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Error estimates of finite volume method for Stokes optimal control problem

Lin Lan¹, Ri-hui Chen¹, Xiao-dong Wang^{1*}, Chen-xia Ma¹ and Hao-nan Fu¹

*Correspondence:

kmustxiaodong@163.com

¹Faculty of Land Resources Engineering, Kunming University of Science and Technology, Kunming, Yunnan 650093, China

Abstract

In this paper, we discuss a priori error estimates for the finite volume element approximation of optimal control problem governed by Stokes equations. Under some reasonable assumptions, we obtain optimal L^2 -norm error estimates. The approximate orders for the state, costate, and control variables are $O(h^2)$ in the sense of L^2 -norm. Furthermore, we derive H^1 -norm error estimates for the state and costate variables. Finally, we give some conclusions and future works.

MSC: 49J20; 65N30

Keywords: Optimal control problem; Stokes equations; Finite volume method; A priori error estimates; Variational discretization

1 Introduction

Flow control problems have been widely used in science and engineering. Finite element [2, 5], finite difference [28], and spectral [7] methods have been employed to numerically solve them. Finite volume method is an effective discretization technique for partial differential equations. Due to its local conservative property and other attractive properties, the finite volume method is widely used in the numerical approximation fluid dynamics. Since the method was proposed, there have been a lot of studies of mathematical theory for the finite volume method in the literature [1, 3, 4, 6, 10, 11, 14, 15]. Bank and Rose obtained some results for elliptic boundary value problems that the finite volume element approximation was comparable with the finite element approximation in H^1 -norm which can be found in [1]. In [15], the authors presented the optimal L^2 -error estimate for second-order elliptic boundary value problems under the assumption that $f \in H^1$, they also obtained the H^1 -norm and maximum-norm error estimates for those problems. In [6], Chatzipantelidis proposed a nonconforming finite volume method and obtained the L^2 -norm and H^1 -norm error estimates for elliptic boundary value problems in two dimensions. The authors discussed a priori estimates for a linear elliptic optimal control problem in [26], they derived the optimal order error estimates in L^2 and L^∞ -norm for the state, costate, and control variables, and the optimal H^1 and $W^{1,\infty}$ -norm error estimates for the state and costate variables. At the same time, there are some other literature works to study optimal control problems [8, 9, 16, 19–25].

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In fact, finite volume methods lie somewhere between finite difference and finite element methods, they have a flexibility similar to that of finite element methods for handling complicated solution domain geometries and boundary conditions, and they have a simplicity for implementation comparable to finite difference methods with triangulations of a simple structure. The finite volume methods and finite element methods are commonly employed in computational fluid mechanics and computational solid dynamics, where the finite volume method is traditionally associated with computational fluid mechanics and the finite element method is associated with computational solid dynamics. In general, two different functional spaces (one for the trial space and one for the test space) are used in the finite volume method. Owing to the two different spaces, the numerical analysis of the finite volume method is more difficult than that of the finite element method and finite difference method. In [18], the authors developed a family of stabilized discontinuous finite volume element methods for the Stokes equations. A priori error estimates are derived for the velocity and pressure in the energy norm, and convergence rates are predicted for velocity in the L_2 -norm under the assumption that the source term was locally in H_1 . In [13], the authors established a general framework for analyzing the class of finite volume methods for the Stokes equations. Under the framework, optimal L_2 error estimates for velocity were obtained for the first time for several different finite volume methods. In recent years, the authors studied the Legendre–Galerkin in spectral approximation of distributed optimal control problems governed by Stokes equations. They derived a priori error estimates in both H_1 and L_2 norms for the Legendre–Galerkin approximation of the unconstrained control problems in [7]. However, a priori error estimates for the finite volume element approximation of optimal control problem governed by Stokes equations have few papers to study.

