Wu et al. *Journal of Inequalities and Applications* (2021) 2021:32
https://doi.org/10.1186/s13660-020-02527-1

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On some Hermite–Hadamard inequalities involving k-fractional operators

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Abstract

The main objective of this paper is to establish some new Hermite–Hadamard type inequalities involving k-Riemann–Liouville fractional integrals. Using the convexity of differentiable functions some related inequalities have been proved, which have deep connection with some known results. At the end, some applications of the obtained results in error estimations of quadrature formulas are also considered.

MSC: Primary 26D15; 26D10; secondary 26A33

Keywords: Convex function; Hermite–Hadamard inequalities; Fractional integrals

1 Introduction

In literature, inequalities are very important for convex functions especially the integral inequalities for convex functions originated form Hermite and Hadamard (see [11, p. 137]). The researchers have worked on Hermite–Hadamard type inequalities since 1893 [4]. The classical Hermite–Hadamard inequality reads as follows: if $f : I \to \mathbb{R}$ is convex on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(\mu)\,d\mu \le \frac{f(a)+f(b)}{2}.$$

We note that the Hermite–Hadamard inequality may be assessed as a treatment of the conception of convexity. The Hermite–Hadamard inequality for convex functions has conferred revived awareness in the latest years and some unusual variations of essential and conclusion have been established (see, for example, [5, 6, 14, 17]). In the last few years, the theory of inequalities has progressed very fast. The evolution of the hypothesis associated with ancient inequalities has developed in a resumption of attentiveness in this field. In many classical inequalities, the Hermite–Hadamard inequality is one of the important inequality of analysis. Such an inequality has been applied for different types of problems of fractional calculus (see [1, 2, 8–10, 15, 16]). In this paper, as a continuation of the study of the Hermite–Hadamard inequality, we establish some results for k-Riemann–Liouville fractional integral by using the definition of convex functions via fractional calculus.

Below, let us recall first some basic concepts and some earlier results.

Definition 1 A function $f : [a,b] \to \mathbb{R}$ is said to be convex on an interval $[a,b] \subseteq \mathbb{R}$, if

$$f\big(\tau\xi + (1-\tau)\eta\big) \le \tau f(\xi) + (1-\tau)f(\eta) \tag{1}$$

holds for $\xi, \eta \in [a,b]$ and $\tau \in [0,1]$.

Definition 2 ([13]) Let $f \in L_1[a,b]$. The Riemann–Liouville integrals $I^{\lambda}_{a^+}f$ and $I^{\lambda}_{b^-}f$ of order $\lambda > 0$ with $a \ge 0$ are defined by

$$I^{\lambda}_{a^+}f(\xi) = \frac{1}{\Gamma(\lambda)}\int_a^{\xi} (\xi-\mu)^{\lambda-1}f(\mu)\,d\mu, \quad \xi > a,$$

and

$$I^{\lambda}_{b^-}f(\xi) = \frac{1}{\Gamma(\lambda)}\int_{\xi}^{b} (\mu-\xi)^{\lambda-1}f(\mu)\,d\mu, \quad \xi < b,$$

respectively, where Γ is the classical Gamma function and $I^0_{a^+}f = I^0_{b^-}f = f(\xi)$.

Theorem 1 ([14]) *Consider $f : [a,b] \to \mathbb{R}$ a positive mapping with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\lambda+1)}{2(b-a)^{\lambda}}\big[I^{\lambda}_{a^+}f(b) + I^{\lambda}_{b^-}f(a)\big] \le \frac{f(a)+f(b)}{2}. \tag{2}$$

Lemma 1.1 ([14]) *Suppose $f : [a,b] \to \mathbb{R}$ is a differentiable function on (a,b) with $a < b$. If $f' \in Ł[a,b]$, then*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\lambda+1)}{2(b-a)^{\lambda}}\big[I^{\lambda}_{a^+}f(b) + I^{\lambda}_{b^-}f(a)\big] \\ &\quad = \frac{b-a}{2}\int_0^1 \big[(1-\mu)^{\lambda} - \mu^{\lambda}\big]f'\big(\mu a + (1-\mu)b\big)\,d\mu. \end{aligned} \tag{3}$$

Theorem 2 ([14]) *Assume that $f : [a,b] \to \mathbb{R}$ is a differentiable function on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\lambda+1)}{2(b-a)^{\lambda}}\big[I^{\lambda}_{a^+}f(b) + I^{\lambda}_{b^-}f(a)\big]\right| \\ &\quad \le \frac{b-a}{2(\lambda+1)}\left(1 - \frac{1}{2^{\lambda}}\right)\big[f'(a) + f'(b)\big]. \end{aligned} \tag{4}$$

Theorem 3 ([12]) *Let $f : [a,b] \to \mathbb{R}$ be a positive mapping with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{2^{\lambda-1}\Gamma(\lambda+1)}{(b-a)^{\lambda}}\big[I^{\lambda}_{(\frac{a+b}{2})^+}f(b) + I^{\lambda}_{(\frac{a+b}{2})^-}f(a)\big] \le \frac{f(a)+f(b)}{2}. \tag{5}$$

Lemma 1.2 ([12]) *Suppose $f:[a,b]\to\mathbb{R}$ is a differentiable function on (a,b) with $a<b$. If $f'\in Ł[a,b]$, then*

$$\begin{aligned}&\frac{2^{\lambda-1}\Gamma(\lambda+1)}{(b-a)^{\lambda}}\left[I^{\lambda}_{(\frac{a+b}{2})^{+}}f(b)+I^{\lambda}_{(\frac{a+b}{2})^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\\&\quad=\frac{b-a}{4}\left[\int_{0}^{1}\mu^{\lambda}f'\left(\frac{\mu}{2}a+\frac{2-\mu}{2}b\right)d\mu-\int_{0}^{1}\mu^{\lambda}f'\left(\frac{\mu}{2}b+\frac{2-\mu}{2}a\right)d\mu\right].\end{aligned}\tag{6}$$

Theorem 4 ([12]) *Consider $f:[a,b]\to\mathbb{R}$, a differentiable function on (a,b) with $a<b$. If $|f'|^{h}$ is convex on $[a,b]$ for $h\geq 1$, then*

$$\begin{aligned}&\left|\frac{2^{\lambda-1}\Gamma(\lambda+1)}{(b-a)^{\lambda}}\left[I^{\lambda}_{(\frac{a+b}{2})^{+}}f(b)+I^{\lambda}_{(\frac{a+b}{2})^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\quad\leq\frac{b-a}{4(\lambda+1)}\left(\frac{1}{2(\lambda+2)}\right)^{\frac{1}{h}}\left\{\left((\lambda+1)\left|f'(a)\right|^{h}+(\lambda+3)\left|f'(b)\right|^{h}\right)^{\frac{1}{h}}\right.\\&\qquad\left.+\left((\lambda+3)\left|f'(a)\right|^{h}+(\lambda+1)\left|f'(b)\right|^{h}\right)^{\frac{1}{h}}\right\}.\end{aligned}\tag{7}$$

2 Hermite–Hadamard's inequalities for *k*-fractional integrals

In [3], the k-gamma function was introduced by Diaz *et al.* as follows.

