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A reverse Hardy–Hilbert-type integral inequality involving one derivative function

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Abstract

In this article, by using weight functions, the idea of introducing parameters, the reverse extended Hardy–Hilbert integral inequality and the techniques of real analysis, a reverse Hardy–Hilbert-type integral inequality involving one derivative function and the beta function is obtained. The equivalent statements of the best possible constant factor related to several parameters are considered. The equivalent form, the cases of non-homogeneous kernel and some particular inequalities are also presented.

MSC: 26D15

Keywords: Weight function; Hardy–Hilbert-type integral inequality; Derivative function; Parameter; Beta function; Reverse

1 Introduction

Assuming that $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, we have the following Hilbert inequality with the best possible constant factor π (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \pi \left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2 \right)^{1/2}. \quad (1)$$

If $0 < \int_0^{\infty} f^2(x) dx < \infty$ and $0 < \int_0^{\infty} g^2(y) dy < \infty$, then we still have the integral analogue of (1) as follows (cf. [1], Theorem 316):

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} dx dy < \pi \left(\int_0^{\infty} f^2(x) dx \int_0^{\infty} g^2(y) dy \right)^{1/2}, \quad (2)$$

where the constant factor π is the best possible. Inequalities (1) and (2) with their extensions play an important role in analysis and its applications (cf. [2–13]).

The following half-discrete Hilbert-type inequality was presented in 1934 (cf. [1], Theorem 351): If $K(x)$ ($x > 0$) is a non-negative decreasing function, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi(s) = \int_0^{\infty} K(x)x^{s-1} dx < \infty$, $f(x) \geq 0$, $0 < \int_0^{\infty} f^p(x) dx < \infty$, then

$$\sum_{n=1}^{\infty} n^{p-2} \left(\int_0^{\infty} K(nx)f(x) dx \right)^p < \phi^p \left(\frac{1}{q} \right) \int_0^{\infty} f^p(x) dx. \quad (3)$$

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In recent years, some new extensions and reverses of (3) were presented by [14–19].

In 2006, by using Euler–Maclaurin summation formula, Krnic et al. [20] gave an extension of (1) with the kernel $\frac{1}{(m+n)^\lambda}$ ($0 < \lambda \leq 14$). In 2019–2020, using the results of [20], A diyasuren et al. [21] considered an extension of (1) involving the partial sums, and Mo et al. [22] gave an extension of (2) involving the upper limit functions. In 2016–2017, by applying the weight functions, Hong et al. [23, 24] considered some equivalent statements of the extensions of (1) and (2) with several parameters. For some similar work, see [25–28].

In this paper, following [21, 23], by the use of weight functions, the idea of introducing parameters, the reverse extension of (1) and the technique of real analysis, a reverse Hardy–Hilbert-type integral inequality with the kernel $\frac{1}{(x+y)^{\lambda+1}}$ ($\lambda > 0$) involving one derivative function and the beta function is given. The equivalent statements of the best possible constant factor related to several parameters are considered. The equivalent form, the cases of non-homogeneous kernel and a few particular inequalities are obtained.

2 Some lemmas

In what follows, we assume that $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 0, \lambda_i \in (0, \lambda) (i = 1, 2), a := \lambda - \lambda_1 - \lambda_2, f(x)$ is a non-negative measurable function in $R_+ = (0, \infty)$, and $g(y)$ is a non-negative increasing differentiable function unless at finite points in R_+ , with $g(y) = o(1) (y \rightarrow 0^+), g(y) = o(e^{ty}) (t > 0; y \rightarrow \infty)$ satisfying

$$0 < \int_0^\infty x^{p(1-\lambda_1)-a-1} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_0^\infty y^{q(1-\lambda_2)-a-1} g'^q(y) dy < \infty.$$

By the definition of the gamma function, for $\lambda, x, y > 0$, the following expression holds (cf. [29]):

$$\frac{1}{(x+y)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-(x+y)t} dt, \tag{4}$$

where the gamma function is defined by

$$\Gamma(\alpha) := \int_0^\infty e^{-t} t^{\alpha-1} dt \quad (\alpha > 0),$$

satisfying

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha) \quad (\alpha > 0).$$

Lemma 1 *For $t > 0$, we have the following expression:*

$$\int_0^\infty e^{-ty} g(y) dy = \frac{1}{t} \int_0^\infty e^{-ty} g'(y) dy. \tag{5}$$