In this paper, we mainly establish finite volume schemes for Stokes optimal control problem and obtain some optimal order error estimates. Firstly, we use the finite volume method to discretize the state and adjoint equation of the optimal control problem. Then, applying the variational discretization concept [17], the control variable is not discretized directly, but discretized by a projection of the discrete costate variable. At last, we obtain some optimal order error estimates under some reasonable assumptions.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|v\|_{m,p}^p$ given by $\|v\|_{m,p}^p = \sum_{|\alpha| \leq m} \|D^\alpha v\|_{L^p(\Omega)}^p$ and the semi-norm $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, $\|\cdot\|_m = \|\cdot\|_{m,2}$, and $\|\cdot\| = \|\cdot\|_{0,2}$. Let $\|\cdot\|_\infty$ denote the maximum norm, $\|f\|_\infty = \text{ess sup}_{x \in \Omega} |f(x)|$. $L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q = 0\}$. As usual, we use (\cdot, \cdot) to denote the $L^2(\Omega)$ -inner product.

In this paper, we consider the following Stokes optimal control problem:

$$\min_{u \in U_{ad}} \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u\|_{L^2(\Omega)}^2, \tag{1.1}$$

$$- \nu \Delta y + \nabla r = f + u, \quad \text{in } \Omega, \tag{1.2}$$

$$\nabla \cdot y = 0, \quad \text{in } \Omega, \tag{1.3}$$

$$y = 0, \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded convex polygon domain with boundary $\partial\Omega$, $f \in L^2(\Omega)^2$ or $H_0^1(\Omega)^2$, $\nu > 0$ is a given constant, and y, u are unknown functions, U_{ad} is denoted by

$$U_{ad} = \{u \in L^2(\Omega)^2 : u(x) \geq 0, \text{ a.e. in } \Omega\}.$$

Let

$$\begin{aligned} a(y, w) &= \nu \int_{\Omega} \nabla y \cdot \nabla w, \quad \forall w \in H_0^1(\Omega)^2, \\ b(w, r) &= \int_{\Omega} r \nabla \cdot w, \quad \forall (w, r) \in H_0^1(\Omega)^2 \times L_0^2(\Omega), \\ (u, w) &= \int_{\Omega} u \cdot w, \quad \forall (u, w) \in L_0^2(\Omega)^2 \times H_0^1(\Omega)^2. \end{aligned}$$

The bilinear form $b(\cdot, \cdot)$ relating the functional spaces for velocity and pressure satisfies the following Babuška–Brezzi condition (see [27] for example): there exists a constant $\varsigma > 0$ such that

$$\inf_{q \in L_0^2(\Omega)} \sup_{v \in H_0^1(\Omega)^2} \frac{b(v, q)}{\|v\|_1} \geq \varsigma \|q\|. \tag{1.5}$$

The weak formulation associated with the state equations (1.1)–(1.4) is given as follows: find $(y, r) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ such that

$$\min_{u \in U_{ad}} \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u\|_{L^2(\Omega)}^2 \tag{1.6}$$

$$a(y, w) - b(w, r) = (f + u, w), \quad \forall w \in H_0^1(\Omega)^2, \tag{1.7}$$

