leq (k+1)h} \left| \int_{kh}^t \sigma(\bar{Y}(s), \bar{Y}(s - \tau)) \, dB(s) \right|^p \right) \right\} \\ &\lesssim \sup_{0 \leq k \leq M-1} \left\{ h^{p-1} \mathbb{E} \int_{kh}^{(k+1)h} |b(\bar{Y}(s), \bar{Y}(s - \tau))|^p \, ds \right. \\ &\quad \left. + h^{\frac{p}{2}-1} \mathbb{E} \int_{kh}^{(k+1)h} \|\sigma(\bar{Y}(s), \bar{Y}(s - \tau))\|^p \, ds \right\} \\ &\lesssim h^{\frac{p}{2}-1} \sup_{0 \leq k \leq M-1} \left\{ \int_{kh}^{(k+1)h} (1 + \mathbb{E} |\bar{Y}(s)|^p + \mathbb{E} |\bar{Y}(s - \tau)|^{p(q+1)}) \, ds \right\} \\ &\lesssim h^{\frac{p}{2}}, \end{aligned}$$

where in the last step we have used (2.6). The desired assertion is therefore proved. □

2.2 Convergence result

The first main result in this paper is stated as follows.

Theorem 2.2 *Under Assumptions (A1)–(A3),*

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^p \right) \lesssim h^{p/2}, \quad p \geq 2. \tag{2.9}$$

So the convergence rate of the EM scheme (i.e., (2.3)) associated with (2.1) is one-half.

With Lemma 2.1 in hand, we are now in the position to finish the proof of Theorem 2.2.

Proof of Theorem 2.2 We follow the Yamada–Watanabe approach (see, e.g., [4]) to complete the proof of Theorem 2.2. For a fixed $\kappa > 1$ and arbitrary $\varepsilon \in (0, 1)$, there exists a continuous nonnegative function $\varphi_{\kappa\varepsilon}(\cdot)$ with the support $[\varepsilon/\kappa, \varepsilon]$ such that

$$\int_{\varepsilon/\kappa}^\varepsilon \varphi_{\kappa\varepsilon}(x) \, dx = 1 \quad \text{and} \quad \varphi_{\kappa\varepsilon}(x) \leq \frac{2}{x \ln \kappa}, \quad x > 0.$$

Set

$$\phi_{\kappa\varepsilon}(x) := \int_0^x \int_0^y \varphi_{\kappa\varepsilon}(z) \, dz \, dy, \quad x > 0.$$

We can see that $\phi_{\kappa\varepsilon}(\cdot)$ is such that

$$x - \varepsilon \leq \phi_{\kappa\varepsilon}(x) \leq x, \quad x > 0. \tag{2.10}$$

Let

$$V_{\kappa\varepsilon}(x) = \phi_{\kappa\varepsilon}(|x|), \quad x \in \mathbb{R}^n. \tag{2.11}$$

By a straightforward calculation, it holds

$$(\nabla V_{\kappa\varepsilon})(x) = \phi'_{\kappa\varepsilon}(|x|)|x|^{-1}x, \quad x \in \mathbb{R}^n$$

and

$$(\nabla^2 V_{\kappa\varepsilon})(x) = \phi'_{\kappa\varepsilon}(|x|)(|x|^2\mathbf{I} - x \otimes x)|x|^{-3} + |x|^{-2}\phi''_{\kappa\varepsilon}(|x|)x \otimes x, \quad x \in \mathbb{R}^n,$$

where ∇ and ∇^2 stand for the gradient and Hessian operators, respectively, \mathbf{I} denotes the identity matrix, and $x \otimes x = xx^*$ with x^* being the transpose of $x \in \mathbb{R}^n$. Moreover, we have

$$|(\nabla V_{\kappa\varepsilon})(x)| \leq 1 \quad \text{and} \quad \|(\nabla^2 V_{\kappa\varepsilon})(x)\| \leq 2n \left(1 + \frac{1}{\ln \kappa}\right) \frac{1}{|x|} \mathbf{1}_{[\varepsilon/\kappa, \varepsilon]}(|x|), \tag{2.12}$$

where $\mathbf{1}_A(\cdot)$ is the indicator function of the subset $A \subset \mathbb{R}_+$.

For notation simplicity, set

$$Z(t) := X(t) - Y(t) \quad \text{and} \quad \Lambda(t) := Z(t) - G(X(t - \tau)) + G(\bar{Y}(t - \tau)). \tag{2.13}$$

In the sequel, let $t \in [0, T]$ be arbitrary and fix $p \geq 2$. Due to $\Lambda(0) = \mathbf{0} \in \mathbb{R}^n$ and $V_{\kappa\varepsilon}(\mathbf{0}) = 0$, an application of Itô's formula gives

$$\begin{aligned} V_{\kappa\varepsilon}(\Lambda(t)) &= \int_0^t \langle (\nabla V_{\kappa\varepsilon})(\Lambda(s)), \Gamma_1(s) \rangle \, ds \\ &\quad + \frac{1}{2} \int_0^t \text{trace}\{(\Gamma_2(s))^* (\nabla^2 V_{\kappa\varepsilon})(\Lambda(s)) \Gamma_2(s)\} \, ds \\ &\quad + \int_0^t \langle \nabla(V_{\kappa\varepsilon})(\Lambda(s)), \Gamma_2(s) \, dB(s) \rangle \\ &=: I_1(t) + I_2(t) + I_3(t), \end{aligned}$$

where

$$\Gamma_1(t) := b(X(t), X(t - \tau)) - b(\bar{Y}(t), \bar{Y}(t - \tau)) \tag{2.14}$$

and

$$\Gamma_2(t) := \sigma(X(t), X(t - \tau)) - \sigma(\bar{Y}(t), \bar{Y}(t - \tau)).$$

Set

$$V(x, y) := 1 + |x|^q + |y|^q, \quad x, y \in \mathbb{R}^n. \tag{2.15}$$

According to (2.4), for any $q \geq 2$ there exists a constant $C_T > 0$ such that

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} V(X(t - \tau), \bar{Y}(t - \tau))^q\right) \leq C_T. \tag{2.16}$$

Noting that

$$X(t) - \bar{Y}(t) = \Lambda(t) + \Gamma(t) + G(X(t - \tau)) - G(\bar{Y}(t - \tau)), \tag{2.17}$$

and using Hölder’s and BDG inequalities, we get from (2.12) and (A1)–(A2) that

$$\begin{aligned} \Theta(t) &:= \mathbb{E}\left(\sup_{0 \leq s \leq t} |I_1(s)|^p\right) + \mathbb{E}\left(\sup_{0 \leq s \leq t} |I_3(s)|^p\right) \\ &\lesssim \int_0^t \{ \mathbb{E}|\Gamma_1(s)|^p + \mathbb{E}\|\Gamma_2(s)\|^p \} ds \\ &\lesssim \int_0^t \mathbb{E}|X(s) - \bar{Y}(s)|^p ds + \int_{-\tau}^{t-\tau} \mathbb{E}(V(X(s), \bar{Y}(s))^p |X(s) - \bar{Y}(s)|^p) ds \\ &\lesssim \int_0^t \mathbb{E}\{ |\Lambda(s)|^p + |\Gamma(s)|^p \} ds + \int_{-\tau}^{t-\tau} \mathbb{E}(V(X(s), \bar{Y}(s))^p |X(s) - \bar{Y}(s)|^p) ds. \end{aligned} \tag{2.18}$$