Proof Since $g(y) = o(1) (y \rightarrow 0^+)$, we find

$$\begin{aligned} \int_0^\infty e^{-ty} g'(y) dy &= \int_0^\infty e^{-ty} dg(y) \\ &= e^{-ty} g(y) \Big|_0^\infty - \int_0^\infty g(y) de^{-ty} = \lim_{y \rightarrow \infty} \frac{g(y)}{e^{ty}} + t \int_0^\infty e^{-ty} g(y) dy. \end{aligned}$$

In view of $g(y) = o(e^{ty})$ ($t > 0; y \rightarrow \infty$), we have $\lim_{y \rightarrow \infty} \frac{g(y)}{e^{ty}} = 0$, and then

$$t \int_0^\infty e^{-ty} g(y) dy = \int_0^\infty e^{-ty} g'(y) dy,$$

namely, Eq. (5) follows.

The lemma is proved. □

Lemma 2 Define the following weight functions:

$$\varpi(\lambda_2, x) := x^{\lambda-\lambda_2} \int_0^\infty \frac{t^{\lambda_2-1}}{(x+t)^\lambda} dt \quad (x \in \mathbb{R}_+), \tag{6}$$

$$\omega(\lambda_1, y) := y^{\lambda-\lambda_1} \int_0^\infty \frac{t^{\lambda_1-1}}{(t+y)^\lambda} dt \quad (y \in \mathbb{R}_+). \tag{7}$$

We have the following expressions:

$$\varpi(\lambda_2, x) = B(\lambda_2, \lambda - \lambda_2) \quad (x \in \mathbb{R}_+), \tag{8}$$

$$\omega(\lambda_1, y) = B(\lambda_1, \lambda - \lambda_1) \quad (y \in \mathbb{R}_+), \tag{9}$$

where $B(u, v) := \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}} dt$ ($u, v > 0$) is the beta function, such that

$$B(u, v) = \frac{1}{\Gamma(u+v)} \Gamma(u) \Gamma(v). \tag{10}$$

Proof Setting $u = \frac{t}{x}$, we find

$$\varpi(\lambda_2, x) = x^{\lambda-\lambda_2} \int_0^\infty \frac{(ux)^{\lambda_2-1}}{(x+ux)^\lambda} x du = \int_0^\infty \frac{u^{\lambda_2-1}}{(1+u)^\lambda} du = B(\lambda_2, \lambda - \lambda_2),$$

namely, (8) follows. In the same way, we have (9).

The lemma is proved. □

Lemma 3 We have the following reverse Hardy–Hilbert integral inequality involving one derivative function:

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g'(y)}{(x+y)^\lambda} dx \\ & > B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \\ & \quad \times \left[\int_0^\infty x^{p(1-\lambda_1)-a-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-a-1} g'^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \tag{11}$$

Proof By the reverse Hölder inequality (cf. [30]), we obtain

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g'(y)}{(x+y)^\lambda} dx dy \\ & = \int_0^\infty \int_0^\infty \frac{1}{(x+y)^\lambda} \left[\frac{y^{(\lambda_2-1)/p}}{x^{(\lambda_1-1)/q}} f(x) \right] \left[\frac{x^{(\lambda_1-1)/q}}{y^{(\lambda_2-1)/p}} g'(y) \right] dx dy \end{aligned}$$

$$\begin{aligned}
 &\geq \left\{ \int_0^\infty \left[\int_0^\infty \frac{1}{(x+y)^\lambda} \frac{y^{\lambda_2-1} dy}{x^{(\lambda_1-1)(p-1)}} \right] f^p(x) dx \right\}^{\frac{1}{p}} \\
 &\quad \times \left\{ \int_0^\infty \left[\int_0^\infty \frac{1}{(x+y)^\lambda} \frac{x^{\lambda_1-1} dx}{y^{(\lambda_2-1)(q-1)}} \right] g^q(y) dy \right\}^{\frac{1}{q}} \\
 &= \left[\int_0^\infty \varpi(\lambda_2, x) x^{p(1-\lambda_1)-a-1} f^p(x) dx \right]^{\frac{1}{p}} \\
 &\quad \times \left[\int_0^\infty \omega(\lambda_1, y) y^{q(1-\lambda_2)-a-1} g^q(y) dy \right]^{\frac{1}{q}}. \tag{12}
 \end{aligned}$$

