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Hardy operators and the commutators on Hardy spaces

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Abstract

In this paper, the boundedness of the classic Hardy operator and its adjoint on Hardy spaces is obtained. We also discuss the boundedness for the commutators generated by the classic Hardy operator and its adjoint with *BMO* and *CMO*($\mathbb{R}^+$) functions on Hardy spaces.

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1 Introduction

Let P and Q be the classical Hardy operator and its adjoint on $\mathbb{R}^+ = (0, +\infty)$,

$$Pf(x) = \frac{1}{x}\int_0^x f(y)\,dy, \qquad Qf(x) = \int_x^\infty \frac{f(y)}{y}\,dy.$$

It is well known that, for $p > 1$, $\|Pf\|_{L^p(\mathbb{R}^+)} \le p'\|f\|_{L^p(\mathbb{R}^+)}$ and $\|Qf\|_{L^p(\mathbb{R}^+)} \le p\|f\|_{L^p(\mathbb{R}^+)}$, where $p' = p/(p-1)$. For the boundedness of P and Q on $L^1(\mathbb{R}^+)$, we have that P is bounded from $L^1(\mathbb{R}^+)$ to $L^{1,\infty}(\mathbb{R}^+)$ but not bounded on $L^1(\mathbb{R}^+)$ and Q is bounded on $L^1(\mathbb{R}^+)$.

For earlier development of this kind of inequality and many applications in analysis, see [2, 4–7, 10].

Let $b \in L_{loc}(\mathbb{R}^+)$, the commutators of Hardy operators P and its adjoint Q are defined by

$$[b,P]f(x) = b(x)Pf(x) - P(bf)(x),$$
$$[b,Q]f(x) = b(x)Qf(x) - Q(bf)(x).$$

Long and Wang [9] established Hardy's integral inequalities for commutators generated by P and Q with *CMO* function. Li, Zhang, and Xue in [8] obtained some two-weight inequalities for commutators generated by P and Q with *CMO* function. Zhao, Fu, and Lu in [12] studied the boundedness on Hardy spaces for n-dimensional Hardy operators and the commutators.

In this paper, we discuss the boundedness on the Hardy spaces for the Hardy operator P, its adjoint operator Q, and their commutators with *BMO* and *CMO* functions.

2 The boundedness of *P* and *Q* on Hardy spaces

Because $\mathbb{R}^+$ is a space of homogeneous type with distance $d(x,y) = |x-y|$ and Lebesgue measure, we can define the Hardy space as Coifman and Weiss in [1]. We begin with the definitions of the atom, the molecule, and the Hardy space on $\mathbb{R}^+$.

A function $a \in L^\infty(\mathbb{R}^+)$ is called $(1,\infty)$-atom if it satisfies the following conditions: (1) $\operatorname{supp}(a) \subset (x_0, x_0+r]$, where $x_0 \geq 0$, $r > 0$; (2) $\|a\|_{L^\infty} \leq r^{-1}$; (3) $\int a(x)\,dx = 0$.

Let $\varepsilon > 0$, we say that a function M on $\mathbb{R}^+$ is a ε-molecule centered at x_0 if

$$\left(\int_{\mathbb{R}^+} \left|M(x)\right|^2 dx\right)\left(\int_{\mathbb{R}^+} \left|M(x)\right|^2 |x-x_0|^{1+\varepsilon}\,dx\right)^{1/\varepsilon} \leq 1$$

and

$$\int_{\mathbb{R}^+} M(x)\,dx = 0.$$

The atomic Hardy space $H^1(\mathbb{R}^+)$ is defined by

$$H^1(\mathbb{R}^+) = \left\{ f \in L^1(\mathbb{R}^+) : f(x) = \sum_k \lambda_k a_k(x), \text{ and } \sum_k |\lambda_k| < \infty \right\},$$

where each a_k is a $(1,\infty)$-atom. We set the $H^1(\mathbb{R}^+)$ norm of f by

$$\|f\|_{H^1} = \inf\left\{ \sum_k^\infty |\lambda_k| \right\},$$

where the infimum is taken over all the decompositions of $f = \sum_k \lambda_k a_k$ as above.