Also, by Hölder’s inequality, it follows from (2.5), (A3), and (2.16) that

$$\begin{aligned} \Theta(t) &\lesssim \int_0^t \{ \mathbb{E}|\Lambda(s)|^p + \mathbb{E}|\Gamma(s)|^p + (\mathbb{E}V(X(s - \tau), \bar{Y}(s - \tau))^{2p})^{1/2} \\ &\quad \times ((\mathbb{E}|Z(s - \tau)|^{2p})^{1/2} + (\mathbb{E}|\Gamma(s - \tau)|^{2p})^{1/2}) \} ds \\ &\lesssim \int_0^t \{ \mathbb{E}|\Lambda(s)|^p + \mathbb{E}|\Gamma(s)|^p + (\mathbb{E}|Z(s - \tau)|^{2p})^{1/2} + (\mathbb{E}|\Gamma(s - \tau)|^{2p})^{1/2} \} ds \\ &\lesssim \int_0^t \{ \mathbb{E}|\Lambda(s)|^p + (\mathbb{E}|Z(s - \tau)|^{2p})^{1/2} + h^{p/2} \} ds. \end{aligned} \tag{2.19}$$

Using (2.12), we derive

$$\begin{aligned} \mathbb{E}\left(\sup_{0 \leq s \leq t} |I_2(s)|^p\right) &\lesssim \mathbb{E} \int_0^t \|\nabla^2 V_{\kappa \varepsilon}\|(\Lambda(s))\|^p \|\Gamma_2(s)\|^{2p} ds \\ &\lesssim \mathbb{E} \int_0^t \frac{1}{|\Lambda(s)|^p} \{ |X(s) - \bar{Y}(s)|^{2p} + V(X(s - \tau), \bar{Y}(s - \tau))^{2p} \\ &\quad \times (|X(s - \tau) - \bar{Y}(s - \tau)|^{2p}) \} \mathbf{I}_{[\varepsilon/\kappa, \varepsilon]}(|\Lambda(s)|) ds. \end{aligned}$$

In the light of (A1) and (2.13)–(2.17), then we have

$$\begin{aligned}
 & \mathbb{E} \left(\sup_{0 \leq s \leq t} |I_2(s)|^p \right) \\
 & \lesssim \mathbb{E} \int_0^t \frac{1}{|\Lambda(s)|^p} \{ |\Lambda(s)|^{2p} + |\Gamma(s)|^{2p} + |G(X(s-\tau)) - G(\bar{Y}(s-\tau))|^{2p} \\
 & \quad + V(X(s-\tau), \bar{Y}(s-\tau))^{2p} (|X(s-\tau) - \bar{Y}(s-\tau)|^{2p}) \} \mathbf{I}_{[\varepsilon/\kappa, \varepsilon]}(|\Lambda(s)|) \, ds \\
 & \lesssim \mathbb{E} \int_0^t \frac{1}{|\Lambda(s)|^p} \{ |\Lambda(s)|^{2p} + |\Gamma(s)|^{2p} \\
 & \quad + V(X(s-\tau), \bar{Y}(s-\tau))^{2p} (|X(s-\tau) - \bar{Y}(s-\tau)|^{2p}) \} \mathbf{I}_{[\varepsilon/\kappa, \varepsilon]}(|\Lambda(s)|) \, ds \tag{2.20} \\
 & \lesssim \mathbb{E} \int_0^t \{ |\Lambda(s)|^p + \varepsilon^{-p} |\Gamma(s)|^{2p} \\
 & \quad + \varepsilon^{-p} V^{2p}(X(s-\tau), \bar{Y}(s-\tau)) (|X(s-\tau) - \bar{Y}(s-\tau)|^{2p}) \} \, ds \\
 & \lesssim \int_0^t \{ \varepsilon^{-p} h^p + \mathbb{E} |\Lambda(s)|^p + \varepsilon^{-p} (\mathbb{E} (|Z(s-\tau)|^{4p}))^{1/2} \} \, ds,
 \end{aligned}$$

where in the last step we have used Hölder’s inequality. Now, according to (2.10), (2.19), and (2.20), one has

$$\begin{aligned}
 \mathbb{E} \left(\sup_{0 \leq s \leq t} |\Lambda(s)|^p \right) & \lesssim \varepsilon^p + \mathbb{E} \left(\sup_{0 \leq s \leq t} V_{\kappa \varepsilon}(\Lambda(s)) \right) \\
 & \lesssim \varepsilon^p + \Theta(t) + \mathbb{E} \left(\sup_{0 \leq s \leq t} |I_3(s)|^p \right) \\
 & \lesssim \varepsilon^p + \int_0^t \left\{ h^{p/2} + \varepsilon^{-p} h^p + \mathbb{E} \left(\sup_{0 \leq r \leq s} |\Lambda(r)|^p \right) \right. \\
 & \quad \left. + (\mathbb{E} (|Z(s-\tau)|^{2p}))^{1/2} + \varepsilon^{-p} (\mathbb{E} (|Z(s-\tau)|^{4p}))^{1/2} \right\} \, ds.
 \end{aligned}$$

Thus, Gronwall’s inequality gives

$$\begin{aligned}
 \mathbb{E} \left(\sup_{0 \leq s \leq t} |\Lambda(s)|^p \right) & \lesssim \varepsilon^p + h^{p/2} + \varepsilon^{-p} h^p \\
 & \quad + \int_0^{(t-\tau) \vee 0} \left\{ (\mathbb{E} (|Z(s)|^{2p}))^{1/2} + \varepsilon^{-p} (\mathbb{E} (|Z(s)|^{4p}))^{1/2} \right\} \, ds \tag{2.21} \\
 & \lesssim h^{p/2} + \int_0^{(t-\tau) \vee 0} \left\{ (\mathbb{E} (|Z(s)|^{2p}))^{1/2} + \varepsilon^{-p} (\mathbb{E} (|Z(s)|^{4p}))^{1/2} \right\} \, ds,
 \end{aligned}$$

by choosing $\varepsilon = h^{1/2}$ and taking $|Z(t)| \equiv 0$ for $t \in [-\tau, 0]$ into account. Next, by (A1) and (2.16), it follows from Hölder’s inequality that

$$\begin{aligned}
 & \mathbb{E} \left(\sup_{0 \leq t \leq T} |Z(t)|^p \right) \\
 & \lesssim \mathbb{E} \left(\sup_{0 \leq t \leq T} |\Lambda(t)|^p \right) + \mathbb{E} \left(\sup_{-\tau \leq t \leq T-\tau} |G(X(t)) - G(\bar{Y}(t))|^p \right) \\
 & \lesssim \mathbb{E} \left(\sup_{0 \leq t \leq T} |\Lambda(t)|^p \right) + \mathbb{E} \left(\sup_{-\tau \leq t \leq T-\tau} (V(X(t), \bar{Y}(t))^p |X(t) - \bar{Y}(t)|^p) \right) \tag{2.22}
 \end{aligned}$$

$$\lesssim \mathbb{E} \left(\sup_{0 \leq t \leq T} |\Lambda(t)|^p \right) + h^{p/2} + \left(\mathbb{E} \left(\sup_{0 \leq t \leq (T-\tau) \vee 0} |Z(t)|^{2p} \right) \right)^{1/2}.$$