Definition 3 Let k and $\mathbb{R}(\nu)$ be positive. Then the k-gamma function is defined by following integral:

$$\Gamma_{k}(\xi)=\int_{0}^{\infty}\nu^{\xi-1}\exp\left(-\frac{\nu^{k}}{k}\right)d\nu.$$

Definition 4 ([7]) If $k>0$, Let $f\in L_{1}(a,b)$, $a\geq 0$, then k-Riemann–Liouville fractional integrals $I^{\lambda}_{a^{+},k}f$ and $I^{\lambda}_{b^{-},k}f$ of order $\lambda>0$ for a real-valued continuous function $f(\mu)$ are defined by

$$I^{\lambda}_{a^{+},k}f(\xi)=\frac{1}{k\Gamma_{k}(\lambda)}\int_{a}^{\xi}(\xi-\mu)^{\frac{\lambda}{k}-1}f(\mu)\,d\mu,\quad \xi>a,$$

and

$$I^{\lambda}_{b^{-},k}f(\xi)=\frac{1}{k\Gamma_{k}(\lambda)}\int_{\xi}^{b}(\mu-\xi)^{\frac{\lambda}{k}-1}f(\mu)\,d\mu,\quad \xi<b,$$

respectively. Here Γ_{k} is the k-Gamma function.

Theorem 5 *Let $k>0$, $I^{\lambda}_{a^{+},k}f$ and $I^{\lambda}_{b^{-},k}f$ be the left and right sided k-Riemann–Liouville fractional integral of order $\lambda>0$. Let $f:[a,b]\to\mathbb{R}$ be positive mapping with $0\leq a<b$, $f\in L_{1}[a,b]$. If f is convex on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma_{k}(\lambda+k)}{2(b-a)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{a^{+},k}f(b)+I^{\lambda}_{b^{-},k}f(a)\right]\leq\frac{f(a)+f(b)}{2}.\tag{8}$$

Proof As f is convex mapping on $[a,b]$, we have, for $\xi,\eta \in [a,b]$ with $\tau = \frac{1}{2}$ in (1),

$$f\left(\frac{\xi+\eta}{2}\right) \leq \frac{f(\xi)+f(\eta)}{2}. \tag{9}$$

Now let $\xi = \mu a + (1-\mu)b$ and $\eta = \mu b + (1-\mu)a$, then (9) becomes

$$2f\left(\frac{a+b}{2}\right) \leq f\bigl(\mu a + (1-\mu)b\bigr) + f\bigl(\mu b + (1-\mu)a\bigr). \tag{10}$$

Multiplying both sides of (10) by $\mu^{\frac{\lambda}{k}-1}$, then integrating with respect to μ over $[0,1]$, we get

$$\begin{aligned}\frac{2k}{\lambda} f\left(\frac{a+b}{2}\right) &\leq \int_0^1 \mu^{\frac{\lambda}{k}-1} f\bigl(\mu a + (1-\mu)b\bigr)\,d\mu + \int_0^1 \mu^{\frac{\lambda}{k}-1} f\bigl(\mu b + (1-\mu)a\bigr)\,d\mu \\ &= I_1 + I_2, \end{aligned} \tag{11}$$

where

$$I_1 = \int_0^1 \mu^{\frac{\lambda}{k}-1} f\bigl(\mu a + (1-\mu)b\bigr)\,d\mu$$

and

$$I_2 = \int_0^1 \mu^{\frac{\lambda}{k}-1} f\bigl(\mu b + (1-\mu)a\bigr)\,d\mu.$$

By taking $\mu a + (1-\mu)b = \phi$ in I_1 and $\mu b + (1-\mu)a = \omega$ in I_2, we get

$$I_1 = \frac{1}{(b-a)^{\frac{\lambda}{k}}} \int_a^b (b-\phi)^{\frac{\lambda}{k}-1} f(\phi)\,d\phi \tag{12}$$

and

$$I_2 = \frac{1}{(b-a)^{\frac{\lambda}{k}}} \int_a^b (\omega-a)^{\frac{\lambda}{k}-1} f(\omega)\,d\omega. \tag{13}$$

Substituting the values of I_1 and I_2 from (12) and (13) in (11), we get

$$\frac{2k}{\lambda} f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)^{\frac{\lambda}{k}}} \left[\int_a^b (b-\phi)^{\frac{\lambda}{k}-1} f(\phi)\,d\phi + \int_a^b (\omega-a)^{\frac{\lambda}{k}-1} f(\omega)\,d\omega\right],$$

which implies that

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma_k(\lambda+k)}{2(b-a)^{\frac{\lambda}{k}}} \bigl[I^{\lambda}_{a^+,k} f(b) + I^{\lambda}_{b^-,k} f(a)\bigr]. \tag{14}$$

This completes the first inequality in (8). To complete the second inequality, we note that if f is convex, then, for $\tau \in [0,1]$, it yields that

$$f\bigl(\mu a + (1-\mu)b\bigr) \leq \mu f(a) + (1-\mu) f(b)$$

and

$$f\big(\mu b + (1-\mu)a\big) \le \mu f(b) + (1-\mu)f(a).$$

By adding the above two inequalities, we get

$$f\big(\mu a + (1-\mu)b\big) + f\big(\mu b + (1-\mu)a\big) \le f(a) + f(b). \tag{15}$$

Multiplying by $\mu^{\frac{\lambda}{k}-1}$ on both sides of (15), then integrating with regard to μ over $[0,1]$, we get

$$\int_0^1 \mu^{\frac{\lambda}{k}-1} f\big(\mu a + (1-\mu)b\big)\,d\mu + \int_0^1 \mu^{\frac{\lambda}{k}-1} f\big(\mu b + (1-\mu)a\big)\,d\mu \le \frac{k}{\lambda}\big[f(a) + f(b)\big]. \tag{16}$$