$$b(y, \phi) = 0, \quad \forall \phi \in L_0^2(\Omega). \tag{1.8}$$

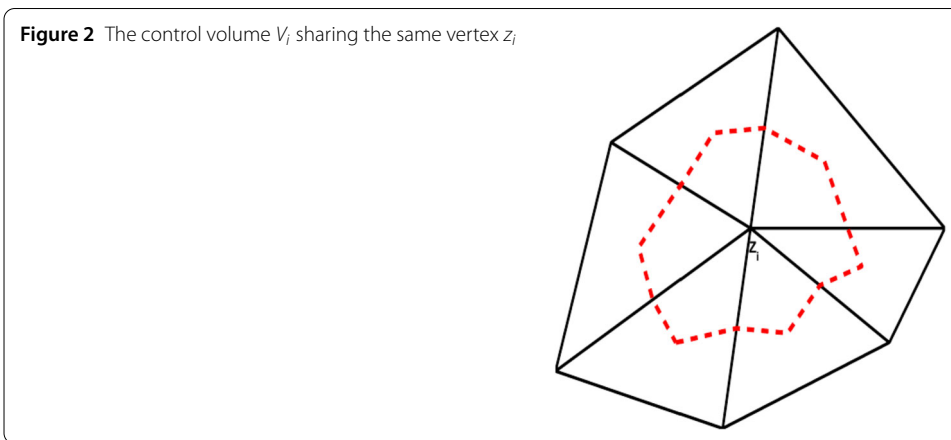
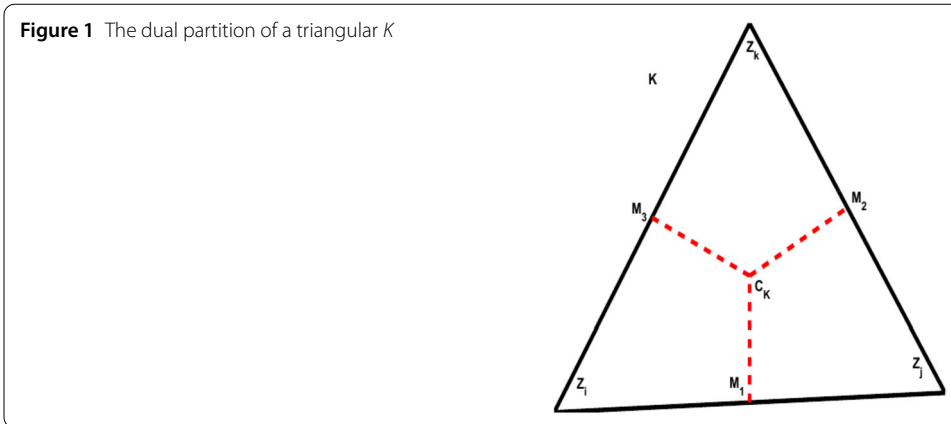
The paper is organized as follows. In Sect. 2, we present some notations and describe the finite volume method briefly. In Sect. 3, we analyze the error estimates between the exact solution and the finite volume element approximation. Finally, we give a conclusion and some possible future work in Sect. 4.

2 Finite volume element approximation

As is shown in [15], the partition \mathcal{T}_h is quasi-uniform, i.e., there exists a positive constant C such that

$$C^{-1}h^2 \leq \text{meas}(V_i) \leq Ch^2, \quad \forall V_i \in \mathcal{T}_h.$$

For the convex polygon Ω , we consider a quasi-uniform triangulation \mathcal{T}_h consisting of closed triangle elements K such that $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$. We use N_h to denote the set of all nodes or vertices of \mathcal{T}_h , N_t denote the number of triangles in the primal partition. To define the dual partition \mathcal{T}_h^* of \mathcal{T}_h , we divide each $K \in \mathcal{T}_h$ into three quadrilaterals by connecting the barycenter C_K of K with line segments to the midpoints of edges of K as is shown in Fig. 1.



The control volume V_i consists of the quadrilaterals sharing the same vertex z_i as is shown in Fig. 2.

The dual partition \mathcal{T}_h^* consists of the union of the control volume V_i . Let $h = \max\{h_K\}$, where h_K is the diameter of the triangle K . The dual partition \mathcal{T}_h^* is also quasi-uniform.

We define the finite dimensional space V_h associated with \mathcal{T}_h for the trial functions by

$$V_h = \{v : v \in L^2(\Omega)^2, v|_K \in P_1(K)^2, \forall K \in \mathcal{T}_h, v|_{\partial\Omega} = 0\},$$

and define the finite dimensional space Q_h associated with the dual partition \mathcal{T}_h^* for the test functions by

$$Q_h = \{q \in L^2(\Omega)^2 : q|_V \in P_0(V)^2, \forall V \in \mathcal{T}_h^*; q|_{V_z} = 0, z \in \partial\Omega\},$$

where V_z is a dual element and $P_l(K)$ or $P_l(V)$ consists of all the polynomials with degree less than or equal to l defined on K or V .

Let R_h be the following finite dimensional space for pressure:

$$R_h = \{r \in L^2_0(\Omega) : r|_K \in P_0(K), \forall k \in \mathcal{T}_h\}.$$

To connect the trial space and the test space, we define a transfer operator $I_h : V_h \rightarrow Q_h$ as follows:

$$I_h v_h = \sum_{z_i \in N_h} v_h(z_i) \chi_i, I_h v_h|_{V_i} = v_h(z_i), \quad \forall V_i \in \mathcal{T}_h^*,$$

where χ_i is the characteristic function of V_i .