Remark 2.1 *P* is not bounded on $H^1(\mathbb{R}^+)$. In fact, if we take $f_0(x) = \frac{1}{4}\chi_{(0,1]} - \frac{1}{4}\chi_{(1,2]}$, then we have

$$Pf_0(x) = \begin{cases} \frac{1}{4}, & 0 < x \leq 1, \\ \frac{1}{2x} - \frac{1}{4}, & 1 < x \leq 2, \\ 0, & 2 < x. \end{cases}$$

Obviously, $f_0 \in H^1(\mathbb{R}^+)$; however $\int_0^{+\infty} Pf_0(x)\,dx \neq 0$, therefore P is not bounded on $H^1(\mathbb{R}^+)$.

Lemma 2.2 ([1]) *If M is a ε-molecule centered at x_0, then $M \in H^1(\mathbb{R}^+)$. Moreover, $\|M\|_{H^1}$ depends only on the constant ε.*

Theorem 2.3 *Q is bounded on $H^1(\mathbb{R}^+)$.*

Proof Assume that a is a $(1,\infty)$-atom of $H^1(\mathbb{R}^+)$ with $\operatorname{supp}(a) \subset (x_0, x_0+r]$, $x_0 \geq 0$, $r > 0$. It is enough to prove that Qa is a 1-molecule.

$$\begin{aligned} &\int_{\mathbb{R}^+} \left|Qa(x)\right|^2 dx \cdot \int_{\mathbb{R}^+} \left|Qa(x)\right|^2 |x-x_0|^2\,dx \\ &\quad\leq \int_{x_0}^{x_0+r} \frac{1}{r^2}\left|\int_x^{x_0+r} \frac{1}{y}\,dy\right|^2 dx \cdot \int_{x_0}^{x_0+r} \frac{1}{r^2}\left|\int_x^{x_0+r} \frac{1}{y}\,dy\right|^2 |x-x_0|^2\,dx \end{aligned}$$

$$\begin{aligned}&\leq \frac{1}{r^2}\int_{x_0}^{x_0+r}\left(\log\frac{x_0+r}{x}\right)^2 dx\cdot\int_{x_0}^{x_0+r}\left(\log\frac{x_0+r}{x}\right)^2 dx\\&\leq 4\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty Qa(x)\,dx &= \int_0^\infty\int_x^\infty\frac{a(y)}{y}\,dy\,dx = \int_0^\infty\int_0^y\frac{a(y)}{y}\,dy\\&= \int_0^\infty a(y)\,dx\,dy = 0.\end{aligned}$$

Hence $Q\tilde{a}$ is a 1-molecule. Using Lemma 2.2, we have $\|Qa\|_{H^1}\leq C$, C depends only on the constant ε. This ends the proof. □

Theorem 2.4 *P and Q are bounded from* $H^1(\mathbb{R}^+)$ *to* $L^1(\mathbb{R}^+)$.

Proof It is sufficient to prove that

$$\|Pa\|_{L^1}\leq C,\qquad \|Qa\|_{L^1}\leq C,$$

where a is a $(1,\infty)$-atom and C is independent of a.

We first prove that $\int_0^\infty|Pa(x)|\,dx\leq C$, where a is a $(1,\infty)$-atom of $H^1(\mathbb{R}^+)$ and C is independent of a. Suppose that $\operatorname{supp} a\subset(x_0,x_0+r]$, $x_0\geq 0$, $r>0$. Using conditions (2) and (3) of a, we have

$$\begin{aligned}&\int_0^\infty\left|Pa(x)\right|dx\\&\quad=\int_0^{x_0}\left|\frac{1}{x}\int_0^x a(y)\,dy\right|dx+\int_{x_0}^{x_0+r}\left|\frac{1}{x}\int_0^x a(y)\,dy\right|dx+\int_{x_0+r}^\infty\left|\frac{1}{x}\int_0^x a(y)\,dy\right|dx\\&\quad\leq\int_{x_0}^{x_0+r}\frac{1}{x}\int_0^x\left|a(y)\right|dy\,dx\\&\quad\leq\int_{x_0}^{x_0+r}\frac{1}{x}\int_0^x\frac{1}{r}\,dy\,dx=1.\end{aligned}$$

For the operator Q, by Lemma 2.2, we can get that Q is bounded from $H^1(\mathbb{R}^+)$ to $L^1(\mathbb{R}^+)$. Here we give a direct proof. Suppose that a is a $(1,\infty)$-atom of $H^1(\mathbb{R}^+)$, $\operatorname{supp} a\subset(x_0,x_0+r]$ and $x_0>0$. If $x_0=0$, the proof is more simple.