We denote

$$K_1 = \int_0^1 \mu^{\frac{\lambda}{k}-1} f\big(\mu a + (1-\mu)b\big)\,d\mu$$

and

$$K_2 = \int_0^1 \mu^{\frac{\lambda}{k}-1} f\big(\mu b + (1-\mu)a\big)\,d\mu.$$

Putting $\phi = \mu a + (1-\mu)b$ in K_1, and $\omega = \mu b + (1-\mu)a$ in K_2, we obtain

$$K_1 = \frac{1}{(b-a)^{\frac{\lambda}{k}}} \int_a^b (b-\phi)^{\frac{\lambda}{k}-1} f(\phi)\,d\phi \tag{17}$$

and

$$K_2 = \frac{1}{(b-a)^{\frac{\lambda}{k}}} \int_a^b (\omega-a)^{\frac{\lambda}{k}-1} f(\omega)\,d\omega. \tag{18}$$

Substituting the values of K_1 and K_2 from (17) and (18) in (16), we get

$$\frac{1}{(b-a)^{\frac{\lambda}{k}}}\bigg[\int_a^b (b-\phi)^{\frac{\lambda}{k}-1} f(\phi)\,d\phi + \int_a^b (\omega-a)^{\frac{\lambda}{k}-1} f(\omega)\,d\omega\bigg] \le \frac{k}{\lambda}\big[f(a) + f(b)\big],$$

which implies that

$$\frac{\Gamma_k(\lambda+k)}{2(b-a)^{\frac{\lambda}{k}}}\big[I^{\lambda}_{a^+,k} f(b) + I^{\lambda}_{b^-,k} f(a)\big] \le \frac{f(a)+f(b)}{2}. \tag{19}$$

By combining (17), and (19), we get (8). □

Lemma 2.1 *Let $k > 0$, $I^{\lambda}_{a^+,k}f$ and $I^{\lambda}_{b^-,k}f$ be defined as Definition 4. Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in Ł[a,b]$, then*

$$\begin{aligned}&\frac{(b-a)}{2}\int_0^1 \bigl[(1-\mu)^{\frac{\lambda}{k}} - \mu^{\frac{\lambda}{k}}\bigr] f'\bigl(\mu a + (1-\mu)b\bigr)\,d\mu \\ &\quad = \frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\lambda+k)}{2(b-a)^{\frac{\lambda}{k}}}\bigl[I^{\lambda}_{a^+,k}f(b) + I^{\lambda}_{b^-,k}f(a)\bigr]. \end{aligned} \tag{20}$$

Proof Let us consider

$$I = \int_0^1 \bigl[(1-\mu)^{\frac{\lambda}{k}} - \mu^{\frac{\lambda}{k}}\bigr] f'\bigl(\mu a + (1-\mu)b\bigr)\,d\mu,$$

which, we can write as

$$\begin{aligned} I &= \biggl[\int_0^1 (1-\mu)^{\frac{\lambda}{k}} f'\bigl(\mu a + (1-\mu)b\bigr)\,d\mu\biggr] + \biggl[-\int_0^1 \mu^{\frac{\lambda}{k}}\bigl(f'\bigl(\mu a + (1-\mu)b\bigr)\,d\mu\bigr)\biggr] \\ &= I_1 + I_2. \end{aligned} \tag{21}$$

Integrating I_1 by parts, we get

$$I_1 = \frac{f(b)}{b-a} - \frac{\lambda}{k(b-a)}\int_0^1 (1-\mu)^{\frac{\lambda}{k}-1} f\bigl(\mu a + (1-\mu)b\bigr)\,d\mu.$$

Setting $\xi = \mu a + (1-\mu)b$, then after some calculation, we get

$$I_1 = \frac{f(b)}{b-a} - \frac{\Gamma_k(\lambda+k)}{(b-a)^{\frac{\lambda}{k}+1}}\bigl[I^{\lambda}_{b^-,k}f(a)\bigr]. \tag{22}$$

Now integrating I_2 by parts to get

$$I_2 = \frac{f(a)}{b-a} - \frac{\lambda}{k(b-a)}\int_0^1 \mu^{\frac{\lambda}{k}-1} f\bigl(\mu a + (1-\mu)b\bigr)\,d\mu.$$

Setting $\xi = (\mu a + (1-\mu)b)$, after some calculation, we get

$$I_2 = \frac{f(a)}{b-a} - \frac{\Gamma_k(\lambda+k)}{(b-a)^{\frac{\lambda}{k}+1}}\bigl[I^{\lambda}_{a^+,k}f(b)\bigr]. \tag{23}$$

Applying (22) and (23) in (21), it follows that

$$I = \frac{f(b)}{b-a} + \frac{f(a)}{b-a} - \frac{\Gamma_k(\lambda+k)}{(b-a)^{\frac{\lambda}{k}+1}}\bigl[I^{\lambda}_{a^+,k}f(b) + I^{\lambda}_{b^-,k}f(a)\bigr],$$

or

$$\begin{aligned}&\int_0^1 \bigl[(1-\mu)^{\frac{\lambda}{k}} - \mu^{\frac{\lambda}{k}}\bigr] f'\bigl(\mu a + (1-\mu)b\bigr)\,d\mu \\ &\quad = \frac{f(b)}{b-a} + \frac{f(a)}{b-a} - \frac{\Gamma_k(\lambda+k)}{(b-a)^{\frac{\lambda}{k}+1}}\bigl[I^{\lambda}_{a^+,k}f(b) + I^{\lambda}_{b^-,k}f(a)\bigr]. \end{aligned} \tag{24}$$

Multiplying both sides of (24) by $\frac{b-a}{2}$ to get the required result. □

Theorem 6 *Let $k > 0$, $I^{\lambda}_{a^+,k}f$ and $I^{\lambda}_{b^-,k}f$ be defined as Definition 4. Let $f : [a,b] \to \mathbb{R}$ be differentiable on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\lambda+k)}{2(b-a)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{a^+,k}f(b) + I^{\lambda}_{b^-,k}f(a)\right]\right| \\ &\quad \le \frac{(b-a)}{2(\frac{\lambda}{k}+1)}\left(1 - \frac{1}{2^{\frac{\lambda}{k}}}\right)\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \end{aligned} \tag{25}$$