It is well known (see [12] for example) that there exists a positive constant C such that, for all $v \in V_h$,

$$\|v - I_h v\| \leq Ch \|v\|_1, \tag{2.1}$$

$$a(v_h, I_h v_h) \geq C \|v_h\|_1^2. \tag{2.2}$$

The finite volume scheme of (1.6)–(1.8) is defined as the solution of the problem: find $(y_h, r_h) \in V_h \times R_h$ such that

$$a(y_h, I_h w_h) - b(I_h w_h, r_h) = (f + u_h, I_h w_h), \quad \forall w_h \in H_0^1(\Omega)^2, \tag{2.3}$$

$$b(I_h y_h, \phi_h) = 0, \quad \forall \phi_h \in R_h, \tag{2.4}$$

where the bilinear forms $a(y_h, I_h w_h)$ and $b(I_h w_h, r_h)$ are defined by

$$a(y_h, I_h w_h) = -A \sum_{z_i \in N_h} w_h(z_i) \int_{\partial V_i} \nabla y_h \cdot \mathbf{n} \, ds, \quad y_h, w_h \in H_0^1(\Omega)^2 \cap V_h$$

and

$$b(I_h w_h, r_h) = \sum_{z_i \in N_h} w_h(z_i) \int_{\partial V_i} r_h \mathbf{n} \, ds, \quad r_h, w_h \in R_h \cap V_h,$$

where \mathbf{n} is the unit outward normal vector to ∂V_i .

The bilinear form $a(\cdot, \cdot)$ is not symmetric though the problem is self-adjoint. Then, for all $w_h, v_h \in V_h$, there exist positive constants C and $h_0 \geq 0$ such that [11], for all $0 < h < h_0$,

$$|a(w_h, I_h v_h) - a(v_h, I_h w_h)| \leq Ch \|w_h\|_1 \|v_h\|_1. \tag{2.5}$$

It is well known (see, e.g., [20]) that the optimal control problem (1.1)–(1.4) has a unique solution (y, r, u) , and that if a triplet (y, r, u) is the solution of (1.1)–(1.4), then there is a co-state $(p, s) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ such that (y, r, p, s, u) satisfies the following optimality conditions:

$$a(y, w) - b(w, r) = (f + u, w), \quad \forall w \in H_0^1(\Omega)^2, \tag{2.6}$$

$$b(y, \phi) = 0, \quad \forall \phi \in L_0^2(\Omega), \tag{2.7}$$

$$a(q, p) + b(q, s) = (y - y_d, q), \quad \forall q \in H_0^1(\Omega)^2, \tag{2.8}$$

$$b(p, \psi) = 0, \quad \forall \psi \in L_0^2(\Omega), \tag{2.9}$$

$$(u + p, v - u) \geq 0, \quad \forall v \in U_{ad}. \tag{2.10}$$

We use the finite volume method to discretize the state and costate equation. Then the optimal control problem (2.6)–(2.10) can be approximated as follows: find $(y_h, r_h, p_h, s_h, u_h) \in V_h \times R_h \times V_h \times R_h \times U_{ad}$ such that

$$a(y_h, I_h w_h) - b(I_h w_h, r_h) = (f + u_h, I_h w_h), \quad \forall w_h \in V_h, \tag{2.11}$$

$$b(I_h y_h, \phi_h) = 0, \quad \forall \phi_h \in R_h, \tag{2.12}$$

$$a(p_h, I_h q_h) + b(I_h q_h, s_h) = (y_h - y_d, I_h q_h), \quad \forall q_h \in V_h, \tag{2.13}$$

$$b(I_h p_h, \psi_h) = 0, \quad \forall \psi_h \in R_h, \tag{2.14}$$

$$(u_h + p_h, v_h - u_h) \geq 0, \quad \forall v_h \in U_{ad}. \tag{2.15}$$

3 L^2 error estimates

In this section, we consider the error analysis of the finite volume element approximation. Let $(y_h(u), r_h(u), p_h(y), s_h(y))$ be the solution of

$$a(y_h(u), I_h w_h) - b(r_h(u), I_h w_h) = (f + u, I_h w_h), \quad \forall w_h \in V_h, \tag{3.1}$$

$$b(I_h y_h(u), \phi_h) = 0, \quad \forall \phi_h \in R_h, \tag{3.2}$$

$$a(p_h(y), I_h q_h) + b(s_h(y), I_h q_h) = (y_h(u) - y_d, I_h q_h), \quad \forall q_h \in V_h. \tag{3.3}$$

$$b(I_h p_h(y), \psi_h) = 0, \quad \forall \psi_h \in R_h. \tag{3.4}$$

For $y_h(u)$ and $p_h(u)$, note that $y_h = y_h(u_h)$ and $p_h = p_h(u_h)$.