$$\begin{aligned}&\int_0^\infty\left|Qa(x)\right|dx\\&\quad=\int_0^{x_0}\left|\int_x^\infty\frac{a(y)}{y}\,dy\right|dx+\int_{x_0}^{x_0+r}\left|\int_x^\infty\frac{a(y)}{y}\,dy\right|dx+\int_{x_0+r}^\infty\left|\int_x^\infty\frac{a(y)}{y}\,dy\right|dx\\&\quad\leq\int_0^{x_0}\left|\int_{x_0}^{x_0+r}\frac{a(y)}{y}\,dy\right|dx+\int_{x_0}^{x_0+r}\left|\int_x^\infty\frac{a(y)}{y}\,dy\right|dx\\&\quad\leq x_0\int_{x_0}^{x_0+r}\frac{|a(y)|}{y}\,dy+\frac{1}{r}\int_{x_0}^{x_0+r}\int_x^{x_0+r}\frac{1}{y}\,dy\,dx\end{aligned}$$

$$\begin{aligned}&\le 1+\frac{1}{r}\int_{x_0}^{x_0+r}\int_{x_0}^{y}\frac{1}{y}\,dx\,dy\\&=1+\frac{1}{r}\int_{x_0}^{x_0+r}\left(1-\frac{x_0}{y}\right)dy\le 2.\end{aligned}$$

The proof is complete. □

3 The boundedness for commutators on Hardy spaces

In this section, we give some results on the boundedness for the commutators generated by P and Q with BMO and CMO functions on the Hardy spaces and other Hardy-type spaces.

Let $b\in L_{loc}(\mathbb{R}^+)$, we say that $b\in BMO(\mathbb{R}^+)$ if, for any interval $I\subset\mathbb{R}^+$,

$$\sup_I\frac{1}{|I|}\int_I\left|b(y)-b_I\right|dy=\|b\|_{BMO(\mathbb{R}^+)}<\infty,$$

where $b_I=\frac{1}{|I|}\int_I b(x)\,dx$ and $|I|$ is the length of I. The definition of BMO function and John–Nirenberg inequality for the BMO function on spaces of homogeneous type imply that, for any $p\ge 1$, the following are true:

$$\begin{aligned}\|b\|_{BMO(\mathbb{R}^+)}&\approx\sup_I\inf_{c\in\mathbb{C}}\left(\frac{1}{|I|}\int_I\left|b(y)-c\right|^p dy\right)^{1/p}\\&\approx\sup_I\left(\frac{1}{|I|}\int_I\left|b(y)-b_I\right|^p dy\right)^{1/p}.\end{aligned}$$

Remark 3.1 Let $b\in BMO(\mathbb{R}^+)$, then $[b,P]$ is not bounded from $H^1(\mathbb{R}^+)$ to $L^1(\mathbb{R}^+)$. In fact, if we take

$$b(x)=\begin{cases}1, & 0<x\le 1,\\-1, & 1<x\le 2,\\0, & 2<x,\end{cases}\qquad f(x)=\begin{cases}1, & 0<x\le 1,\\-1, & 1<x\le 2,\\0, & 2<x,\end{cases}$$

it is easy to see that $b\in BMO(\mathbb{R}^+)$ for any $1\le p<\infty$ and $f\in H^1(\mathbb{R}^+)$, then

$$[b,P]f(x)=\frac{1}{x}\int_0^x\left[b(x)-b(y)\right]f(y)\,dy=\begin{cases}0, & 0<x\le 1,\\-\frac{2}{x}, & 1<x.\end{cases}$$

We have

$$\int_0^{+\infty}\left|[b,P]f(x)\right|dx=\int_1^{+\infty}\frac{2}{x}\,dx=+\infty.$$

So, the commutator $[b,P]$ is not bounded from $H^1(\mathbb{R}^+)$ to $L^1(\mathbb{R}^+)$.

Theorem 3.2 *Let $b\in BMO(\mathbb{R}^+)$, then $[b,P]$ is bounded from $H^1(\mathbb{R}^+)$ to $L^{1,\infty}(\mathbb{R}^+)$. More precisely,*

$$\left\|[b,P]f\right\|_{L^{1,\infty}(\mathbb{R}^+)}\le C\|b\|_{BMO(\mathbb{R}^+)}\|f\|_{H^1(\mathbb{R}^+)}.$$

Proof It is sufficient to prove that, for any $\lambda > 0$,

$$\lambda\big|\{x \in \mathbb{R}^+ : \big|[b,P]a(x)\big| > \lambda\}\big| \le C\|b\|_{BMO},$$

where a is a $(1,\infty)$-atom of $H^1(\mathbb{R}^+)$, C is independent of a.