If (12) keeps the form of an equality, then there exist constants A and B , such that they are not all zero, satisfying

$$A \frac{y^{\lambda_2-1}}{x^{(\lambda_1-1)(p-1)}} f^p(x) = B \frac{x^{\lambda_1-1}}{y^{(\lambda_2-1)(q-1)}} g^q(y) \quad \text{a.e. in } (0, \infty) \times (0, \infty).$$

We assume that $A \neq 0$. For fixed a.e. $y \in (0, \infty)$, we have

$$x^{p(1-\lambda_1)-a-1} f^p(x) = \left(\frac{B}{A} y^{q(1-\lambda_2)} g^q(y) \right) x^{-1-a} \quad \text{a.e. in } (0, \infty).$$

Integration in the above expression, since for any $a = \lambda - \lambda_1 - \lambda_2 \in \mathbf{R}$, $\int_0^\infty x^{-1-a} dx = \infty$, which contradicts the fact that

$$0 < \int_0^\infty x^{p(1-\lambda_1)-a-1} f^p(x) dx < \infty.$$

Therefore, by (8) and (9), we have (11).

The lemma is proved. □

3 Main results

Theorem 1 *We have the following reverse Hardy–Hilbert-type integral inequality involving one derivative function:*

$$\begin{aligned}
 I &:= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+1}} dx dy \\
 &> \frac{1}{\lambda} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \\
 &\quad \times \left[\int_0^\infty x^{p(1-\lambda_1)-a-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-a-1} g^q(y) dy \right]^{\frac{1}{q}}. \tag{13}
 \end{aligned}$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$ (or $a = 0$), we reduce (13) to the following:

$$\begin{aligned}
 &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+1}} dx dy \\
 &> \frac{1}{\lambda} B(\lambda_1, \lambda_2) \left(\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{q(1-\lambda_2)-1} g^q(y) dy \right)^{\frac{1}{q}}, \tag{14}
 \end{aligned}$$

where the constant factor $\frac{1}{\lambda} B(\lambda_1, \lambda_2)$ is the best possible.

Proof Using (4) and (5), in view of the Fubini theorem (cf. [31]), we find

$$\begin{aligned}
 I &= \frac{1}{\Gamma(\lambda + 1)} \int_0^\infty \int_0^\infty f(x)g(y) \left[\int_0^\infty t^{(\lambda+1)-1} e^{-(x+y)t} dt \right] dx dy \\
 &= \frac{1}{\Gamma(\lambda + 1)} \int_0^\infty t^{(\lambda+1)-1} \left(\int_0^\infty e^{-xt} f(x) dx \right) \left(\int_0^\infty e^{-yt} g(y) dy \right) dt \\
 &= \frac{1}{\Gamma(\lambda + 1)} \int_0^\infty t^{(\lambda+1)-1} \left(\int_0^\infty e^{-xt} f(x) dx \right) \left(\int_0^\infty t^{-1} e^{-yt} g'(y) dy \right) dt \\
 &= \frac{1}{\lambda \Gamma(\lambda)} \int_0^\infty \int_0^\infty f(x)g'(y) \left[\int_0^\infty t^{\lambda-1} e^{-(x+y)t} dt \right] dx dy \\
 &= \frac{\Gamma(\lambda)}{\lambda \Gamma(\lambda)} \int_0^\infty \int_0^\infty \frac{f(x)g'(y)}{(x + y)^\lambda} dx dy. \tag{15}
 \end{aligned}$$

Then by (11), we have (13).

For $a = 0$ in (13), we have (14). For any $\varepsilon > 0$, we set

$$\tilde{f}(x) := \begin{cases} 0, & 0 < x \leq 1, \\ x^{\lambda_1 - \frac{\varepsilon}{p} - 1}, & x > 1, \end{cases} \quad \tilde{g}(y) := \begin{cases} 0, & 0 < y \leq 1, \\ y^{\lambda_2 - \frac{\varepsilon}{q}}, & y > 1. \end{cases}$$