Firstly, we give some intermediate error estimates.

Lemma 3.1 *Let (y, r, p, s, u) and $(y_h, r_h, p_h, s_h, u_h) \in V_h \times R_h \times V_h \times R_h \times U_{ad}$ be the solutions of (2.6)–(2.10) and (2.11)–(2.15), respectively. Assume that $(y_h(u), r_h(u), p_h(y), s_h(y))$ are the solutions of (3.1)–(3.4), respectively. Then we have*

$$\|y_h(u) - y_h\|_1 + \|r_h(u) - r_h\| \leq C \|u - u_h\|, \tag{3.5}$$

$$\|p_h(y) - p_h\|_1 + \|s_h(y) - s_h\| \leq C \|y - y_h\|. \tag{3.6}$$

Proof Subtracting (2.11)–(2.12) from (3.1)–(3.2), we have

$$a(y_h(u) - y_h, I_h w_h) - b(r_h(u) - r_h, I_h w_h) = (u - u_h, I_h w_h), \quad \forall w_h \in V_h,$$

$$b(I_h (y_h(u) - y_h), \phi_h) = 0, \quad \forall \phi_h \in R_h.$$

Let $w_h = y_h(u) - y_h$ and $\phi_h = r_h(u) - r_h$. Note that

$$b(r_h(u) - r_h, I_h (y_h(u) - y_h)) = 0.$$

Then we can obtain

$$a(y_h(u) - y_h, I_h (y_h(u) - y_h)) = (u - u_h, I_h (y_h(u) - y_h)). \tag{3.7}$$

By using (2.2), we have

$$\begin{aligned} C\|y_h(u) - y_h\|_1^2 &\leq (u - u_h, I_h(y_h(u) - y_h)) \\ &\leq C\|u_h - u\| \cdot \|y_h(u) - y_h\|_1. \end{aligned} \tag{3.8}$$

It is clear that we obtain

$$\|y_h(u) - y_h\|_1 \leq C\|u - u_h\|. \tag{3.9}$$

Then, we deal with this term $\|r_h(u) - r_h\|$ by using (1.5)

$$\begin{aligned} \|r_h(u) - r_h\| &\leq \frac{1}{S} \sup_{w_h \in V_h} \frac{b(I_h v_h, r_h(u) - r_h)}{\|w_h\|_1} \\ &= \frac{1}{S} \sup_{w_h \in V_h} \frac{a(y_h(u) - y_h, I_h w_h) + (u_h - u, I_h w_h)}{\|w_h\|_1} \\ &\leq \|u - u_h\|. \end{aligned} \tag{3.10}$$

Similarly, we can obtain

$$\|p_h(u) - p_h\|_1 + \|s_h(u) - s_h\| \leq C\|y - y_h\|. \tag{3.11}$$

This completes the proof. □

Lemma 3.2 *Let (y, r, p, s, u) and $(y_h, r_h, p_h, s_h, u_h) \in V_h \times R_h \times V_h \times R_h \times U_{ad}$ be the solutions of (2.6)–(2.10) and (2.11)–(2.15), respectively. Assume $A \in W^{2,\infty}(\Omega)$ and $f, y_d \in H^1(\Omega)^2$. Then we have*

$$\|p_h(u) - p\| + \|y_h(u) - y\| \leq Ch^2, \tag{3.12}$$

$$\|r_h(u) - r\| + \|s_h(u) - s\| \leq Ch. \tag{3.13}$$

Proof Similar to the proof of Theorem 4.1 in [29], directly apply Lemma 3.1 to readily derive the following estimates:

$$\|y_h(u) - y\|_1 + \|r_h(u) - r\| \leq Ch, \tag{3.14}$$

$$\|p_h(u) - p\|_1 + \|s_h(u) - s\| \leq Ch. \tag{3.15}$$

Then we will estimate the derivation of L^2 -estimates for $y - y_h(u)$ and $p - p_h(y)$. Let us consider the dual problem: find (η, ρ) such that