Proof By using Lemma 2.1 and the definition of a convex function of $|f'|$, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\lambda+k)}{2(b-a)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{a^+,k}f(b) + I^{\lambda}_{b^-,k}f(a)\right]\right| \\ &\quad \le \frac{(b-a)}{2}\int_0^1 \left|(1-\mu)^{\frac{\lambda}{k}} - \mu^{\frac{\lambda}{k}}\right|\left|f'\bigl(\mu a + (1-\mu)b\bigr)\right|\,d\mu \\ &\quad \le \frac{(b-a)}{2}\int_0^1 \left|(1-\mu)^{\frac{\lambda}{k}} - \mu^{\frac{\lambda}{k}}\right|\left[\mu\left|f'(a)\right| + (1-\mu)\left|f'(b)\right|\right]d\mu \\ &\quad = \frac{(b-a)}{2}\left[\int_0^1 \left|(1-\mu)^{\frac{\lambda}{k}} - \mu^{\frac{\lambda}{k}}\right|\mu\left|f'(a)\right|\,d\mu + \int_0^1 \left|(1-\mu)^{\frac{\lambda}{k}} - \mu^{\frac{\lambda}{k}}\right|(1-\mu)\left|f'(b)\right|\,d\mu\right] \\ &\quad = \frac{(b-a)}{2}(K_1 + K_2), \end{aligned} \tag{26}$$

where

$$\begin{aligned}K_1 &= \left|f'(a)\right|\left[\int_0^{\frac{1}{2}} \mu(1-\mu)^{\frac{\lambda}{k}}\,d\mu - \int_0^{\frac{1}{2}} \mu^{\frac{\lambda}{k}+1}\,d\mu\right] \\ &\quad + \left|f'(b)\right|\left[\int_0^{\frac{1}{2}} (1-\mu)^{\frac{\lambda}{k}+1}\,d\mu - \int_0^{\frac{1}{2}} (1-\mu)\mu^{\frac{\lambda}{k}}\,d\mu\right]\end{aligned}$$

and

$$\begin{aligned}K_2 &= \left|f'(a)\right|\left[\int_{\frac{1}{2}}^1 \mu^{\frac{\lambda}{k}+1}\,d\mu - \int_{\frac{1}{2}}^1 \mu(1-\mu)^{\frac{\lambda}{k}}\,d\mu\right] \\ &\quad + \left|f'(b)\right|\left[\int_{\frac{1}{2}}^1 (1-\mu)\mu^{\frac{\lambda}{k}}\,d\mu - \int_{\frac{1}{2}}^1 (1-\mu)^{\frac{\lambda}{k}+1}\,d\mu\right].\end{aligned}$$

We calculate K_1 to get

$$K_1 = \left|f'(a)\right|\left[\frac{1}{(\frac{\lambda}{k}+1)(\frac{\lambda}{k}+2)} - \frac{(\frac{1}{2})^{\frac{\lambda}{k}+1}}{(\frac{\lambda}{k}+1)}\right] - \left|f'(b)\right|\left[\frac{1}{(\frac{\lambda}{k}+2)} - \frac{(\frac{1}{2})^{\frac{\lambda}{k}+1}}{(\frac{\lambda}{k}+1)}\right]. \tag{27}$$

Similarly we can calculate K_2 and get

$$K_2 = \left|f'(a)\right|\left[\frac{1}{(\frac{\lambda}{k}+2)} - \frac{(\frac{1}{2})^{\frac{\lambda}{k}+1}}{(\frac{\lambda}{k}+1)}\right] + \left|f'(b)\right|\left[\frac{1}{(\frac{\lambda}{k}+1)(\frac{\lambda}{k}+2)} - \frac{(\frac{1}{2})^{\frac{\lambda}{k}+1}}{(\frac{\lambda}{k}+1)}\right]. \tag{28}$$

Substituting the values of K_1 and K_2 in (26) and after some calculations, we get (25). □

3 Some more fractional inequalities for convex functions

Definition 5 Let $f \in L_1[a,b]$. The k-Riemann–Liouville integrals $I^{\lambda}_{(\frac{a+b}{2})^+,k}f$ and $I^{\lambda}_{(\frac{a+b}{2})^-,k}f$ of order $\lambda > 0$ and $k > 0$ with $a \geq 0$ are defined by

$$I^{\lambda}_{(\frac{a+b}{2})^+,k}f(\xi) = \frac{1}{k\Gamma_k(\lambda)} \int_{\frac{a+b}{2}}^{\xi} (\xi - \mu)^{\frac{\lambda}{k}-1} f(\mu)\, d\mu, \quad \xi > \frac{a+b}{2},$$

and

$$I^{\lambda}_{(\frac{a+b}{2})^-,k}f(\xi) = \frac{1}{k\Gamma_k(\lambda)} \int_{\xi}^{\frac{a+b}{2}} (\mu - \xi)^{\frac{\lambda}{k}-1} f(\mu)\, d\mu, \quad \xi < \frac{a+b}{2},$$

respectively. Here $\Gamma_k(\lambda)$ is the k-Gamma function.

Theorem 7 *Let $k > 0$, $I^{\lambda}_{(\frac{a+b}{2})^+,k}f$ and $I^{\lambda}_{(\frac{a+b}{2})^-,k}f$ be defined in Definition 5. Let $f : [a,b] \to \mathbb{R}$ be positive mapping with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\frac{\lambda}{k}-1}\Gamma_k(\lambda+k)}{(b-a)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{(\frac{a+b}{2})^+,k}f(b) + I^{\lambda}_{(\frac{a+b}{2})^-,k}f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{29}$$

Proof As f is convex function on $[a,b]$, we have, for $\xi, \eta \in [a,b]$ with $\tau = \frac{1}{2}$,

$$f\left(\frac{\xi+\eta}{2}\right) \leq \frac{f(\xi)+f(\eta)}{2}. \tag{30}$$

Putting $\xi = \frac{\mu a}{2} + \frac{(2-\mu)b}{2}$ and $\eta = \frac{\mu b}{2} + \frac{(2-\mu)a}{2}$, then (30) becomes

$$2f\left(\frac{a+b}{2}\right) \leq f\left(\frac{\mu a}{2} + \frac{(2-\mu)b}{2}\right) + f\left(\frac{\mu b}{2} + \frac{(2-\mu)a}{2}\right). \tag{31}$$