Let $\operatorname{supp}(a) \subset (x_0, x_0+r]$ for $r > 0$. Then

$$\begin{aligned}[b,P]a(x) &= \frac{1}{x}\int_0^x \big(b(x) - b(y)\big)a(y)\,dy \\ &= \frac{1}{x}\int_0^x \big(b(x) - b_{(x_0,x_0+r]}\big)a(y)\,dy + \frac{1}{x}\int_0^x \big(b_{(x_0,x_0+r]} - b(y)\big)a(y)\,dy \\ &= \mathrm{J}_1 + \mathrm{J}_2.\end{aligned}$$

So we have

$$\big|\{x \in \mathbb{R}^+ : \big|[b,P]a(x)\big| > \lambda\}\big| \le \left|\left\{x \in \mathbb{R}^+ : |\mathrm{J}_1| \ge \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in \mathbb{R}^+ : |\mathrm{J}_2| \ge \frac{\lambda}{2}\right\}\right|.$$

Using Chebyshev's inequality yields

$$\begin{aligned}\frac{\lambda}{2}\left|\left\{x \in \mathbb{R}^+ : |\mathrm{J}_1| > \frac{\lambda}{2}\right\}\right| &\le \int_0^{+\infty} |\mathrm{J}_1|\,dx \\ &= \int_0^{x_0+r} |\mathrm{J}_1|\,dx + \int_{x_0+r}^{+\infty} |\mathrm{J}_1|\,dx \\ &= \mathrm{J}_{11} + \mathrm{J}_{12}.\end{aligned}$$

Since P is bounded on $L^p(\mathbb{R})$ for all $1 < p < \infty$, using Hölder's inequality, we obtain

$$\begin{aligned}\mathrm{J}_{11} &= \int_{x_0}^{x_0+r} \big|\big(b(x) - b_{(x_0,x_0+r]}\big)Pa(x)\big|\,dx \\ &\le \left(\int_{x_0}^{x_0+r} \big|b(x) - b_{(x_0,x_0+r]}\big|^p\,dx\right)^{1/p} \left(\int_{x_0}^{x_0+r} \big|Pa(x)\big|^{p'}\,dx\right)^{1/p'} \\ &\le p^{p'} r^{1/p}\left(\frac{1}{r}\int_{x_0}^{x_0+r} \big|b(x) - b_{(x_0,x_0+r]}\big|^p\,dx\right)^{1/p} \left(\int_{x_0}^{x_0+r} \big|a(x)\big|^{p'}\,dx\right)^{1/p'} \\ &\le C\|b\|_{BMO},\end{aligned}$$

where we exploited condition (2) of the $(1,\infty)$-atom a.

If $x \in (x_0+r,\infty)$, using condition (3) of the $(1,\infty)$-atom a, we have

$$Pa(x) = \frac{1}{x}\int_0^x a(y)\,dy = \frac{1}{x}\int_0^{x_0+r} a(y)\,dy = 0.$$

So, $\mathrm{J}_{12} = 0$.

We next estimate the term $|\{x \in \mathbb{R}^+ : |J_2| \geq \lambda/2\}|$, which is divided into two parts: $|\{x \in (0, x_0 + r] : |J_2| \geq \lambda/2\}|$ and $|\{x \in (x_0 + r, \infty) : |J_2| \geq \lambda/2\}|$. Moreover,

$$\begin{aligned}
&\frac{\lambda}{2}\left|\left\{x \in (0, x_0 + r] : |J_2| \geq \frac{\lambda}{2}\right\}\right| \\
&\quad \leq \int_0^{x_0+r} |J_2|\,dx \\
&\quad = \int_{x_0}^{x_0+r} \left|\frac{1}{x}\int_{x_0}^{x} \bigl(b_{(x_0,x_0+r]} - b(y)\bigr)a(y)\,dy\right| dx \\
&\quad \leq \int_{x_0}^{x_0+r} \frac{1}{x}\left(\int_{x_0}^{x_0+r} \left|b_{(x_0,x_0+r]} - b(y)\right|^p dy\right)^{\frac{1}{p}} \left(\int_{x_0}^{x} \left|a(y)\right|^{p'} dy\right)^{\frac{1}{p'}} dx \\
&\quad \leq r^{1/p-1} \int_{x_0}^{x_0+r} \frac{1}{(x-x_0)^{1/p}} \left(r^{-1}\int_{x_0}^{x_0+r} \left|b_{(x_0,x_0+r]} - b(y)\right|^p dy\right)^{\frac{1}{p}} dx \\
&\quad \leq C\|b\|_{BMO}.
\end{aligned}$$