Substituting (2.21) into (2.22) yields

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq T} |Z(t)|^p \right) &\lesssim h^{p/2} + \left(\mathbb{E} \left(\sup_{0 \leq t \leq (T-\tau) \vee 0} |Z(t)|^{2p} \right) \right)^{1/2} \\ &\quad + \int_0^{(T-\tau) \vee 0} \left\{ (\mathbb{E}(|Z(t)|^{2p}))^{1/2} + \varepsilon^{-p} (\mathbb{E}(|Z(t)|^{4p}))^{1/2} \right\} dt. \end{aligned} \tag{2.23}$$

Hence, we have

$$\mathbb{E} \left(\sup_{0 \leq t \leq \tau} |Z(t)|^p \right) \lesssim h^{p/2}$$

and

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq 2\tau} |Z(t)|^p \right) &\lesssim h^{p/2} + \left(\mathbb{E} \left(\sup_{0 \leq t \leq \tau} |Z(t)|^{2p} \right) \right)^{1/2} \\ &\quad + \int_0^\tau \left\{ (\mathbb{E}(|Z(t)|^{2p}))^{1/2} + \varepsilon^{-p} (\mathbb{E}(|Z(t)|^{4p}))^{1/2} \right\} dt \\ &\lesssim h^{p/2} \end{aligned}$$

by taking $\varepsilon = h^{1/2}$. Thus, the desired assertion (2.9) follows from an inductive argument. \square

3 The NSDDE driven by pure jump processes

Next, we move to consider the convergence rate of EM scheme corresponding to a class of SDDEs of neutral type driven by pure jump processes. More precisely, we consider an SDDEs of neutral type

$$d\{X(t) - G(X(t - \tau))\} = b(X(t), X(t - \tau)) dt + \int_U g(X(t-), X((t - \tau)-), u) \tilde{N}(du, dt) \tag{3.1}$$

with the initial data $X(\theta) = \xi(\theta), \theta \in [-\tau, 0]$. Herein, G and b are given as in (2.1), $g : \mathbb{R}^n \times \mathbb{R}^n \times U \mapsto \mathbb{R}^m$, where $U \in \mathcal{B}(\mathbb{R})$; $\tilde{N}(dt, du) := N(dt, du) - dt\lambda(du)$ is the compensated Poisson measure associated with the Poisson counting measure $N(dt, du)$ generated by a stationary \mathcal{F}_t -Poisson point process $\{p(t)\}_{t \geq 0}$ on \mathbb{R} with characteristic measure $\lambda(\cdot)$, i.e., $N(t, U) = \sum_{s \in D(p), s \leq t} I_U(p(s))$ for $U \in \mathcal{B}(\mathbb{R})$; $X(t-) := \lim_{s \uparrow t} X(s)$.

We assume that b and G are such that (A1) and (A2) hold with $\sigma \equiv \mathbf{0}_{n \times m}$ therein. We further suppose that there exist $L_0, r > 0$ such that for any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ and $u \in U$,

$$(A4) \quad |g(x, y, u) - g(\bar{x}, \bar{y}, u)| \leq L_0(|x - \bar{x}| + (1 + |y|^q + |\bar{y}|^q)|y - \bar{y}|)|u|^r \text{ and } |g(0, 0, u)| \leq |u|^r,$$

where $q \geq 1$ is the same as that in (A1).

$$(A5) \quad \int_U |u|^p \lambda(du) < \infty \text{ for any } p \geq 2.$$

Remark 3.1 The jump coefficient may also be highly nonlinear with respect to the delay argument, for example, $x, y \in \mathbb{R}, u \in U$ and $q \geq 1, g(x, y, u) = (x + y^q)u$ satisfies (A5).

By carrying out a similar argument to that of [12, Theorem 3.1, p. 210], (3.1) admits a unique strong solution $\{X(t)\}$ according to [20, Theorem 117, p. 79].

By following the procedures of (2.2) and (2.3), the discrete-time EM scheme and the continuous-time EM approximation associated with (3.1) are defined respectively as follows:

$$\begin{aligned}
 & Y_h^{(n+1)} - G(Y_h^{(n+1-m)}) \\
 &= Y_h^{(n)} - G(Y_h^{(n-m)}) + b(Y_h^{(n)}, Y_h^{(n-m)})h + g(Y_h^{(n)}, Y_h^{(n-m)}, u) \Delta \tilde{N}_{nh},
 \end{aligned}
 \tag{3.2}$$

where $\Delta \tilde{N}_{nh} := \tilde{N}((n + 1)h, U) - \tilde{N}(nh, U)$, and

$$\begin{aligned}
 Y(t) &= G(\bar{Y}(t - \tau)) + \xi(0) - G(\xi(-\tau)) + \int_0^t b(\bar{Y}(s), \bar{Y}(s - \tau)) ds \\
 &+ \int_0^t \int_U g(\bar{Y}(s-), \bar{Y}((s - \tau)-), u) \tilde{N}(du, ds),
 \end{aligned}
 \tag{3.3}$$

where \bar{Y} is defined similarly as in (2.3).

3.1 Pth moment bound

Hereinafter, $(X(t))$ is the strong solution to (3.1) and $(Y(t))$ is the continuous-time EM scheme (i.e., (3.3)) associated with (3.1).

The lemma below plays a crucial role in revealing convergence rate of the EM scheme.

Lemma 3.1 *Under (A1)–(A5) with $\sigma \equiv \mathbf{0}_{n \times m}$, for any $p \geq 2$, there exists a constant C_T such that*

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |X(t)|^p \right) \vee \mathbb{E} \left(\sup_{0 \leq t \leq T} |Y(t)|^p \right) \leq C_T
 \tag{3.4}$$

and

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |\Gamma(t)|^p \right) \lesssim h,
 \tag{3.5}$$

where $\Gamma(t) := Y(t) - \bar{Y}(t)$.

Proof On the other hand, the proof of (3.1) is similar to that of (2.4) except for some technical details. To make this paper self-contained, the key steps will be sketched below.

Again, we only focus on the p th moment estimation of $Y(t)$,

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |Y(t)|^p \right) \leq C_T,
 \tag{3.6}$$

since the uniform p th moment of $Y(t)$ in a finite time interval can be replicated similarly.

According to (A1), (A2), (A4), and (A5), one has

$$|G(y)| \lesssim 1 + |y|^{1+q},
 \tag{3.7}$$

$$|b(x, y)| + \|\sigma(x, y)\| \lesssim 1 + |x| + |y|^{1+q},
 \tag{3.8}$$

and

$$|g(x, y, u)| \lesssim (1 + |x| + |y|^{1+q})|u|^r,
 \tag{3.9}$$

where $x, y \in \mathbb{R}^n, u \in U$.