$$-v\Delta\eta + \nabla\rho = y - y_h(u), \quad \text{in } \Omega, \tag{3.16}$$

$$\nabla \cdot \eta = 0, \quad \text{in } \Omega, \tag{3.17}$$

$$\eta = 0, \quad \text{on } \partial\Omega, \tag{3.18}$$

which is uniquely solvable; moreover, the following $H^2(\Omega) \times H^1(\Omega)$ -regularity is satisfied:

$$\|\eta\|_2 + \|\rho\|_1 \leq c\|y - y_h(u)\|. \tag{3.19}$$

Let $\eta_I \in V_h$ be the usual continuous piecewise linear interpolant; it is not hard to see that there exists a constant c independent of h such that

$$\|\eta - \eta_I\|_2 \leq ch\|\eta\|_2. \tag{3.20}$$

$$\|\eta - \eta_I\| \leq ch^2\|\eta\|_2. \tag{3.21}$$

Let Π denote the L^2 -projection from $L^2_0(\Omega)$ to Q_h , we can get

$$\|\rho - \Pi\rho\| \leq ch\|\rho\|_1. \tag{3.22}$$

Since $\eta_I \in V_h$ is a continuous interpolant of η ,

$$a(y - y_h(u), \eta_I) + b(\eta_I, r - r_h(u)) = 0, \tag{3.23}$$

$$b(y - y_h(u), \Pi\lambda) = 0. \tag{3.24}$$

Multiplying (3.16) by $y - y_h(u)$, integrating by parts, we can get

$$\|y - y_h(u)\|^2 = a(y - y_h(u), \eta) - b(y - y_h(u), \rho). \tag{3.25}$$

Note that

$$b(\eta_I, r - r_h(u)) = -\operatorname{div} \eta_I, r - r_h(u) - (\nabla r, \eta_I - I_h\eta_I) \tag{3.26}$$

and

$$a(y - y_h(u), \eta_I) = (\nabla(y - y_h(u)), \nabla\eta_I) - (\Delta y, \eta_I - I_h\eta_I). \tag{3.27}$$

Subtracting (3.23) from (3.25), we have

$$\begin{aligned} & \|y - y_h(u)\|^2 \\ &= a(y - y_h(u), \eta) - a(y - y_h(u), \eta_I) - b(y - y_h(u), \rho) - b(\eta_I, r - r_h(u)) \\ &= a(y - y_h(u), \eta) - a(y - y_h(u), \eta_I) + (\operatorname{div} \eta_I, r - r_h(u)) - (\Delta y, \eta_I - I_h\eta_I) \\ &\quad + (\nabla r, \eta_I - I_h\eta_I) - b(y - y_h(u), \rho) + b(y - y_h(u), \Pi\rho) \\ &= a(y - y_h(u), \eta - \eta_I) + (\operatorname{div} \eta_I, r - r_h(u)) \\ &\quad + (-\Delta y + \nabla r, \eta_I - I_h\eta_I) - b(y - y_h(u), \rho - \Pi\rho) \\ &\equiv E_1 + E_2 + E_3 + E_4. \end{aligned} \tag{3.28}$$

For the first term of (3.28), we can obtain

$$E_1 = a(y - y_h(u), \eta - \eta_I)$$

$$\begin{aligned} &\leq Ch^2 \|y\|_2 \|\eta\|_2 \\ &\leq Ch^2 \|y\|_2 \|y - y_h(u)\|. \end{aligned} \tag{3.29}$$

Then we estimate E_2 as follows:

$$\begin{aligned} E_2 &= (\operatorname{div} \eta_I, r - r_h(u)) \\ &\leq |(\operatorname{div}(\eta - \eta_I), r - r_h(u))| \\ &\leq Ch^2 \|r\|_1 \|y - y_h(u)\|. \end{aligned} \tag{3.30}$$

By using (1.2), we can obtain

$$\begin{aligned} E_3 &= (-\Delta y + \nabla r, \eta_I - I_h \eta_I) \\ &= (f + u, \eta_I - I_h \eta_I) \\ &\leq Ch^2 (\|f\|_1 + \|u\|_1 + \|y\|_2) \|y - y_h(u)\|. \end{aligned} \tag{3.31}$$

According to the quality of Π , we have

$$\begin{aligned} E_4 &= b(y - y_h(u), \rho - \Pi\rho) \\ &\leq Ch^2 \|y\|_2 \|y - y_h(u)\|. \end{aligned} \tag{3.32}$$

Putting (3.29)–(3.32) into (3.28), we can prove

$$\|y - y_h(u)\| \leq Ch^2. \tag{3.33}$$

In the same way, we can obtain

$$\|p - p_h(u)\| \leq Ch^2. \tag{3.34}$$

□

Now, we estimate the error of the approximate control in L^2 -norm.