We obtain $\tilde{g}(y) = o(1)$ ($y \rightarrow 0^+$), $\tilde{g}(y) = o(e^{ty})$ ($t > 0; y \rightarrow \infty$), $\tilde{g}'(y) \equiv 0$ ($0 < y < 1$), and

$$\tilde{g}'(y) = \left(\lambda_2 - \frac{\varepsilon}{q} \right) y^{\lambda_2 - \frac{\varepsilon}{q} - 1} \quad (y > 1).$$

If there exists a constant $M (\geq \frac{1}{\lambda} B(\lambda_1, \lambda_2))$, such that (14) is valid when replacing $\frac{1}{\lambda} B(\lambda_1, \lambda_2)$ by M , then in particular, by substitution of $f(x) = \tilde{f}(x), g(y) = \tilde{g}(y)$ and $g'(y) = \tilde{g}'(y)$, we have

$$\begin{aligned}
 \tilde{I} &:= \int_0^\infty \int_0^\infty \frac{\tilde{f}(x)\tilde{g}(y)}{(x + y)^{\lambda+1}} dx dy \\
 &> M \left[\int_0^\infty x^{p(1-\lambda_1)-1} \tilde{f}^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1} \tilde{g}'^q(y) dy \right]^{\frac{1}{q}}. \tag{16}
 \end{aligned}$$

We obtain

$$\begin{aligned}
 \tilde{J} &:= \left[\int_0^\infty x^{p(1-\lambda_1)-1} \tilde{f}^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1} \tilde{g}'^q(y) dy \right]^{\frac{1}{q}} \\
 &= \left(\lambda_2 - \frac{\varepsilon}{q} \right) \left(\int_1^\infty x^{-\varepsilon-1} dx \right)^{\frac{1}{p}} \left(\int_1^\infty y^{-\varepsilon-1} dy \right)^{\frac{1}{q}} \\
 &= \left(\lambda_2 - \frac{\varepsilon}{q} \right) \int_1^\infty x^{-\varepsilon-1} dx = \frac{1}{\varepsilon} \left(\lambda_2 - \frac{\varepsilon}{q} \right).
 \end{aligned}$$

In view of the Fubini theorem (cf. [31]), it follows that

$$\tilde{I} = \int_1^\infty \left[\int_1^\infty \frac{y^{\lambda_2 - \frac{\varepsilon}{q}}}{(x + y)^{\lambda+1}} dy \right] x^{\lambda_1 - \frac{\varepsilon}{p} - 1} dx = \int_1^\infty x^{-\varepsilon-1} \left[\int_{\frac{1}{x}}^\infty \frac{u^{\lambda_2 - \frac{\varepsilon}{q}}}{(1 + u)^{\lambda+1}} du \right] dx$$

$$\begin{aligned}
 &= \int_1^\infty x^{-\varepsilon-1} \left[\int_{\frac{1}{x}}^1 \frac{u^{\lambda_2 - \frac{\varepsilon}{q}}}{(1+u)^{\lambda+1}} du \right] dx + \int_1^\infty x^{-\varepsilon-1} \left[\int_1^\infty \frac{u^{\lambda_2 - \frac{\varepsilon}{q}}}{(1+u)^{\lambda+1}} du \right] dx \\
 &= \int_0^1 \left(\int_{\frac{1}{u}}^\infty x^{-\varepsilon-1} dx \right) \frac{u^{\lambda_2 - \frac{\varepsilon}{q}}}{(1+u)^{\lambda+1}} du + \frac{1}{\varepsilon} \int_1^\infty \frac{u^{\lambda_2 - \frac{\varepsilon}{q}}}{(1+u)^{\lambda+1}} du \\
 &= \frac{1}{\varepsilon} \left[\int_0^1 \frac{u^{\lambda_2 + \frac{\varepsilon}{p}}}{(1+u)^{\lambda+1}} du + \int_1^\infty \frac{u^{\lambda_2 - \frac{\varepsilon}{q}}}{(1+u)^{\lambda+1}} du \right].
 \end{aligned}$$

By (16), we obtain

$$\int_0^1 \frac{u^{\lambda_2 + \frac{\varepsilon}{p}}}{(1+u)^{\lambda+1}} du + \int_1^\infty \frac{u^{\lambda_2 - \frac{\varepsilon}{q}}}{(1+u)^{\lambda+1}} du > \varepsilon \tilde{I} > \varepsilon M \tilde{J} > M \left(\lambda_2 - \frac{\varepsilon}{q} \right).$$