Then, by applying the BDG and Hölder's inequalities, one can derive from (3.7)–(3.9) that

$$\begin{aligned} & \mathbb{E} \left(\sup_{-\tau \leq s \leq t} |Y(s)|^p \right) \\ & \lesssim 1 + \|\xi\|_\infty^{p(1+q)} + \mathbb{E} \left(\sup_{-\tau \leq s \leq t-\tau} |\bar{Y}(s)|^{p(1+q)} \right) \\ & \quad + \int_0^t \left\{ \mathbb{E} |\bar{Y}(s)|^p + \mathbb{E} |\bar{Y}(s-\tau)|^{p(1+q)} \right\} ds \\ & \quad + \int_0^t \int_U \left\{ [1 + \mathbb{E} |\bar{Y}(s)|^p + \mathbb{E} |\bar{Y}(s-\tau)|^{p(1+q)}] |u|^{pr} \right\} du ds, \\ & \lesssim 1 + \|\xi\|_\infty^{p(1+q)} + \mathbb{E} \left(\sup_{-\tau \leq s \leq t-\tau} |Y(s)|^{p(1+q)} \right) + \int_0^T \mathbb{E} \left(\sup_{-\tau \leq r \leq s} |Y(r)|^p \right) ds, \end{aligned}$$

where we have used $Y(kh) = \bar{Y}(kh)$ in the last display. This, together with Gronwall's inequality, yields

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} |Y(s)|^p \right) \lesssim 1 + \|\xi\|_\infty^{p(1+q)} + \mathbb{E} \left(\sup_{0 \leq s \leq (t-\tau) \vee 0} |Y(s)|^{p(1+q)} \right).$$

The rest of the proof leading to (3.6) can be done in an identical way as for its Brownian motion counterpart, so we omit the details here.

In the sequel, we aim to show (3.5). From (A4), by applying BDG (see, e.g., [14, Theorem 1]) and Hölder's inequalities, we derive that

$$\begin{aligned} & \mathbb{E} \left(\sup_{0 \leq t \leq T} |\Gamma(t)|^p \right) \\ & \lesssim \sup_{0 \leq k \leq M-1} \left\{ \mathbb{E} \left(\sup_{kh \leq t \leq (k+1)h} \left| \int_{kh}^t b(\bar{Y}(s), \bar{Y}(s-\tau)) ds \right|^p \right) \right. \\ & \quad \left. + \mathbb{E} \left(\sup_{kh \leq t \leq (k+1)h} \left| \int_{kh}^t \int_U g(\bar{Y}(s-), \bar{Y}((s-\tau)-), u) \tilde{N}(ds, du) \right|^p \right) \right\} \\ & \lesssim \sup_{0 \leq k \leq M-1} \left\{ \int_{kh}^{(k+1)h} \left(h^{p-1} \mathbb{E} |b(\bar{Y}(s), \bar{Y}(s-\tau))|^p \right. \right. \\ & \quad \left. \left. + \int_U \mathbb{E} |g(\bar{Y}(s), \bar{Y}(s-\tau), u)|^p \lambda(du) \right) ds \right\} \\ & \lesssim \sup_{0 \leq k \leq M-1} \left\{ \int_{kh}^{(k+1)h} \left(1 + \mathbb{E} \left(\sup_{-\tau \leq r \leq s} |Y(r)|^{p(1+q)} \right) \right) \right. \\ & \quad \left. \times \left(h^{p-1} + \int_U |u|^{pr} \lambda(du) \right) ds \right\} \\ & \lesssim h^p + h \\ & \lesssim h, \end{aligned}$$

where we have used (A2) with $\sigma \equiv \mathbf{0}_{n \times m}$ and (3.9) in the third step, and (3.4) and (A5) in the last two steps, respectively. So (3.5) follows as required. \square

3.2 Convergence results

Our second main result in this paper is presented as follows.

Theorem 3.2 *Under (A1)–(A5) with $\sigma \equiv \mathbf{0}_{n \times m}$ therein, for any $p \geq 2$ and $\theta \in (0, 1)$,*

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^p \right) \lesssim h^{\frac{1}{(1+\theta)(T/\tau)}}. \tag{3.10}$$

So the best convergence rate of EM scheme (i.e., (3.3)) associated with (3.1) is close to one-half.

Remark 3.2 By a close inspection of the proof for Theorem 3.2, the conditions (A4) and (A5) can be replaced by the following: For any $p > 2$ there exists $K_p, K_0 > 0$ and $q > 1$ such that

$$\begin{aligned} \int_U |g(x, y, u)|^p \lambda(du) &\leq K_p (1 + |x|^p + |y|^q), \\ \int_U |g(x, y, u) - g(\bar{x}, \bar{y}, u)|^p \lambda(du) &\leq K_p [|x - \bar{x}|^p + (1 + |y|^q + |\bar{y}|^q) |y - \bar{y}|^p], \\ \int_U |g(x, y, u)|^2 \lambda(du) &\leq K_0 (1 + |x|^2 + |y|^q), \\ \int_U |g(x, y, u) - g(\bar{x}, \bar{y}, u)|^2 \lambda(du) &\leq K_0 [|x - \bar{x}|^2 + (1 + |y|^q + |\bar{y}|^q) |y - \bar{y}|^2] \end{aligned}$$

for any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$.

Next, we go back to finish the proof of Theorem 3.2.

Proof of Theorem 3.2 We follow the idea of the proof for Theorem 2.2 to complete the proof. Set

$$\Gamma_3(t, u) := g(X(t), X(t - \tau), u) - g(\bar{Y}(t), \bar{Y}(t - \tau), u).$$

Applying Itô’s formula, as well as the Lagrange mean value theorem to $V_{\kappa\varepsilon}(\cdot)$, defined by (2.11), gives

$$\begin{aligned} V_{\kappa\varepsilon}(\Lambda(t)) &= \int_0^t \langle (\nabla V_{\kappa\varepsilon})(\Lambda(s)), \Gamma_1(s) \rangle ds \\ &\quad + \int_0^t \int_U \{ V_{\kappa\varepsilon}(\Lambda(s) + \Gamma_3(s)) - V_{\kappa\varepsilon}(\Lambda(s)) - \langle (\nabla V_{\kappa\varepsilon})(\Lambda(s)), \Gamma_3(s) \rangle \} \lambda(du) ds \\ &\quad + \int_0^t \int_U \{ V_{\kappa\varepsilon}(\Lambda(s-) + \Gamma_3(s-)) - V_{\kappa\varepsilon}(\Lambda(s-)) \} \tilde{N}(du, ds) \\ &= V_{\kappa\varepsilon}(\Lambda(0)) + \int_0^t \langle (\nabla V_{\kappa\varepsilon})(\Lambda(s)), \Gamma_1(s) \rangle ds \\ &\quad + \int_0^t \int_U \left\{ \int_0^1 \langle \nabla V_{\kappa\varepsilon}(\Lambda(s) + r\Gamma_3(s)) - \nabla V_{\kappa\varepsilon}(\Lambda(s)), \Gamma_3(s) \rangle dr \right\} \lambda(du) ds \\ &\quad + \int_0^t \int_U \left\{ \int_0^1 \langle \nabla V_{\kappa\varepsilon}(\Lambda(s-) + r\Gamma_3(s-)), \Gamma_3(s-) \rangle dr \right\} \tilde{N}(du, ds) \end{aligned}$$