For the last term, we have

$$\begin{aligned}
&\left|\left\{x \in (x_0 + r, \infty) : |J_2| \geq \frac{\lambda}{2}\right\}\right| \\
&\quad = \left|\left\{x \in (x_0 + r, \infty) : \frac{1}{x}\left|\int_{x_0}^{x_0+r} \bigl(b_{(x_0,x_0+r]} - b(y)\bigr)a(y)\,dy\right| \geq \frac{\lambda}{2}\right\}\right| \\
&\quad \leq \left|\left\{x : x_0 + r < x \leq \frac{2}{\lambda}\left|\int_{x_0}^{x_0+r} \bigl(b_{(x_0,x_0+r]} - b(y)\bigr)a(y)\,dy\right|\right\}\right| \\
&\quad \leq \frac{2}{\lambda}\left|\int_{x_0}^{x_0+r} \bigl(b_{(x_0,x_0+r]} - b(y)\bigr)a(y)\,dy\right| \\
&\quad \leq \frac{2}{\lambda}\left(\int_{x_0}^{x_0+r} \left|b_{(x_0,x_0+r]} - b(y)\right|^p dy\right)^{\frac{1}{p}} \left(\int_{x_0}^{x_0+r} \left|a(y)\right|^{p'} dy\right)^{\frac{1}{p'}} \\
&\quad \leq \frac{2}{\lambda}\cdot\|b\|_{BMO}.
\end{aligned}$$

Combining all the above estimates, we complete the proof. □

Theorem 3.3 *Let $b \in BMO(\mathbb{R}^+)$, then $[b, Q]$ is bounded from $H^1(\mathbb{R}^+)$ to $L^1(\mathbb{R}^+)$. More precisely, there exists a constant $C > 0$ such that, for any $f \in H^1(\mathbb{R}^+)$,*

$$\bigl\|[b, Q]f\bigr\|_{L^1} \leq C\|b\|_{BMO}\|f\|_{H^1}.$$

Proof It is sufficient to prove that

$$\bigl\|[b, Q]a\bigr\|_{L^1} \leq C\|b\|_{BMO},$$

where a is a $(1, \infty)$-atom and C is independent of a.

Using the condition $\operatorname{supp}(a) \subset (x_0, x_0 + r]$, $x_0 \geq 0$, $r > 0$, we have

$$\begin{aligned}
&\int_0^{+\infty} \left|[b,Q]a\right| dx \\
&= \int_0^{x_0} \left| \int_x^{\infty} \big(b(x) - b(y)\big) \frac{a(y)}{y} \, dy \right| dx \\
&\quad + \int_{x_0}^{x_0+r} \left| \int_x^{+\infty} \big(b(x) - b(y)\big) \frac{a(y)}{y} \, dy \right| dx + \int_{x_0+r}^{\infty} \left| \int_x^{\infty} \big(b(x) - b(y)\big) \frac{a(y)}{y} \, dy \right| dx \\
&= \int_0^{x_0} \left| \int_x^{\infty} \big(b(x) - b(y)\big) \frac{a(y)}{y} \, dy \right| dx + \int_{x_0}^{x_0+r} \left| \int_x^{+\infty} \big(b(x) - b(y)\big) \frac{a(y)}{y} \, dy \right| dx \\
&= \mathrm{K}_1 + \mathrm{K}_2.
\end{aligned}$$