Theorem 3.1 *Let (y, r, p, s, u) and $(y_h, r_h, p_h, s_h, u_h) \in V_h \times R_h \times V_h \times R_h \times U_{ad}$ be the solutions of (2.6)–(2.10) and (2.11)–(2.15), respectively. We assume $A \in W^{2,\infty}(\Omega)$ and $f, y_d \in H^1(\Omega)^2$. Then we have the following error estimate:*

$$\|u - u_h\| \leq Ch^2. \tag{3.35}$$

Proof Let $v = u_h$ in (2.10) and $v = u$ in (2.15), then we have

$$(u - p, u_h - u) \geq 0, \tag{3.36}$$

$$(u_h - p_h, u - u_h) \geq 0. \tag{3.37}$$

By using (3.36) and (3.37), we obtain

$$c\|u - u_h\|^2 \leq (p - p_h, u_h - u)$$

$$\begin{aligned}
 &= (p - p_h(y), u_h - u) + (p_h(y) - p_h, u_h - u) \\
 &= (p - p_h(y), u_h - u) + (I_h(p_h(y) - p_h), u_h - u) \\
 &\quad + ((p_h(y) - p_h) - I_h(p_h(y) - p_h), u_h - u) \\
 &\equiv E_1 + E_2 + E_3. \tag{3.38}
 \end{aligned}$$

Now, we estimate all terms on the right-hand side of (3.38). From Lemma 3.2 and δ -Cauchy inequality, we have

$$\begin{aligned}
 E_1 &= (p - p_h(y), u_h - u) \\
 &\leq C \|p - p_h(y)\| \cdot \|u - u_h\| \\
 &\leq Ch^2 \|u - u_h\|. \tag{3.39}
 \end{aligned}$$

Note that $b(r_h - r_h(u), I_h(p_h(y) - p_h)) = 0$ and $b(s_h(y) - s_h, I_h(y_h - y_h(u))) = 0$, we have

$$\begin{aligned}
 E_2 &= (I_h(p_h(y) - p_h), u_h - u) \\
 &= a(y_h - y_h(u), I_h(p_h(y) - p_h)) - b(r_h - r_h(u), I_h(p_h(y) - p_h)) \\
 &= a(y_h - y_h(u), I_h(p_h(y) - p_h)) - a(p_h(y) - p_h, I_h(y_h - y_h(u))) \\
 &\quad + a(p_h(y) - p_h, I_h(y_h - y_h(u))) - b(r_h - r_h(u), I_h(p_h(y) - p_h)) \\
 &= a(y_h - y_h(u), I_h(p_h(y) - p_h)) - a(p_h(y) - p_h, I_h(y_h - y_h(u))) \\
 &\quad + (y_h(u) - y_h, I_h(y_h - y_h(u))) - b(s_h(y) - s_h, I_h(y_h - y_h(u))). \tag{3.40}
 \end{aligned}$$

By applying $(y_h(u) - y_h, I_h(y_h - y_h(u))) \leq 0$ and Lemma 3.2, it is clear that

$$\begin{aligned}
 E_2 &= a(y_h - y_h(u), I_h(p_h(y) - p_h)) - a(p_h(y) - p_h, I_h(y_h - y_h(u))) \\
 &\leq Ch \|y_h - y_h(u)\|_1 \cdot \|p_h(y) - p_h\|_1 \\
 &\leq Ch \|u - u_h\| \cdot \|y - y_h\| \\
 &\leq Ch \|u - u_h\| \cdot (Ch^2 + \|u - u_h\|). \tag{3.41}
 \end{aligned}$$

According to (2.1) and Lemma 3.2, we obtain

$$\begin{aligned}
 E_3 &= ((p_h(y) - p_h) - I_h(p_h(y) - p_h), u_h - u) \\
 &\leq Ch \|p_h(y) - p_h\|_1 \cdot \|u - u_h\| \\
 &\leq Ch \|y - y_h\| \cdot \|u - u_h\| \\
 &\leq Ch \|u - u_h\| \cdot (Ch^2 + \|u - u_h\|). \tag{3.42}
 \end{aligned}$$