Putting $\varepsilon \rightarrow 0^+$ in the above inequality, in view of the continuity of the beta function, we find

$$\frac{\lambda_2}{\lambda} B(\lambda_1, \lambda_2) = \frac{\lambda_2 \Gamma(\lambda_1) \Gamma(\lambda_2)}{\lambda \Gamma(\lambda)} = B(\lambda_1, \lambda_2 + 1) \geq M \lambda_2,$$

namely, $\frac{1}{\lambda} B(\lambda_1, \lambda_2) \geq M$. Hence, $M = \frac{1}{\lambda} B(\lambda_1, \lambda_2)$ is the best possible constant factor in (14).

The theorem is proved. □

Remark 1 We set $\hat{\lambda}_1 := \lambda_1 + \frac{a}{p} = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \lambda_2 + \frac{a}{q} = \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}$. It follows that $\hat{\lambda}_1 + \hat{\lambda}_2 = \lambda$. For

$$a = \lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1)),$$

we find $0 < \hat{\lambda}_1 < \lambda$, and then $0 < \hat{\lambda}_2 = \lambda - \hat{\lambda}_1 < \lambda$. So we rewrite (13) as follows:

$$\begin{aligned}
 I &:= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+1}} dx dy \\
 &> \frac{1}{\lambda} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \\
 &\quad \times \left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\hat{\lambda}_2)-1} g'^q(y) dy \right]^{\frac{1}{q}}. \tag{17}
 \end{aligned}$$

Theorem 2 *If the constant factor*

$$\frac{1}{\lambda} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

in (13) (or (17)) is the best possible, then $\lambda_1 + \lambda_2 = \lambda$.

Proof By (14) (for $\lambda_i = \hat{\lambda}_i$ ($i = 1, 2$)), since

$$\frac{1}{\lambda} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

is the best possible constant factor in (17), we have the following inequality:

$$\frac{1}{\lambda} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \geq \frac{1}{\lambda} B(\hat{\lambda}_1, \hat{\lambda}_2) \quad (\in \mathbb{R}_+),$$

namely,

$$B(\hat{\lambda}_1, \hat{\lambda}_2) \leq B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1).$$

By the reverse Hölder inequality (cf. [30]), we obtain

$$\begin{aligned} B(\hat{\lambda}_1, \hat{\lambda}_2) &= \int_0^\infty \frac{u^{\hat{\lambda}_1-1}}{(1+u)^\lambda} du \\ &= \int_0^\infty \frac{1}{(1+u)^\lambda} u^{\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} - 1} du = \int_0^\infty \frac{1}{(1+u)^\lambda} (u^{\frac{\lambda-\lambda_2-1}{p}}) (u^{\frac{\lambda_1-1}{q}}) du \\ &\geq \left[\int_0^\infty \frac{u^{\lambda-\lambda_2-1}}{(1+u)^\lambda} du \right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\lambda_1-1}}{(1+u)^\lambda} du \right]^{\frac{1}{q}} \\ &= B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1). \end{aligned} \tag{18}$$

It follows that (18) keeps the form of an equality.

We observe that (18) keeps the form of an equality if and only if there exist constants A and B , such that they are not all zero and

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \quad \text{a.e. in } R_+$$

(cf. [30]). Assuming that $A \neq 0$, it follows that

$$u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A} \quad \text{a.e. in } R_+.$$

We have $a = \lambda - \lambda_1 - \lambda_2 = 0$, namely, $\lambda_1 + \lambda_2 = \lambda$.

The theorem is proved. □

Theorem 3 *The following statements (i), (ii), (iii) and (iv) are equivalent:*

- (i) Both $B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$ and $B(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}, \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})$ are finite and independent of p, q ;
- (ii) $B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$ is equal to a single convergent integral

$$B(\hat{\lambda}_1, \hat{\lambda}_2) = \int_0^\infty \frac{u^{\hat{\lambda}_1-1}}{(1+u)^\lambda} du;$$

- (iii) if $a = \lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1))$, then $\lambda_1 + \lambda_2 = \lambda$;
- (iv) the constant factor

$$\frac{1}{\lambda} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

is the best possible in (13).