Suppose that $x_0 > 0$. If $x_0 = 0$, the proof is similar and simple. For K_1, we have the following estimate:

$$\begin{aligned}
\mathrm{K}_1 &\leq \int_0^{x_0} \left| \int_{x_0}^{x_0+r} \big(b(x) - c\big) \frac{a(y)}{y} \, dy \right| dx + \int_0^{x_0} \left| \int_{x_0}^{x_0+r} \big(c - b(y)\big) \frac{a(y)}{y} \, dy \right| dx \\
&\leq \int_0^{x_0} \left|b(x) - c\right| \left| \int_{x_0}^{x_0+r} \frac{|a(y)|}{y} \, dy \right| dx + r^{-1} \int_{x_0}^{x_0+r} \left|c - b(y)\right| dy \\
&\leq \frac{1}{x_0} \left(\int_0^{x_0} \left|b(x) - c\right| dx \right) \int_{x_0}^{x_0+r} \left|a(x)\right| dx + r^{-1} \int_{x_0}^{x_0+r} \left|c - b(y)\right| dy \\
&\leq \frac{1}{x_0} \int_0^{x_0} \left|b(x) - c\right| dx + r^{-1} \int_{x_0}^{x_0+r} \left|c - b(y)\right| dy,
\end{aligned}$$

where $c \in \mathbb{C}$ is an arbitrary complex number. Hence

$$\begin{aligned}
\mathrm{K}_1 &\leq \inf_{c \in \mathbb{C}} \frac{1}{x_0} \int_0^{x_0} \left|b(x) - c\right| dx + \inf_{c \in \mathbb{C}} r^{-1} \int_{x_0}^{x_0+r} \left|c - b(y)\right| dy \\
&\leq 2\|b\|_{BMO}.
\end{aligned}$$

For K_2, using the L^p-boundedness of Q and condition (2) of a, we get

$$\begin{aligned}
\mathrm{K}_2 &= \int_{x_0}^{x_0+r} \left| \int_x^{x_0+r} \big(b(x) - b(y)\big) \frac{a(y)}{y} \, dy \right| dx \\
&\leq \int_{x_0}^{x_0+r} \left| \big(b(x) - b_{(x_0,x_0+r]}\big) Qa(x) \right| dx \\
&\quad + \int_{x_0}^{x_0+r} \int_x^{x_0+r} \left|b_{(x_0,x_0+r]} - b(y)\right| \frac{|a(y)|}{y} \, dy \, dx \\
&\leq \left(\int_{x_0}^{x_0+r} \left|b(x) - b_{(x_0,x_0+r]}\right|^p dy \right)^{1/p} \left(\int_{x_0}^{x_0+r} \left|Qa(x)\right|^{p'} dy \right)^{1/p'} \\
&\quad + \int_{x_0}^{x_0+r} \int_{x_0}^{y} \left|b_{(x_0,x_0+r]} - b(y)\right| \frac{|a(y)|}{y} \, dx \, dy \\
&\leq \left(\int_{x_0}^{x_0+r} \left|b(x) - b_{(x_0,x_0+r]}\right|^p dy \right)^{1/p} \left(\int_{x_0}^{x_0+r} \left|a(x)\right|^{p'} dy \right)^{1/p'}
\end{aligned}$$

$$+\frac{1}{r}\int_{x_0}^{x_0+r}\left|b_{(x_0,x_0+r]}-b(y)\right|dy$$
$$\leq C\|b\|_{BMO}.$$

Combining all the above estimates, we complete the proof. □

Perez in [11] introduced a kind of Hardy spaces associated with *BMO* functions and proved the boundedness on these Hardy spaces for the commutators of singular integrals with *BMO* functions. Now we define similar Hardy spaces on $\mathbb{R}^+$. Let $b \in BMO(\mathbb{R}^+)$, the function a is called b-atom if there is an interval $(x_0, x_0 + r]$, $x_0 \geq 0$, $r > 0$, satisfying

$$\text{(i)}\quad \operatorname{supp}(a) \subset (x_0, x_0 + r], \qquad \text{(ii)}\quad \|a\|_{L^\infty} \leq r^{-1},$$
$$\text{(iii)}\quad \int a(y)\,dy = 0 \quad \text{and} \quad \text{(iv)}\quad \int a(y)b(y)\,dy = 0.$$

The space $H_b^1(\mathbb{R}^+)$ consists of the subspace of $L^1(\mathbb{R}^+)$ of functions f, which can be written as $f = \sum_j \lambda_j a_j$, where a_j are b-atoms and λ_j are complex numbers with $\sum_j |\lambda_j| < \infty$. The $H_b^1(\mathbb{R}^+)$ norm of f is defined by

$$\|f\|_{H_b^1(\mathbb{R}^+)} = \inf\left\{\sum_j |\lambda_j|\right\},$$

where the infimum has taken over all the decompositions of $f = \sum_j \lambda_j a_j$ as above.