$$=: J_1(t) + J_2(t) + J_3(t),$$

in which Γ_1 is defined as in (2.14). By BDG inequality (see, e.g., [14, Theorem 1]), we obtain from (2.12), (2.18) with $\sigma \equiv \mathbf{0}_{n \times m}$ therein, (A4) and (A5) that

$$\begin{aligned} \Upsilon(t) &:= \sum_{i=1}^3 \mathbb{E} \left(\sup_{0 \leq s \leq t} |J_i(s)|^p \right) \\ &\lesssim \int_0^t \mathbb{E} \{ |\Lambda(s)|^p + |\Gamma(s)|^p \} ds + \int_{-\tau}^{t-\tau} \mathbb{E} (V(X(s), \bar{Y}(s))^p |X(s) - \bar{Y}(s)|^p) ds, \end{aligned}$$

where $V(\cdot, \cdot)$ is introduced in (2.15). Observe from Hölder’s inequality that

$$\begin{aligned} &\mathbb{E} V(X(s), \bar{Y}(s))^p |Y(s) - \bar{Y}(s)|^p \\ &\lesssim (\mathbb{E} V(X(s), \bar{Y}(s))^{\frac{p(1+\theta)}{\theta}})^{\frac{\theta}{1+\theta}} (\mathbb{E} |X(s) - \bar{Y}(s)|^{p(1+\theta)})^{\frac{1}{1+\theta}} \\ &\lesssim (\mathbb{E} V(X(s), \bar{Y}(s))^{\frac{p(1+\theta)}{\theta}})^{\frac{\theta}{1+\theta}} (\mathbb{E} |Z(s)|^{p(1+\theta)} + \mathbb{E} |\Gamma(s)|^{p(1+\theta)})^{\frac{1}{1+\theta}} \\ &\lesssim (\mathbb{E} (1 + |X(s)|^{\frac{pq(1+\theta)}{\theta}} + |\bar{Y}(s)|^{\frac{pq(1+\theta)}{\theta}})^{\frac{\theta}{1+\theta}} (\mathbb{E} |Z(s)|^{p(1+\theta)} + \mathbb{E} |\Gamma(s)|^{p(1+\theta)})^{\frac{1}{1+\theta}} \\ &\lesssim (\mathbb{E} |Z(s)|^{p(1+\theta)})^{\frac{1}{1+\theta}} + (\mathbb{E} |\Gamma(s)|^{p(1+\theta)})^{\frac{1}{1+\theta}} \\ &\lesssim h^{\frac{1}{1+\theta}} + (\mathbb{E} |Z(s)|^{p(1+\theta)})^{\frac{1}{1+\theta}}, \quad \theta > 0, \end{aligned} \tag{3.11}$$

in which we have used (3.4) in the penultimate display and (3.5) in the last display, respectively. So we arrive at

$$\Upsilon(t) \lesssim h^{\frac{1}{1+\theta}} + \int_0^t \mathbb{E} \{ |\Lambda(s)|^p + |\Gamma(s)|^p \} ds + \int_{-\tau}^{t-\tau} (\mathbb{E} |Z(s)|^{p(1+\theta)})^{\frac{1}{1+\theta}} ds.$$

This, together with (2.10) and (3.5), implies

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq t} |\Lambda(t)|^p \right) &\lesssim \epsilon^p + \mathbb{E} \left(\sup_{0 \leq s \leq t} V_{\kappa \epsilon}(\Lambda(s)) \right) \\ &\lesssim \epsilon^p + h^{\frac{1}{1+\theta}} + \int_0^t \mathbb{E} \{ |\Lambda(s)|^p + |\Gamma(s)|^p \} ds + \int_{-\tau}^{t-\tau} (\mathbb{E} |Z(s)|^{p(1+\theta)})^{\frac{1}{1+\theta}} ds \\ &\lesssim h^{\frac{1}{1+\theta}} + \int_0^t \mathbb{E} |\Lambda(s)|^p ds + \int_{-\tau}^{t-\tau} (\mathbb{E} |Z(s)|^{p(1+\theta)})^{\frac{1}{1+\theta}} ds \end{aligned}$$

by taking $\epsilon = h^{\frac{1}{p(1+\theta)}}$ in the last display. Using Gronwall’s inequality, due to $Z(\theta) = 0$ for $\theta \in [-\tau, 0]$, one has

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |\Lambda(t)|^p \right) \lesssim h^{\frac{1}{1+\theta}} + \int_0^{(T-\tau) \vee 0} (\mathbb{E} |Z(t)|^{p(1+\theta)})^{\frac{1}{1+\theta}} dt. \tag{3.12}$$

Next, observe from (A1) and Hölder’s inequality that

$$\begin{aligned}
 & \mathbb{E} \left(\sup_{0 \leq t \leq T} |Z(t)|^p \right) \\
 & \lesssim \mathbb{E} \left(\sup_{0 \leq t \leq T} |\Lambda(t)|^p \right) + \mathbb{E} \left(\sup_{-\tau \leq t \leq T-\tau} |G(X(t)) - G(\bar{Y}(t))|^p \right) \\
 & \lesssim \mathbb{E} \left(\sup_{0 \leq t \leq T} |\Lambda(t)|^p \right) + \mathbb{E} \left(\sup_{-\tau \leq t \leq T-\tau} (V(X(t), \bar{Y}(t))^p |X(t) - \bar{Y}(t)|^p) \right) \\
 & \lesssim \mathbb{E} \left(\sup_{0 \leq t \leq T} |\Lambda(t)|^p \right) \\
 & \quad + \left\{ \left(1 + \mathbb{E} \left(\sup_{-\tau \leq t \leq T} |X(t)|^{\frac{pq(1+\theta)}{\theta}} \right) + \left(\sup_{-\tau \leq t \leq T} |Y(t)|^{\frac{pq(1+\theta)}{\theta}} \right) \right)^{\frac{\theta}{1+\theta}} \right. \\
 & \quad \times \left. \left\{ \mathbb{E} \left(\sup_{-\tau \leq t \leq T-\tau} |Z(t)|^{p(1+\theta)} \right) + \mathbb{E} \left(\sup_{-\tau \leq t \leq T} |\Gamma(t)|^{p(1+\theta)} \right) \right\}^{\frac{1}{1+\theta}} \right. \\
 & \lesssim h^{\frac{1}{1+\theta}} + \left(\mathbb{E} \left(\sup_{0 \leq t \leq (T-\tau) \vee 0} |Z(t)|^{p(1+\theta)} \right) \right)^{\frac{1}{1+\theta}},
 \end{aligned} \tag{3.13}$$

where in the last step we have utilized (3.4) and (3.5). So we find that

$$\mathbb{E} \left(\sup_{0 \leq t \leq \tau} |Z(t)|^p \right) \lesssim h^{\frac{1}{1+\theta}},$$

which, in addition to (3.13), further yields that

$$\begin{aligned}
 \mathbb{E} \left(\sup_{0 \leq t \leq 2\tau} |Z(t)|^p \right) & \lesssim h^{\frac{1}{1+\theta}} + \left(\mathbb{E} \left(\sup_{0 \leq t \leq \tau} |Z(t)|^{p(1+\theta)} \right) \right)^{\frac{1}{1+\theta}} \\
 & \lesssim h^{\frac{1}{(1+\theta)^2}} + h^{\frac{1}{1+\theta}} \\
 & \lesssim h^{\frac{1}{(1+\theta)^2}}.
 \end{aligned}$$

Thus, the desired assertion follows from an inductive argument. □

4 Main results

In this section, we investigate the generalized SDDEs of neutral type, by considering the following SDDE of neutral type:

$$\begin{aligned}
 d\{X(t) - G(X(t - \tau))\} & = b(X(t), X(t - \tau)) dt + \sigma(X(t), X(t - \tau)) dB(t) \\
 & \quad + \int_U g(X(t-), X((t - \tau)-), u) \tilde{N}(du, dt)
 \end{aligned} \tag{4.1}$$

with the initial data $X(\theta) = \xi(\theta), \theta \in [-\tau, 0]$. Herein, G, b and σ are given as in (2.1), while g is given as in (3.1).