Proof (i) \Rightarrow (ii). In view of the assumption and the continuity of the beta function, we find

$$B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

$$\begin{aligned}
 &= \lim_{p \rightarrow 1^-} \lim_{q \rightarrow -\infty} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) = B(\lambda_2, \lambda - \lambda_2), \\
 B(\hat{\lambda}_1, \hat{\lambda}_2) &= \lim_{p \rightarrow 1^-} \lim_{q \rightarrow -\infty} B\left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}, \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}\right) = B(\lambda_2, \lambda - \lambda_2).
 \end{aligned}$$

Hence, $B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$ is equal to $B(\hat{\lambda}_1, \hat{\lambda}_2)$, which is a single convergent integral.

(ii) \Rightarrow (iii). Suppose that $B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$ is equal to a single convergent integral $\int_0^\infty \frac{1}{(1+u)^\lambda} u^{\lambda_1-1} du (\in \mathbb{R}_+)$. Then (18) keeps the form of an equality. By the proof of Theorem 2, we have $\lambda_1 + \lambda_2 = \lambda$.

(iii) \Rightarrow (iv). If $\lambda_1 + \lambda_2 = \lambda$, then by Theorem 1, the constant factor

$$\frac{1}{\lambda} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \left(= \frac{1}{\lambda} B(\lambda_1, \lambda_2) \right)$$

in (13) is the best possible.

(iv) \Rightarrow (i). By Theorem 2, we have $\lambda_1 + \lambda_2 = \lambda$, and then

$$\begin{aligned}
 &B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) = B(\lambda_1, \lambda_2), \\
 &B\left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}, \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}\right) = B(\lambda_1, \lambda_2).
 \end{aligned}$$

It follows that both of them are finite and independent of p, q .

Hence, the statements (i), (ii), (iii) and (iv) are equivalent.

The theorem is proved. □

Remark 2 For $a = 0$ in (11), we have

$$\begin{aligned}
 &\int_0^\infty \int_0^\infty \frac{f(x)g'(y)}{(x+y)^\lambda} dx \\
 &> B(\lambda_1, \lambda_2) \left[\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1} g'^q(y) dy \right]^{\frac{1}{q}}. \tag{19}
 \end{aligned}$$

We conform that the constant factor $B(\lambda_1, \lambda_2)$ in (19) is the best possible. Otherwise, we would reach a contradiction by (15) (for $a = 0$): the constant factor in (14) is not the best possible.

4 Equivalent form and some particular inequalities

Theorem 4 *Inequality (13) is equivalent to the following reverse Hardy–Hilbert-type integral inequality involving one derivative function:*

$$\begin{aligned}
 J &:= \left\{ \int_0^\infty x^{q(\lambda_1+a)-a-1} \left[\int_0^\infty \frac{g(y)}{(x+y)^{\lambda+1}} dy \right]^q dx \right\}^{\frac{1}{q}} \\
 &> \frac{1}{\lambda} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \left[\int_0^\infty y^{q(1-\lambda_2)-a-1} g'^q(y) dy \right]^{\frac{1}{q}}. \tag{20}
 \end{aligned}$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$ (or $a = 0$), we reduce (20) to the equivalent form of (14) as follows:

$$\left\{ \int_0^\infty x^{q\lambda_1-1} \left[\int_0^\infty \frac{g(y)}{(x+y)^{\lambda+1}} dy \right]^q dx \right\}^{\frac{1}{q}} > \frac{1}{\lambda} B(\lambda_1, \lambda_2) \left[\int_0^\infty y^{q(1-\lambda_2)-1} g'^q(y) dy \right]^{\frac{1}{q}}, \tag{21}$$

where the constant factor $\frac{1}{\lambda} B(\lambda_1, \lambda_2)$ is the best possible.

Proof Suppose that (20) is valid. By the reverse Hölder integral inequality (cf. [30]), we have

$$\begin{aligned} I &= \int_0^\infty \left[x^{\frac{1}{q}-\lambda_1-\frac{a}{p}} f(x) \right] \left[x^{-\frac{1}{q}+\lambda_1+\frac{a}{p}} \int_0^\infty \frac{g(y)}{(x+y)^{\lambda+1}} dy \right] dx \\ &\geq \left\{ \int_0^\infty x^{p(1-\lambda_1)-a-1} f^p(x) dx \right\}^{\frac{1}{p}} J. \end{aligned} \tag{22}$$

Then by (20), we have (13).