Theorem 3.4 *Let $b \in BMO(\mathbb{R}^+)$, then $[b, P]$ is bounded from $H_b^1(\mathbb{R}^+)$ to $L^1(\mathbb{R}^+)$.*

Proof We only need to prove that, for any b-atom a, we have

$$\left\|[b, P]a\right\|_{L^1} \leq C\|b\|_{BMO}.$$

Using conditions (iii), (iv) and $\operatorname{supp}(a) \subset (x_0, x_0 + r]$, $x_0 \geq 0$, $r > 0$, we have

$$\begin{aligned}
&\int_0^{+\infty}\left|[b,P]a\right|dx \\
&\quad= \int_0^{x_0}\left|\frac{1}{x}\int_0^x \big(b(x)-b(y)\big)a(y)\,dy\right|dx \\
&\qquad+ \int_{x_0}^{x_0+r}\left|\frac{1}{x}\int_0^x \big(b(x)-b(y)\big)a(y)\,dy\right|dx + \int_{x_0+r}^{\infty}\left|\frac{1}{x}\int_0^x \big(b(x)-b(y)\big)a(y)\,dy\right|dx \\
&\quad= \int_{x_0}^{x_0+r}\left|\frac{1}{x}\int_0^x \big(b(x)-b(y)\big)a(y)\,dy\right|dx \\
&\quad\leq \int_{x_0}^{x_0+r}\left|\frac{1}{x}\int_0^x \big(b(x)-c\big)a(y)\,dy\right|dx + \int_{x_0}^{x_0+r}\left|\frac{1}{x}\int_0^x \big(c-b(y)\big)a(y)\,dy\right|dx \\
&\quad= \mathrm{L}_1 + \mathrm{L}_2,
\end{aligned}$$

where $c \in \mathbb{C}$ is an arbitrary complex number.

For L_1, let $p > 1$, using Hölder's inequality, the $L^{p'}$ boundedness of the Hardy operator P, and conditions (ii) of b-atom a, we have

$$\begin{aligned}\mathrm{L}_1 &\le \int_{x_0}^{x_0+r} \left|\bigl(b(x)-c\bigr)Pa(x)\right| dx \\ &\le \left(\int_{x_0}^{x_0+r} \left|b(x)-c\right|^p dx\right)^{1/p} \left(\int_{x_0}^{x_0+r} \left|Pa(x)\right|^{p'} dx\right)^{1/p'} \\ &\le p'\left(\int_{x_0}^{x_0+r} \left|b(x)-c\right|^p dx\right)^{1/p} \left(\int_{x_0}^{x_0+r} \left|a(x)\right|^{p'} dx\right)^{1/p'} \\ &\le p'\left(\frac{1}{r}\int_{x_0}^{x_0+r} \left|b(x)-c\right|^p dx\right)^{1/p}.\end{aligned}$$

For L_2, using Hölder's inequality and conditions (ii) of b-atom a, we obtain

$$\begin{aligned}\mathrm{L}_2 &\le \int_{x_0}^{x_0+r} \left(\frac{1}{x}\int_0^x \left|b(y)-c\right|^p dy\right)^{1/p} \left(\frac{1}{x}\int_{x_0}^x \left|a(y)\right|^{p'} dy\right)^{1/p'} dx \\ &\le \frac{1}{r}\int_{x_0}^{x_0+r} \left(\frac{1}{x}\int_0^x \left|b(y)-c\right|^p dy\right)^{1/p} dx.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{L}_1+\mathrm{L}_2 &\le p'\inf_{c\in\mathbb{C}}\left(\frac{1}{r}\int_{x_0}^{x_0+r} \left|b(x)-c\right|^p dx\right)^{1/p} + \frac{1}{r}\int_{x_0}^{x_0+r} \inf_{c\in\mathbb{C}}\left(\frac{1}{x}\int_0^x \left|b(y)-c\right|^p dy\right)^{1/p} dx \\ &\le C\|b\|_{BMO}.\end{aligned}$$

This ends the proof. □

Let $1 \le p < \infty$, we say that $b \in CMO^p(\mathbb{R}^+)$ if

$$\|b\|_{CMO^p} = \sup_{r>0}\left(\frac{1}{r}\int_0^r \left|b(y)-b_{(0,r]}\right|^p dy\right)^{1/p} < \infty.$$