By generalizing the procedures of (2.2), (2.3), (3.2), and (3.3), the discrete-time EM scheme and the continuous-time EM approximation associated with (4.1) are respectively defined as follow: Without loss of generality, we assume that $h = T/M = \tau/m \in (0, 1)$ for some integers $M, m > 1$. For every integer $k = -m, \dots, 0$, set $Y_h^{(k)} := \xi(kh)$, and for each

integer $k = 1, \dots, M - 1$,

$$\begin{aligned}
 Y_h^{(k+1)} - G(Y_h^{(k+1-m)}) &= Y_h^{(k)} - G(Y_h^{(k-m)}) + b(Y_h^{(k)}, Y_h^{(k-m)})h \\
 &\quad + \sigma(Y_h^{(k)}, Y_h^{(k-m)})\Delta B_h^{(k)} + g(Y_h^{(k)}, Y_h^{(k-m)}, u)\Delta\tilde{N}_{kh},
 \end{aligned}
 \tag{4.2}$$

where $\Delta B_h^{(k)} := B((k + 1)h) - B(kh)$ while $\Delta\tilde{N}_{kh} := \tilde{N}((k + 1)h, U) - \tilde{N}(kh, U)$, and for any $t \in [kh, (k + 1)h)$, set $\bar{Y}(t) := Y_h^{(k)}$, and

$$\begin{aligned}
 Y(t) &= G(\bar{Y}(t - \tau)) + \xi(0) - G(\xi(-\tau)) + \int_0^t b(\bar{Y}(s), \bar{Y}(s - \tau)) ds \\
 &\quad + \int_0^t \sigma(\bar{Y}(s), \bar{Y}(s - \tau)) dB(s) + \int_0^t \int_U g(\bar{Y}(s-), \bar{Y}((s - \tau)-), u)\tilde{N}(du, ds).
 \end{aligned}
 \tag{4.3}$$

Note that, in the rest of this paper, we denote by $X(t)$ the strong solution to (4.1), while $Y(t)$, defined in (4.3), is the continuous-time EM scheme associated with (4.1).

4.1 Pth moment bound

The following lemma is a generalization of Lemma (2.1) and (3.1).

Lemma 4.1 *Under (A1)–(A5), for any $p \geq 2$, there exists a constant C_T such that*

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |X(t)|^p\right) \vee \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y(t)|^p\right) \leq C_T
 \tag{4.4}$$

and

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Gamma(t)|^p\right) \lesssim h^{(\frac{p}{2}-1) \wedge 1},
 \tag{4.5}$$

where $\Gamma(t) := Y(t) - \bar{Y}(t)$.

Proof Here, only key steps are outlined, so that redundant calculation are omitted. To estimate a bound of $Y(t)$, the continuous-time EM scheme associated with (4.1), a simple generalization of two special cases will be sufficient.

For (4.5), an application of BDG and Hölder’s inequalities yields

$$\begin{aligned}
 &\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Gamma(t)|^p\right) \\
 &\lesssim \sup_{0 \leq k \leq M-1} \left\{ \mathbb{E}\left(\sup_{kh \leq t \leq (k+1)h} \left| \int_{kh}^t b(\bar{Y}(s), \bar{Y}(s - \tau)) ds \right|^p\right) \right. \\
 &\quad \left. + \mathbb{E}\left(\sup_{nh \leq t \leq (k+1)h} \left| \int_{kh}^t \sigma(\bar{Y}(s), \bar{Y}(s - \tau)) dB(s) \right|^p\right) \right\} \\
 &\quad + \mathbb{E}\left(\sup_{kh \leq t \leq (k+1)h} \left| \int_{kh}^t \int_U g(\bar{Y}(s-), \bar{Y}((s - \tau)-), u)\tilde{N}(ds, du) \right|^p\right) \\
 &\lesssim \sup_{0 \leq k \leq M-1} \left\{ \int_{kh}^{(k+1)h} \left(h^{p-1} \mathbb{E} |b(\bar{Y}(s), \bar{Y}(s - \tau))|^p \right) \right.
 \end{aligned}$$

$$\begin{aligned}
 & + h^{\frac{p}{2}-1} \mathbb{E} \int_{kh}^{(k+1)h} \|\sigma(\bar{Y}(s), \bar{Y}(s-\tau))\|^p ds \\
 & + \int_U \mathbb{E} |g(\bar{Y}(s), \bar{Y}(s-\tau), u)|^p \lambda(du) ds \Big\} \\
 \lesssim & \sup_{0 \leq k \leq M-1} \left\{ \int_{kh}^{(k+1)h} \left(1 + \mathbb{E} \left(\sup_{-\tau \leq r \leq s} |Y(r)|^{p(1+q)} \right) \right) \right. \\
 & \times \left. \left(h^{\frac{p}{2}-1} + \int_U |u|^{pr} \lambda(du) \right) ds \right\} \\
 \lesssim & h^{\frac{p}{2}-1} + h \\
 \lesssim & h^{(\frac{p}{2}-1) \wedge 1}. \quad \square
 \end{aligned}$$

4.2 Convergence results

The convergence rate of the general SDDEs of neutral type is given as follows.

Theorem 4.2 *Under (A1)–(A5), for any $p \geq 2$ and $\theta \in (0, 1)$,*

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^p \right) \lesssim h^{\frac{1}{(1+\theta)\lceil T/\tau \rceil}}. \tag{4.6}$$

So the best convergence rate of EM scheme (i.e., (4.3)) associated with (4.1) is smaller than the classic convergence rate one-half.

Remark 4.1 The proof of (4.3) is not intuitive by combining two special cases, it requires a more technical approach. Therefore, some key steps will be highlighted in the proof.