Multiplying by $\mu^{\frac{\lambda}{k}-1}$ on both sides of (31), then integrating with respect to μ over $[0,1]$, we get

$$\begin{aligned}\frac{2k}{\lambda}f\left(\frac{a+b}{2}\right) \leq{} & \int_0^1 \mu^{\frac{\lambda}{k}-1} f\left(\frac{\mu a}{2} + \frac{(2-\mu)b}{2}\right) d\mu \\ & + \int_0^1 \mu^{\frac{\lambda}{k}-1} f\left(\frac{\mu b}{2} + \frac{(2-\mu)a}{2}\right) d\mu.\end{aligned} \tag{32}$$

We set

$$I_1 = \int_0^1 \mu^{\frac{\lambda}{k}-1} f\left(\frac{\mu a}{2} + \frac{(2-\mu)b}{2}\right) d\mu. \tag{33}$$

Taking $\phi = \frac{\mu a}{2} + \frac{(2-\mu)b}{2}$, after some calculations we get

$$I_1 = \frac{2^{\frac{\lambda}{k}}}{(b-a)^{\frac{\lambda}{k}}} \int_{\frac{a+b}{2}}^{b} (b-\phi)^{\frac{\lambda}{k}-1} f(\phi)\, d\phi, \tag{34}$$

and we set

$$I_2=\int_0^1 \mu^{\frac{\lambda}{k}-1} f\left(\frac{\mu b}{2}+\frac{(2-\mu)a}{2}\right)d\mu.$$

Putting $\omega=\frac{\mu b}{2}+\frac{(2-\mu)a}{2}$ to get

$$I_2=\frac{2^{\frac{\lambda}{k}}}{(b-a)^{\frac{\lambda}{k}}}\int_a^{\frac{a+b}{2}}(\omega-a)^{\frac{\lambda}{k}-1}f(\omega)\,d\omega. \tag{35}$$

Substituting the values of I_1 and I_2 from (33) and (35) in (32), we get

$$f\left(\frac{a+b}{2}\right)\le \frac{2^{\frac{\lambda}{k}-1}\Gamma_k(\lambda+k)}{(b-a)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{(\frac{a+b}{2})^+,k}f(b)+I^{\lambda}_{(\frac{a+b}{2})^-,k}f(a)\right]. \tag{36}$$

The first part of the inequality is proved. To complete the second inequality, we note that if f is convex function, then, for $\tau\in[0,1]$, showing

$$f\left(\frac{\mu}{2}a+\frac{2-\mu}{2}b\right)\le \frac{\mu}{2}f(a)+\frac{2-\mu}{2}f(b)$$

and

$$f\left(\frac{\mu}{2}b+\frac{2-\mu}{2}a\right)\le \frac{\mu}{2}f(b)+\frac{2-\mu}{2}f(a).$$

By adding the above two inequalities, we get

$$f\left(\frac{\mu}{2}a+\frac{2-\mu}{2}b\right)+f\left(\frac{\mu}{2}b+\frac{2-\mu}{2}a\right)\le f(a)+f(b). \tag{37}$$

Multiplying by $\mu^{\frac{\lambda}{k}-1}$ on both sides of (37) and integrating inequalities with respect to μ over $[0,1]$, we get

$$\int_0^1 \mu^{\frac{\lambda}{k}-1} f\left(\frac{\mu}{2}a+\frac{2-\mu}{2}b\right)d\mu+\int_0^1 \mu^{\frac{\lambda}{k}-1} f\left(\frac{\mu}{2}b+\frac{2-\mu}{2}a\right)d\mu\le \frac{k}{\lambda}\left[f(a)+f(b)\right]. \tag{38}$$

We take

$$L_1=\int_0^1 \mu^{\frac{\lambda}{k}-1} f\left(\frac{\mu a}{2}+\frac{(2-\mu)b}{2}\right)d\mu$$

and choose $\phi=\frac{\mu a}{2}+\frac{(2-\mu)b}{2}$, we get after some simple calculations

$$L_1=\frac{2^{\frac{\lambda}{k}}}{(b-a)^{\frac{\lambda}{k}}}\int_{\frac{a+b}{2}}^{b}(b-\phi)^{\frac{\lambda}{k}-1}f(\phi)\,d\phi. \tag{39}$$

Likewise we take

$$L_2=\int_0^1 \mu^{\frac{\alpha}{k}-1} f\left(\frac{\mu b}{2}+\frac{(2-\mu)a}{2}\right)d\mu,$$

and choose $\omega = \frac{\mu b}{2} + \frac{(2-\mu)a}{2}$, we get

$$L_2 = \frac{2^{\frac{\lambda}{k}}}{(b-a)^{\frac{\lambda}{k}}} \int_a^{\frac{a+b}{2}} (\omega - a)^{\frac{\lambda}{k}-1} f(\omega)\, d\omega. \tag{40}$$

Substituting the values of L_1 and L_2 from (39) and (40) in (38), we get

$$\frac{2^{\frac{\lambda}{k}}}{(b-a)^{\frac{\lambda}{k}}} \left[\int_{\frac{a+b}{2}}^{b} (b-\phi)^{\frac{\lambda}{k}-1} f(\phi)\, d\phi + \int_a^{\frac{a+b}{2}} (\omega - a)^{\frac{\lambda}{k}-1} f(\omega)\, d\omega \right] \le \frac{k}{\lambda} \big[f(a) + f(b) \big].$$

This implies that

$$\frac{2^{\frac{\lambda}{k}-1} \lambda \Gamma_k(\lambda)}{(b-a)^{\frac{\lambda}{k}}} \big[I^{\lambda}_{(\frac{a+b}{2})^+,k} f(b) + I^{\lambda}_{(\frac{a+b}{2})^-,k} f(a) \big] \le \frac{f(a)+f(b)}{2}. \tag{41}$$