On the other hand, assuming that (13) is valid, we set

$$f(x) := x^{q(\lambda_1+a)-a-1} \left[\int_0^\infty \frac{g(y)}{(x+y)^{\lambda+1}} dy \right]^{q-1}, \quad x \in \mathbb{R}_+.$$

If $J = \infty$, then (20) is naturally valid; if $J = 0$, then it is impossible to make (20) valid, namely $J > 0$. Suppose that $0 < J < \infty$. By (13), we have

$$\begin{aligned} &\int_0^\infty x^{p(1-\lambda_1)-a-1} f^p(x) dx \\ &= J^q = I > \frac{1}{\lambda} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) J^{q-1} \left[\int_0^\infty y^{q(1-\lambda_2)-a-1} g'^q(y) dy \right]^{\frac{1}{q}}, \\ J &= \left[\int_0^\infty x^{p(1-\lambda_1)-a-1} f^p(x) dx \right]^{\frac{1}{q}} \\ &> \frac{1}{\lambda} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \left[\int_0^\infty y^{q(1-\lambda_2)-a-1} g'^q(y) dy \right]^{\frac{1}{q}}, \end{aligned}$$

namely, (20) follows, which is equivalent to (13).

The constant factor $\frac{1}{\lambda} B(\lambda_1, \lambda_2)$ is the best possible in (21). Otherwise, by (22) (for $a = 0$), we would reach a contradiction: that the constant factor in (14) is not the best possible.

The theorem is proved. □

Replacing x by $\frac{1}{x}$, and then replacing $x^{\lambda-1} f(\frac{1}{x})$ by $f(x)$ in (13) and (20), by calculation, we have the following.

Corollary 1 *The following reverse Hardy–Hilbert-type integral inequalities with the non-homogeneous kernel involving one derivative function are equivalent:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(1+xy)^{\lambda+1}} dx dy$$

$$\begin{aligned}
 &> \frac{1}{\lambda} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \\
 &\quad \times \left[\int_0^\infty x^{p(\lambda_1 - \lambda) + a - 1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1 - \lambda_2) - a - 1} g'^q(y) dy \right]^{\frac{1}{q}}, \tag{23}
 \end{aligned}$$

$$\begin{aligned}
 &\left\{ \int_0^\infty x^{q(\lambda - \lambda_1 - a + 1) + a - 1} \left[\int_0^\infty \frac{g(y)}{(1 + xy)^{\lambda + 1}} dy \right]^q dx \right\}^{\frac{1}{q}} \\
 &> \frac{1}{\lambda} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \left[\int_0^\infty y^{q(1 - \lambda_2) - a - 1} g'^q(y) dy \right]^{\frac{1}{q}}. \tag{24}
 \end{aligned}$$

Moreover, $\lambda_1 + \lambda_2 = \lambda$ (or $a = 0$) if and only if the constant factor $\frac{1}{\lambda} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$ in (23) and (24) is the best possible.

For $\lambda_1 + \lambda_2 = \lambda$ (or $a = 0$), we have the following reverse equivalent inequalities with the non-homogeneous kernel and the best possible constant factor $\frac{1}{\lambda} B(\lambda_1, \lambda_2)$:

$$\begin{aligned}
 &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(1 + xy)^{\lambda + 1}} dx dy \\
 &> \frac{1}{\lambda} B(\lambda_1, \lambda_2) \times \left(\int_0^\infty x^{-p\lambda_2 - 1} f^p(x) dx \right)^{\frac{1}{p}} \left[\int_0^\infty y^{q(1 - \lambda_2) - 1} g'^q(y) dy \right]^{\frac{1}{q}}, \tag{25}
 \end{aligned}$$

$$\begin{aligned}
 &\left\{ \int_0^\infty x^{q(\lambda_2 + 1) - 1} \left[\int_0^\infty \frac{g(y)}{(1 + xy)^{\lambda + 1}} dy \right]^q dx \right\}^{\frac{1}{q}} \\
 &> \frac{1}{\lambda} B(\lambda_1, \lambda_2) \left[\int_0^\infty y^{q(1 - \lambda_2) - 1} g'^q(y) dy \right]^{\frac{1}{q}}. \tag{26}
 \end{aligned}$$