Proof of Theorem 4.2

Define

$$Z(t) := X(t) - Y(t) \quad \text{and} \quad \Lambda(t) := Z(t) - G(X(t-\tau)) + G(\bar{Y}(t-\tau)).$$

Then, an application of Yamada–Watanabe approach yields

$$\begin{aligned}
 & V_{\kappa\varepsilon}(\Lambda(t)) \\
 & = \int_0^t \langle (\nabla V_{\kappa\varepsilon})(\Lambda(s)), \Gamma_1(s) \rangle ds + \frac{1}{2} \int_0^t \text{trace}\{(\Gamma_2(s))^* (\nabla^2 V_{\kappa\varepsilon})(\Lambda(s)) \Gamma_2(s)\} ds \\
 & \quad + \int_0^t \langle \nabla(V_{\kappa\varepsilon})(\Lambda(s)), \Gamma_2(s) dB(s) \rangle \\
 & \quad + \int_0^t \int_U \{V_{\kappa\varepsilon}(\Lambda(s) + \Gamma_3(s)) - V_{\kappa\varepsilon}(\Lambda(s)) - \langle (\nabla V_{\kappa\varepsilon})(\Lambda(s)), \Gamma_3(s) \rangle\} \lambda(du) ds \\
 & \quad + \int_0^t \int_U \{V_{\kappa\varepsilon}(\Lambda(s-) + \Gamma_3(s-)) - V_{\kappa\varepsilon}(\Lambda(s-))\} \tilde{N}(du, ds) \\
 & = V_{\kappa\varepsilon}(\Lambda(0)) + \int_0^t \langle (\nabla V_{\kappa\varepsilon})(\Lambda(s)), \Gamma_1(s) \rangle ds \\
 & \quad + \frac{1}{2} \int_0^t \text{trace}\{(\Gamma_2(s))^* (\nabla^2 V_{\kappa\varepsilon})(\Lambda(s)) \Gamma_2(s)\} ds
 \end{aligned}$$

$$\begin{aligned}
 &+ \int_0^t \langle \nabla(V_{\kappa\varepsilon})(\Lambda(s)), \Gamma_2(s) dB(s) \rangle \\
 &+ \int_0^t \int_U \left\{ \int_0^1 \langle \nabla V_{\kappa\varepsilon}(\Lambda(s) + r\Gamma_3(s)) - \nabla V_{\kappa\varepsilon}(\Lambda(s)), \Gamma_3(s) \rangle dr \right\} \lambda(d\mu) ds \\
 &+ \int_0^t \int_U \left\{ \int_0^1 \langle \nabla V_{\kappa\varepsilon}(\Lambda(s-) + r\Gamma_3(s-)), \Gamma_3(s-) \rangle dr \right\} \tilde{N}(d\mu, ds)
 \end{aligned}$$

where

$$\Gamma_1(t) := b(X(t), X(t - \tau)) - b(\bar{Y}(t), \bar{Y}(t - \tau)),$$

$$\Gamma_2(t) := \sigma(X(t), X(t - \tau)) - \sigma(\bar{Y}(t), \bar{Y}(t - \tau)),$$

and

$$\Gamma_3(t, u) := g(X(t), X(t - \tau), u) - g(\bar{Y}(t), \bar{Y}(t - \tau), u).$$

Recall (2.21) and (3.12) that

$$\begin{aligned}
 \mathbb{E} \left(\sup_{0 \leq t \leq T} |\Lambda(t)|^p \right) &\lesssim h^{p/2} + \int_0^{(T-\tau) \vee 0} \left\{ (\mathbb{E}(|Z(t)|^{2p}))^{1/2} + \varepsilon^{-p} (\mathbb{E}(|Z(t)|^{4p}))^{1/2} \right\} dt, \\
 &+ h^{\frac{1}{1+\theta}} + \int_0^{(T-\tau) \vee 0} (\mathbb{E}|Z(t)|^{p(1+\theta)})^{\frac{1}{1+\theta}} dt.
 \end{aligned} \tag{4.7}$$

By replicating the procedure in (2.23) and (3.13), it yields that

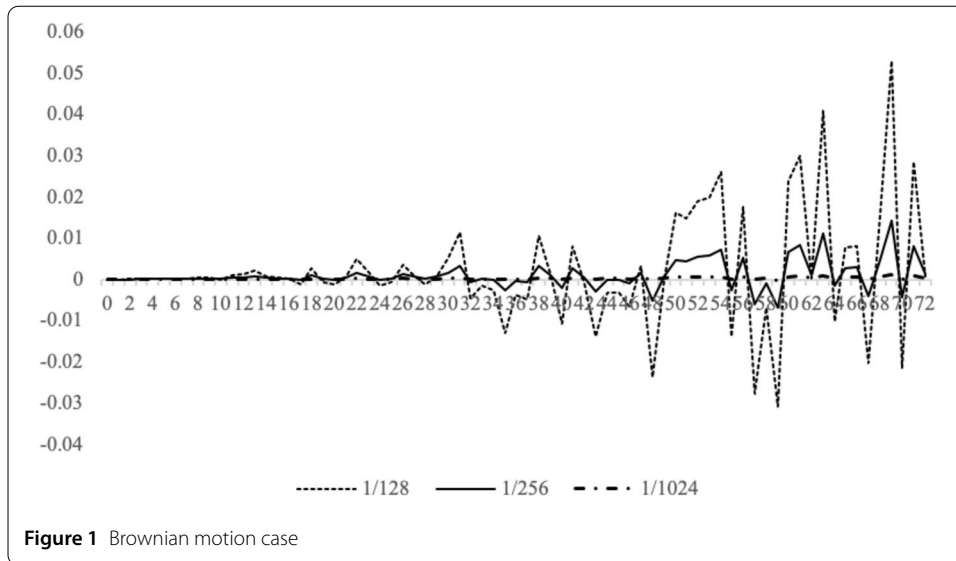
$$\begin{aligned}
 \mathbb{E} \left(\sup_{0 \leq t \leq T} |Z(t)|^p \right) &\lesssim h^{p/2} + \left(\mathbb{E} \left(\sup_{0 \leq t \leq (T-\tau) \vee 0} |Z(t)|^{2p} \right) \right)^{1/2} \\
 &+ \int_0^{(T-\tau) \vee 0} \left\{ (\mathbb{E}(|Z(t)|^{2p}))^{1/2} + \varepsilon^{-p} (\mathbb{E}(|Z(t)|^{4p}))^{1/2} \right\} dt \\
 &+ h^{\frac{1}{1+\theta}} + \left(\mathbb{E} \left(\sup_{0 \leq t \leq (T-\tau) \vee 0} |Z(t)|^{p(1+\theta)} \right) \right)^{\frac{1}{1+\theta}},
 \end{aligned} \tag{4.8}$$

which implies that

$$\mathbb{E} \left(\sup_{0 \leq t \leq \tau} |Z(t)|^p \right) \lesssim h^{\frac{1}{1+\theta}}$$

and

$$\begin{aligned}
 \mathbb{E} \left(\sup_{0 \leq t \leq 2\tau} |Z(t)|^p \right) &\lesssim h^{p/2} + \left(\mathbb{E} \left(\sup_{0 \leq t \leq \tau} |Z(t)|^{2p} \right) \right)^{1/2} \\
 &+ \int_0^\tau \left\{ (\mathbb{E}(|Z(t)|^{2p}))^{1/2} + \varepsilon^{-p} (\mathbb{E}(|Z(t)|^{4p}))^{1/2} \right\} dt \\
 &+ h^{\frac{1}{1+\theta}} + \left(\mathbb{E} \left(\sup_{0 \leq t \leq \tau} |Z(t)|^{p(1+\theta)} \right) \right)^{\frac{1}{1+\theta}} \\
 &\lesssim h^{p/2} + h^{\frac{1}{(1+\theta)^2}} + h^{\frac{1}{1+\theta}} \\
 &\lesssim h^{\frac{1}{(1+\theta)^2}},
 \end{aligned}$$



where we take $\epsilon = h^{1/2}$ and use the fact that $h^{\frac{1}{1+\theta}} \geq h^{p/2}$, for any $p \geq 2, \theta > 0$. Therefore, the desired assertion follows from an inductive argument. \square

5 Examples

In this section, three numerical examples will be discussed to demonstrate the convergence results established in the previous sections, which shows the theoretical convergence rates agree with the numerical simulation very well.