By the definition of CMO^p function, for any $p \ge 1$, we have

$$\|b\|_{CMO^p} \approx \sup_{r>0}\inf_{c\in\mathbb{C}}\left(\frac{1}{r}\int_0^r \left|b(y)-c\right|^p dy\right)^{1/p}.$$

It is easy to see $BMO(\mathbb{R}^+) \subsetneqq CMO^p(\mathbb{R}^+)$, where $1 \le p < \infty$. $CMO^q(\mathbb{R}^+) \subsetneqq CMO^p(\mathbb{R}^+)$ for $1 \le p < q < \infty$.

Let $1 < p \le \infty$, a function a is called central $(1,p)$-atom, if it satisfies the following conditions: (1) $\operatorname{supp}(a) \subset (0,r]$, where $r > 0$; (2) $\|a\|_{L^p} \le r^{1/p-1}$; (3) $\int a(x)\,dx = 0$.

The central atomic Hardy space $H_c^{1,p}(\mathbb{R}^+)$ is defined by

$$H_c^{1,p}(\mathbb{R}^+) = \left\{f \in L^1(\mathbb{R}^+) : f(x) = \sum_k \lambda_k a_k(x), \text{ and } \sum_k |\lambda_k| < \infty\right\},$$

where each a_k is a central $(1,p)$-atom. We set the $H_c^{1,p}(\mathbb{R}^+)$ norm of f by

$$\|f\|_{H_c^{1,p}} = \inf\left\{\sum_k^{\infty} |\lambda_k|\right\},$$

where the infimum is taken over all the decompositions of $f = \sum_k \lambda_k a_k$ as above.

Similar to the proof in García-Cuerva [3], we can obtain that the dual space of $H_c^{1,p}(\mathbb{R}^+)$ is $CMO^{p'}(\mathbb{R}^+)$ for $1 < p \leq \infty$.

Taking the same example in Remark 3.1, we can show that, for $1 < p < \infty$ and $b \in CMO^p(\mathbb{R}^+)$, the commutator $[b,P]$ is not bounded from $H_c^{1,p'}(\mathbb{R}^+)$ to $L^1(\mathbb{R}^+)$. Similar to the proof of Theorem 3.2 and Theorem 3.3, we can prove the following theorem and omit the details here.

Theorem 3.5 *Let $1 < p < \infty$ and $b \in CMO^p(\mathbb{R}^+)$, then $[b,P]$ is bounded from $H_c^{1,p'}(\mathbb{R}^+)$ to $L^{1,\infty}(\mathbb{R}^+)$ and $[b,Q]$ bounded from $H_c^{1,p'}(\mathbb{R}^+)$ to $L^1(\mathbb{R}^+)$.*

Let $p \geq 1$ and $b \in CMO^p(\mathbb{R}^+)$, the function a is called central $(1,p,b)$-atom if there exists $r > 0$, satisfying (i). $\operatorname{supp}(a) \subset (0,r]$, (ii). $\|a\|_{L^p} \leq r^{1/p-1}$, (iii). $\int a(y)\,dy = 0$, (iv). $\int a(y)b(y)\,dy = 0$.

The space $H_{b,c}^{1,p}(\mathbb{R}^+)$ consists of the subspace of $L^1(\mathbb{R}^+)$ of functions f, which can be written as $f = \sum_j \lambda_j a_j$, where a_j are central $(1,p,b)$-atoms and λ_j are complex numbers with $\sum_j |\lambda_j| < \infty$. The $H_{b,c}^{1,p}(\mathbb{R}^+)$ norm of f is defined by

$$\|f\|_{H_{b,c}^{1,p}} = \inf\left\{\sum_j |\lambda_j|\right\},$$

where the infimum has taken over all the decompositions of $f = \sum_j \lambda_j a_j$ as above.

It is easy to see $H_{b,c}^{1,p}(\mathbb{R}^+) \subsetneqq H_c^{1,p}(\mathbb{R}^+)$. Similar to the proof of Theorem 3.4, we can obtain the following and omit the details here.

Theorem 3.6 *Let $1 < p < \infty$ and $b \in CMO^p(\mathbb{R}^+)$, then $[b,P]$ is bounded from $H_{b,c}^{1,p'}(\mathbb{R}^+)$ to $L^1(\mathbb{R}^+)$.*

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