Finally, we can derive the result (3.35) from (3.38)–(3.42). □

Theorem 3.2 *Let (y, r, p, s, u) and $(y_h, r_h, p_h, s_h, u_h) \in V_h \times R_h \times V_h \times R_h \times U_{ad}$ be the solutions of (2.6)–(2.10) and (2.11)–(2.15), respectively. We assume $A \in W^{2,\infty}(\Omega)$ and*

$f, y_d \in H^1(\Omega)^2$. Then there exists $h_0 > 0$ such that, for all $0 < h \leq h_0$,

$$\|u - u_h\| + \|y - y_h\| + \|p - p_h\| \leq Ch^2, \tag{3.43}$$

$$\|r - r_h\| + \|s - s_h\| \leq Ch. \tag{3.44}$$

Proof Using the triangle inequality, we have

$$\|y - y_h\| \leq \|y - y_h(u)\| + \|y_h(u) - y_h\|,$$

$$\|p - p_h\| \leq \|p - p_h(u)\| + \|p_h(u) - p_h\|.$$

Lemma 3.1 implies that

$$\|y - y_h\| \leq \|y - y_h(u)\| + C\|u - u_h\|, \tag{3.45}$$

$$\|p - p_h\| \leq \|p - p_h(u)\| + C\|y - y_h\|. \tag{3.46}$$

By using Lemma 3.2, we can easily obtain

$$\|y - y_h\| \leq Ch^2. \tag{3.47}$$

By using (3.47) and Lemma 3.2, we derive

$$\|p - p_h\| \leq Ch^2. \tag{3.48}$$

From (3.47)–(3.48), we can immediately obtain (3.43). In the same way, we can obtain (3.44). □

Next, we will discuss the error estimates of the numerical solutions of the state and costate in H^1 -norm.

Theorem 3.3 *Assume that $A \in W^{2,\infty}(\Omega)$ and $f, y_d \in L^2(\Omega)^2$. Let (y, r, p, s, u) and $(y_h, r_h, p_h, s_h, u_h) \in V_h \times R_h \times V_h \times R_h \times U_{ad}$ be the solutions of (2.6)–(2.10) and (2.11)–(2.15), respectively. Then there exists $h_0 > 0$ such that, for all $0 < h \leq h_0$,*

$$\|y - y_h\|_1 + \|p - p_h\|_1 \leq Ch. \tag{3.49}$$

Proof Using the triangle inequality, we have

$$\|y - y_h\|_1 \leq \|y - y_h(u)\|_1 + \|y_h(u) - y_h\|_1,$$

$$\|p - p_h\|_1 \leq \|p - p_h(u)\|_1 + \|p_h(u) - p_h\|_1.$$

Lemma 3.1 implies that

$$\|y - y_h\|_1 \leq \|y - y_h(u)\|_1 + C\|u - u_h\|, \tag{3.50}$$

$$\|p - p_h\|_1 \leq \|p - p_h(u)\|_1 + C\|y - y_h\|. \tag{3.51}$$

From Theorem 3.2, (3.14)–(3.15), and (3.50)–(3.51), we can easily obtain (3.49). □

4 Conclusion and future works

In this paper, we considered a priori error estimates for the finite volume element approximation of Stokes optimal control problem. Then we used the finite volume method to discretize the state and adjoint equation of the system. Under some reasonable assumptions, we obtained some optimal order error estimates. The approximate orders for the state, costate, and control variables were $O(h^2)$, and the approximate orders for the state and costate variables was $O(h)$ in the sense of L^2 -norm and H^1 -norm. To our best knowledge, in the context of optimal control problems, these a priori error estimates of the finite volume method for Stokes optimal control problem are new.

In future, we shall consider a posteriori error estimates and superconvergence of the finite volume element solutions for Stokes optimal control problem.

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The authors declare that they have no competing interests.

Authors' contributions

LL and XW have participated in the sequence alignment and drafted the manuscript. RC, CM, and HF have made substantial contributions to the conception and design. All authors read and approved the final manuscript.

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