From (36) and (41), we get the required result. □

Lemma 3.1 *Let $k > 0$, $I^{\lambda}_{(\frac{a+b}{2})^+,k} f$ and $I^{\lambda}_{(\frac{a+b}{2})^-,k} f$ be defined as Definition 5. Let $f : [a,b] \to \mathbb{R}$ be differentiable on (a,b) with $a < b$. If $f' \in Ł[a,b]$, then*

$$\begin{aligned} &\frac{2^{\frac{\lambda}{k}-1} \Gamma_k(\lambda + k)}{(b-a)^{\frac{\lambda}{k}}} \big[I^{\lambda}_{(\frac{a+b}{2})^+,k} f(b) + I^{\lambda}_{(\frac{a+b}{2})^-,k} f(a) \big] - f\left(\frac{a+b}{2}\right) \\ &\quad = \frac{b-a}{4} \left[\int_0^1 \mu^{\frac{\lambda}{k}} f'\left(\frac{\mu}{2} a + \frac{2-\mu}{2} b\right) d\mu - \int_0^1 \mu^{\frac{\lambda}{k}} f'\left(\frac{\mu}{2} b + \frac{2-\mu}{2} a\right) d\mu \right]. \end{aligned} \tag{42}$$

Proof Let

$$\begin{aligned} I &= \left[\int_0^1 \mu^{\frac{\lambda}{k}} f'\left(\frac{\mu}{2} a + \frac{2-\mu}{2} b\right) d\mu - \int_0^1 \mu^{\frac{\lambda}{k}} f'\left(\frac{\mu}{2} b + \frac{2-\mu}{2} a\right) d\mu \right] \\ &= I_1 - I_2. \end{aligned}$$

Note that

$$\begin{aligned} I_1 &= \int_0^1 \mu^{\frac{\lambda}{k}} f'\left(\frac{\mu}{2} a + \frac{2-\mu}{2} b\right) d\mu \\ &= -f\left(\frac{a+b}{2}\right)\left(\frac{2}{b-a}\right) + \frac{\lambda}{k}\left(\frac{2}{b-a}\right) \int_0^1 \mu^{\frac{\lambda}{k}-1} f\left(\frac{\mu}{2} a + \frac{2-\mu}{2} b\right) d\mu. \end{aligned}$$

Substituting $\xi = \frac{\mu a}{2} + \frac{(2-\mu)b}{2}$, we get after some computations

$$I_1 = -f\left(\frac{a+b}{2}\right)\left(\frac{2}{b-a}\right) + \frac{2^{\frac{\lambda}{k}+1} \Gamma_k(\lambda + k)}{(b-a)^{\frac{\lambda}{k}+1}} \big[I^{\lambda}_{(\frac{a+b}{2})^+,k} f(b) \big]. \tag{43}$$

Similarly we can write for I_2

$$\begin{aligned} I_2 &= \int_0^1 \mu^{\frac{\lambda}{k}} f'\left(\frac{\mu}{2} b + \frac{2-\mu}{2} a\right) d\mu \\ &= f\left(\frac{a+b}{2}\right)\left(\frac{2}{b-a}\right) - \frac{\lambda}{k} \int_0^1 \mu^{\frac{\lambda}{k}-1} f\left(\frac{\mu}{2} b + \frac{2-\mu}{2} a\right)\left(\frac{2}{b-a}\right) d\mu. \end{aligned}$$

Taking $\xi = \frac{\mu b}{2} + \frac{(2-\mu)a}{2}$, we get

$$I_2 = f\left(\frac{a+b}{2}\right)\left(\frac{2}{b-a}\right) - \frac{2^{\frac{\lambda}{k}+1}\Gamma_k(\lambda+k)}{(b-a)^{\frac{\lambda}{k}+1}}\left[I^{\lambda}_{(\frac{a+b}{2})^-,k}f(a)\right]. \tag{44}$$

By using (43) and (44), it follows that

$$I_1 - I_2 = \frac{2^{\frac{\lambda}{k}+1}\Gamma_k(\lambda+k)}{(b-a)^{\frac{\lambda}{k}+1}}\left[I^{\lambda}_{(\frac{a+b}{2})^+,k}f(b) + I^{\lambda}_{(\frac{a+b}{2})^-,k}f(a)\right] - f\left(\frac{a+b}{2}\right)\left(\frac{4}{b-a}\right).$$

Thus, multiplying $\frac{b-a}{4}$ on both sides of the above, we get (42). □

Theorem 8 *Let $k > 0$, $I^{\lambda}_{(\frac{a+b}{2})^+,k}f$ and $I^{\lambda}_{(\frac{a+b}{2})^-,k}f$ be defined as Definition 5. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f'|^h$ is convex on $[a,b]$ for $h \geq 1$, then*

$$\begin{aligned}
&\left|\frac{2^{\frac{\lambda}{k}-1}\Gamma_k(\lambda+k)}{(b-a)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{(\frac{a+b}{2})^+,k}f(b) + I^{\lambda}_{(\frac{a+b}{2})^-,k}f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \\
&\quad \leq \left(\frac{b-a}{4}\right)\left(\frac{1}{\frac{\lambda}{k}+1}\right)\left(\frac{1}{2(\frac{\lambda}{k}+2)}\right)^{\frac{1}{h}}\left[\left(\left(\frac{\lambda}{k}+1\right)\left|f'(a)\right|^h + \left(\frac{\lambda}{k}+3\right)\left|f'(b)\right|^h\right)^{\frac{1}{h}}\right. \\
&\qquad \left. + \left(\left(\frac{\lambda}{k}+3\right)\left|f'(a)\right|^h + \left(\frac{\lambda}{k}+1\right)\left|f'(b)\right|^h\right)^{\frac{1}{h}}\right].
\end{aligned} \tag{45}$$

Proof First, we consider the case of $h = 1$. By using Lemma 3.1, and the definition of convex function of $|f'|$, we obtain