Example 5.1 Let $B(t)$ be a scalar Brownian motion, consider a one-dimensional nonlinear SDDE of neutral type driven by Brownian motion

$$d\{X(t) - X^2(t - 1)\} = \{X(t) + 0.5X^3(t - 1)\} dt + X^2(t - 1) dB(t), \quad t \geq 0,$$

with the initial data $X = 0$, for $t \in [-1, 0]$.

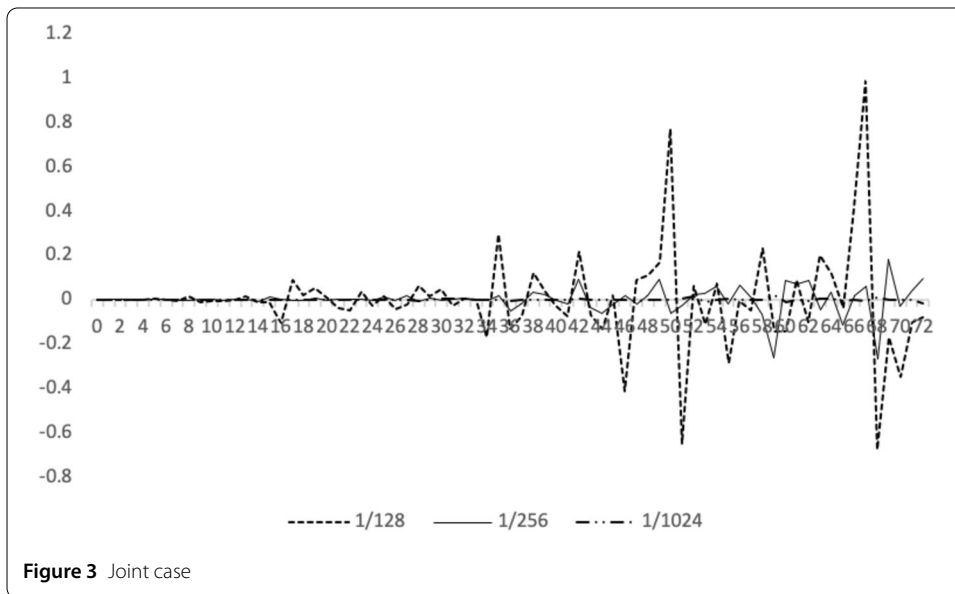
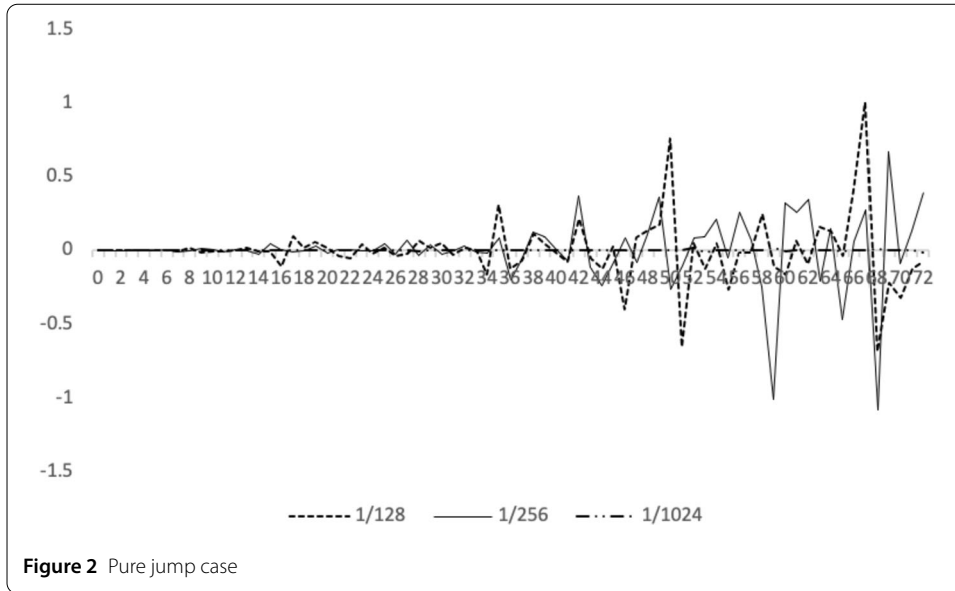
In Fig. 1, the EM scheme results for stepsizes $h = 1/256$, $h = 1/512$, and $h = 1/1024$ are plotted, respectively. The figure shows that the convergence rate is consistent with the result obtained in Sect. 2.

Example 5.2 Let $\tilde{N}(t)$ be a pure jump process with intensity $\lambda = 1$. Consider a one-dimensional nonlinear SDDE of neutral type driven by pure jump process

$$d\{X(t) - X^2(t - 1)\} = \{X(t) + 0.5X^3(t - 1)\} dt + 0.8X^2(t - 1) d\tilde{N}(t), \quad t \geq 0,$$

with the initial data $X = 0$, for $t \in [-1, 0]$.

In Fig. 2, the EM scheme results for stepsizes $h = 1/256$, $h = 1/512$ and $h = 1/1024$ are plotted, respectively. The figure shows that the convergence rate obtained from Sect. 3 is much slower than its counterpart obtained in Sect. 2.



Example 5.3 Let $B(t)$ be a scalar Brownian motion, consider a one-dimensional nonlinear SDDE of neutral type driven by Brownian motion and pure jump process

$$\begin{aligned}
 & d\{X(t) - X^2(t - 1)\} \\
 & = \{X(t) + 0.5X^3(t - 1)\} dt + X^2(t - 1) dB(t) + 0.8X^2(t - 1) d\tilde{N}(t), \quad t \geq 0,
 \end{aligned}$$

with the initial data $X = 0$, for $t \in [-1, 0]$ and intensity $\lambda = 1$.

In Fig. 3, the EM scheme results for stepsizes $h = 1/256$, $h = 1/512$ and $h = 1/1024$ are plotted, respectively. The figure shows that the convergence rate is dominated by the jump process, which verified our theoretical result obtained in the Sect. 4.

6 Conclusion

In this paper, the convergence rate of EM scheme for SDDEs of neutral type is studied under a more general polynomial condition. In the Brownian motion case, the convergence rate is consistent with the classic result of one-half. Meanwhile, in the pure jump case, the convergence rate is much slower than one-half. As a result, in the general SDDEs of neutral type, the convergence is dominated by the slower rate.

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