Remark 3 For $\lambda_1 = \frac{\lambda}{r}, \lambda_2 = \frac{\lambda}{s}$ ($r > 1, \frac{1}{r} + \frac{1}{s} = 1$) in (14), (21), (25) and (26), we have the following two couples of reverse equivalent integral inequalities with the same best possible constant factor $\frac{1}{\lambda} B(\frac{\lambda}{r}, \frac{\lambda}{s})$:

$$\begin{aligned}
 &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x + y)^{\lambda + 1}} dx dy \\
 &> \frac{1}{\lambda} B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \left[\int_0^\infty x^{p(1 - \frac{\lambda}{r}) - 1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1 - \frac{\lambda}{s}) - 1} g'^q(y) dy \right]^{\frac{1}{q}}, \tag{27}
 \end{aligned}$$

$$\begin{aligned}
 &\left\{ \int_0^\infty x^{\frac{q\lambda}{r} - 1} \left[\int_0^\infty \frac{g(y)}{(x + y)^{\lambda + 1}} dy \right]^q dx \right\}^{\frac{1}{q}} \\
 &> \frac{1}{\lambda} B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \left[\int_0^\infty y^{q(1 - \frac{\lambda}{s}) - 1} g'^q(y) dy \right]^{\frac{1}{q}}; \tag{28}
 \end{aligned}$$

$$\begin{aligned}
 &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(1 + xy)^{\lambda + 1}} dx dy \\
 &> \frac{1}{\lambda} B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \left(\int_0^\infty x^{-\frac{p\lambda}{s} - 1} f^p(x) dx \right)^{\frac{1}{p}} \left[\int_0^\infty y^{q(1 - \frac{\lambda}{s}) - 1} g'^q(y) dy \right]^{\frac{1}{q}}, \tag{29}
 \end{aligned}$$

$$\begin{aligned}
 &\left\{ \int_0^\infty x^{q(\frac{\lambda}{s} + 1) - 1} \left[\int_0^\infty \frac{g(y)}{(1 + xy)^{\lambda + 1}} dy \right]^q dx \right\}^{\frac{1}{q}} \\
 &> \frac{1}{\lambda} B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \left[\int_0^\infty y^{q(1 - \frac{\lambda}{s}) - 1} g'^q(y) dy \right]^{\frac{1}{q}}. \tag{30}
 \end{aligned}$$

In particular, for $\lambda = 1, r = s = 2$, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^2} dx dy > \pi \left(\int_0^\infty x^{\frac{p}{2}-1} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{\frac{q}{2}-1} g'^q(y) dy \right)^{\frac{1}{q}}, \quad (31)$$

$$\left\{ \int_0^\infty x^{\frac{q}{2}-1} \left[\int_0^\infty \frac{g(y)}{(x+y)^2} dy \right]^q dx \right\}^{\frac{1}{q}} > \pi \left(\int_0^\infty y^{\frac{q}{2}-1} g'^q(y) dy \right)^{\frac{1}{q}}; \quad (32)$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(1+xy)^2} dx dy > \pi \left(\int_0^\infty x^{-\frac{p}{2}-1} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{\frac{q}{2}-1} g'^q(y) dy \right)^{\frac{1}{q}}, \quad (33)$$

$$\left\{ \int_0^\infty x^{\frac{3q}{2}-1} \left[\int_0^\infty \frac{g(y)}{(1+xy)^2} dy \right]^q dx \right\}^{\frac{1}{q}} > \pi \left(\int_0^\infty y^{\frac{q}{2}-1} g'^q(y) dy \right)^{\frac{1}{q}}. \quad (34)$$

5 Conclusions

In this paper, following [21, 23], by the use of weight functions, the idea of introducing parameters, the reverse extension of (1) and the technique of real analysis, a reverse Hardy–Hilbert-type integral inequality with the kernel $\frac{1}{(x+y)^{\lambda+1}}$ ($\lambda > 0$) involving one derivative function and the beta function is given in Theorem 1. The equivalent statements of the best possible constant factor related to several parameters are considered in Theorem 3. The equivalent form, the cases of non-homogeneous kernel and a few particular inequalities are obtained in Theorem 4, Corollary 1 and Remark 3. The lemmas and theorems provide an extensive account of this type of inequalities.

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Availability of data and materials

We declare that the data and material in this paper can be used publicly.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. QC participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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