$$\begin{aligned}
&\left|\frac{2^{\frac{\lambda}{k}-1}\Gamma_k(\lambda+k)}{(b-a)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{(\frac{a+b}{2})^+,k}f(b) + I^{\lambda}_{(\frac{a+b}{2})^-,k}f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \\
&\quad \leq \frac{b-a}{4}\int_0^1 \mu^{\frac{\lambda}{k}}\left[\left|f'\left(\frac{\mu}{2}a + \frac{2-\mu}{2}b\right)\right| + \left|f'\left(\frac{\mu}{2}b + \frac{2-\mu}{2}a\right)\right| d\mu\right] \\
&\quad \leq \frac{b-a}{4}\left(\int_0^1 \mu^{\frac{\lambda}{k}}\left[\frac{\mu}{2}\left|f'(a)\right| + \frac{2-\mu}{2}\left|f'(b)\right| + \frac{\mu}{2}\left|f'(b)\right| + \frac{2-\mu}{2}\left|f'(a)\right|\right] d\mu\right) \\
&\quad = \frac{b-a}{4}\left(\int_0^1 \left[\frac{\mu^{\frac{\lambda}{k}+1}}{2}\left|f'(a)\right| + \frac{2\mu^{\frac{\lambda}{k}} - \mu^{\frac{\lambda}{k}+1}}{2}\left|f'(b)\right|\right.\right. \\
&\qquad \left.\left. + \frac{\mu^{\frac{\lambda}{k}+1}}{2}\left|f'(b)\right| + \frac{2\mu^{\frac{\lambda}{k}} - \mu^{\frac{\lambda}{k}+1}}{2}\left|f'(a)\right|\right] d\mu\right) \\
&\quad = \frac{b-a}{4}\left(\left[\frac{|f'(a)|}{2(\frac{\lambda}{k}+2)} + \frac{2(\frac{\lambda}{k}+2) - (\frac{\lambda}{k}+1)}{(\frac{\lambda}{k}+1)(\frac{\lambda}{k}+2)2}\left|f'(b)\right|\right.\right. \\
&\qquad \left.\left. + \frac{2(\frac{\lambda}{k}+2) - (\frac{\lambda}{k}+1)}{(\frac{\lambda}{k}+1)(\frac{\lambda}{k}+2)2}\left|f'(b)\right| + \frac{f'(b)}{2(\frac{\lambda}{k}+2)}\right]\right) \\
&\quad = \frac{b-a}{4}\left(\left[\frac{|f'(a)|}{(\frac{\lambda}{k}+2)} + \frac{(\frac{\lambda}{k}+3)}{2(\frac{\lambda}{k}+1)(\frac{\lambda}{k}+2)}\left|f'(b)\right|\right.\right. \\
&\qquad \left.\left. + \frac{(\frac{\lambda}{k}+3)}{2(\frac{\lambda}{k}+1)(\frac{\lambda}{k}+2)}\left|f'(a)\right| + \frac{|f'(b)|}{(\frac{\lambda}{k}+2)}\right]\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{b-a}{4}\Bigg[\left(\frac{1}{2(\frac{\lambda}{k}+2)} + \frac{(\frac{\lambda}{k}+3)}{2(\frac{\lambda}{k}+1)(\frac{\lambda}{k}+2)}\right)\left|f'(a)\right| \\
&\quad + \left(\frac{(\frac{\lambda}{k}+3)}{2(\frac{\lambda}{k}+1)(\frac{\lambda}{k}+2)} + \frac{1}{2(\frac{\lambda}{k}+2)}\right)\left|f'(b)\right|\Bigg] \\
&= \frac{b-a}{4}\left(\frac{2(\frac{\lambda}{k}+2)}{2(\frac{\lambda}{k}+1)(\frac{\lambda}{k}+2)}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]\right) \\
&= \frac{b-a}{4(\frac{\lambda}{k}+1)}\left[\left|f'(a)\right| + \left|f'(b)\right|\right].
\end{aligned}
$$

Now we consider the case of $h > 1$. By using Lemma 3.1, the Holder inequality and the definition of convex function of $|f'|^h$, we get

$$
\begin{aligned}
&\left|\frac{2^{\frac{\lambda}{k}-1}\Gamma_k(\lambda+k)}{(b-a)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{(\frac{a+b}{2})^+,k}f(b) + I^{\lambda}_{(\frac{a+b}{2})^-,k}f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \\
&\le \frac{b-a}{4}\left[\int_0^1 \mu^{\frac{\lambda}{k}} f'\left(\frac{\mu}{2}a + \frac{2-\mu}{2}b\right)d\mu \right.\\
&\quad \left. + \int_0^1 \mu^{\frac{\lambda}{k}} f'\left(\frac{\mu}{2}b + \frac{2-\mu}{2}a\right)d\mu\right] \\
&\le \frac{b-a}{4}\left[\left(\int_0^1 \mu^{\frac{\lambda}{k}}\left|f'\left(\frac{\mu}{2}a + \frac{2-\mu}{2}b\right)\right|^h d\mu\right)^{\frac{1}{h}} \right.\\
&\quad \left. + \left(\int_0^1 \mu^{\frac{\lambda}{k}}\left|f'\left(\frac{\mu}{2}b + \frac{2-\mu}{2}a\right)\right|^h d\mu\right)^{\frac{1}{h}}\right]\left(\int_0^1 \mu^{\frac{\lambda}{k}}\,d\mu\right)^{1-\frac{1}{h}} \\
&\le \frac{b-a}{4}\left[\left(\int_0^1 \mu^{\frac{\lambda}{k}}\left(\frac{\mu}{2}\left|f'(a)\right|^h + \frac{2-\mu}{2}\left|f'(b)\right|^h\right)d\mu\right)^{\frac{1}{h}} \right.\\
&\quad \left. + \left(\int_0^1 \mu^{\frac{\lambda}{k}}\left(\frac{\mu}{2}\left|f'(b)\right|^h + \frac{2-\mu}{2}\left|f'(a)\right|^h\right)d\mu\right)^{\frac{1}{h}}\right]\left(\int_0^1 \mu^{\frac{\lambda}{k}}\,d\mu\right)^{1-\frac{1}{h}} \\
&= \frac{b-a}{4}\left[\left(\int_0^1 \left(\frac{\mu^{\frac{\lambda}{k}+1}}{2}\left|f'(a)\right|^h + \frac{2\mu^{\frac{\lambda}{k}} - \mu^{\frac{\lambda}{k}+1}}{2}\left|f'(b)\right|^h\right)d\mu\right)^{\frac{1}{h}} \right.\\
&\quad \left. + \left(\int_0^1 \left(\frac{\mu^{\frac{\lambda}{k}+1}}{2}\left|f'(b)\right|^h + \frac{2\mu^{\frac{\lambda}{k}} - \mu^{\frac{\lambda}{k}+1}}{2}\left|f'(a)\right|^h\right)d\mu\right)^{\frac{1}{h}}\right]\left(\int_0^1 \mu^{\frac{\lambda}{k}}\,d\mu\right)^{1-\frac{1}{h}} \\
&= \frac{b-a}{4}\left[\left(\frac{|f'(a)|^h}{2(\frac{\lambda}{k}+2)} + \frac{(\frac{\lambda}{k}+3)}{2(\frac{\lambda}{k}+1)(\frac{\lambda}{k}+2)}\left|f'(b)\right|^h\right)^{\frac{1}{h}} \right.\\
&\quad \left. + \left(\frac{|f'(b)|^h}{2(\frac{\lambda}{k}+2)} + \frac{(\frac{\lambda}{k}+3)}{2(\frac{\lambda}{k}+1)(\frac{\lambda}{k}+2)}\left|f'(a)\right|^h\right)^{\frac{1}{h}}\right]\left(\int_0^1 \mu^{\frac{\lambda}{k}}\,d\mu\right)^{1-\frac{1}{h}} \\
&\le \frac{b-a}{4}\left(\frac{1}{2(\frac{\lambda}{k}+1)(\frac{\lambda}{k}+2)}\right)^{\frac{1}{h}}\left[\left(\left(\frac{\lambda}{k}+1\right)\left|f'(a)\right|^h + \left(\frac{\lambda}{k}+3\right)\left|f'(b)\right|^h\right)^{\frac{1}{h}} \right.\\
&\quad \left. + \left(\left(\frac{\lambda}{k}+1\right)\left|f'(b)\right|^h + \left(\frac{\lambda}{k}+3\right)\left|f'(a)\right|^h\right)^{\frac{1}{h}}\right]\left(\frac{1}{\frac{\lambda}{k}+1}\right)^{1-\frac{1}{h}}.
\end{aligned}
$$

This completes the proof. □

Remark 1 Our results described in the above theorems coincide with the results of [14] and [12] by replacing $I^{\lambda}_{a^+,k}f(\xi)$ by $I^{\lambda}_{a^+}f(\xi)$.

4 Applications to quadrature formulas

In this section we apply the obtained results in to the error estimations of quadrature formulas. It is shown that our main results contain as special cases results such as mid-point inequality and trapezoid inequality. Also, the Hermite–Hadamard inequality can be deduced directly from our main results.

Proposition 1 (Hermite–Hadamard inequality) *By using the assumptions of Theorem 5 with* $\lambda = 1$ *and* $k = 1$, *we get the following*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(\mu)\,d\mu \leq \frac{f(a)+f(b)}{2}.$$

Proposition 2 (Mid-point inequality) *By using the assumptions of Theorem 8 with* $\lambda = 1$, $h = 1$ *and* $k = 1$, *we get the following mid-point type inequality*:

$$\left|\frac{1}{(b-a)}\int_a^b f(\mu)\,d\mu - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right].$$

Proposition 3 (Trapezoid inequality) *By using the assumptions of Theorem 6 with* $\lambda = 1$ *and* $k = 1$, *we get the following trapezoid inequality*:

$$\left|\frac{f(a)+(b)}{2} - \frac{1}{b-a}\int_a^b f(\mu)\,d\mu\right| \leq \frac{b-a}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right].$$

5 Conclusion

We have discussed some Hermite–Hadamard type inequalities for k-Riemann–Liouville fractional integral using the convexity of differentiable functions. We stated our main results by Theorems 5, 6, 7 and 8, and showed that our results contain some existing results as special cases. As applications we have established two inequalities involving the error estimates of quadrature formulas.

Acknowledgements
All authors are thankful to the careful referee and editor for their suggestions which help us to improve the final version of paper.

Funding
This work was supported by the Natural Science Foundation of Fujian Province of China (No. 2016J01023).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in the preparation of this paper. All authors read and approved the final manuscript.

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Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 26 February 2020 Accepted: 7 December 2020 Published online: 16 February 2021

References

1. Bakula, M.K., Pečrić, J., Ödemir, M.E.: Hadamard type inequalities for m-convex and (α,m)-convex functions. J. Inequal. Pure Appl. Math. **9**, 1–9 (2008)
2. Dahmani, Z.: New inequalities in fractional integrals. Int. J. Nonlinear Sci. **9**(4), 493–497 (2010)
3. Diaz, R., Pariguan, E.: On hypergeometric functions and Pochhammer k-symbol. Divulg. Mat. **15**(2), 179–192 (2007)
4. Hadamard, J.: Etude sur les proprietes des fonctions entieres et en particulier d'une fonction consideree par Riemann. J. Math. Pures Appl. **58**, 171–215 (1893)
5. Killbas, A.M., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
6. Kirmaci, U.S., Bakula, M., Ozdemir, M., Pečarić, J.: Hadamard-type inequalities for s-convex functions. Appl. Math. Comput. **193**, 26–35 (2007)
7. Mubeen, S., Habibullah, G.M.: k-Fractional integrals and applications. Int. J. Contemp. Math. Sci. **7**, 89–94 (2012)
8. Ödemir, M.E., Avci, M., Kavurmaci, H.: Hermite–Hadamard-type inequalities via (α,m)-convexity. Comput. Math. Appl. **61**, 2614–2620 (2011)
9. Ödemir, M.E., Avci, M., Set, E.: On some inequalities of Hermite–Hadamard-type via m-convexity. Appl. Math. Lett. **23**, 1065–1070 (2010)
10. Ödemir, M.E., Set, E., Sarikaya, M.Z.: Some new Hadamard type inequalities for co-ordinated m-convex and (α,m)-convex functions. Hacet. J. Math. Stat. **40**, 219–229 (2011)
11. Pečarić, J., Proschan, F., Tong, Y.L.: Convex Functions, Partial Orderings and Statistical Application. Academic Press, San Diego (1992)
12. Sarikaya, M.Z., Hüseyin, Y.: On Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals. Miskolc Math. Notes **17**(2), 1049–1059 (2017)
13. Sarikaya, M.Z., Ogunmez, H.: On new inequalities via Riemann–Liouville fractional integration. Abstr. Appl. Anal. **2012**, Article ID 428983 (2012)
14. Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N.: Hermite–Hadamard inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**(9), 2403–2407 (2013)
15. Set, E., Ozdemir, M.O., Dragomir, S.S.: On the Hadamard-type of inequalities involving several kinds of convexity. J. Inequal. Appl. **2010**, Article ID 286845 (2010)
16. Set, E., Ozdemir, M.O., Dragomir, S.S.: On the Hermite–Hadamard inequality and other integral inequalities involving two functions. J. Inequal. Appl. **2010**, Article ID 148102 (2010)
17. Wang, J., Li, X., Fečkan, M., Zhou, Y.: Hermite–Hadamard-type inequalities for Riemann–Liouville fractional integrals via two kinds of convexity. Appl. Anal. **92**(11), 2241–2253 (2013)