Zhou *Journal of Inequalities and Applications* (2020) 2020:247
https://doi.org/10.1186/s13660-020-02514-6

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Norm inequalities for submultiplicative functions involving contraction sector 2×2 block matrices

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Abstract
In this article, we show unitarily invariant norm inequalities for sector 2×2 block matrices which extend and refine some recent results of Bourahli, Hirzallah, and Kittaneh (Positivity, 2020, https://doi.org/10.1007/s11117-020-00770-w).

MSC: 15A18; 15A45

Keywords: Contraction sector block matrices; Unitarily invariant norms; Submultiplicative functions

1 Introduction

Let $\mathbb{M}_n$ be a set of all $n \times n$ complex matrices. A matrix $A \in \mathbb{M}_n$ is said to be positive semidefinite if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$. If the eigenvalues $\lambda_1(A), \dots, \lambda_n(A)$ of A are all real, we arrange them in nonincreasing order $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. Singular values of A are the eigenvalues of $|A|$ and are arranged in nonincreasing order $s_1(A) \geq \cdots \geq s_n(A)$. For $A \in \mathbb{M}_n$, we denote by $|A| = (A^*A)^{\frac{1}{2}}$, A^*, $\|A\|$, and $\|A\|_\infty = s_1(A)$ the absolute value, the conjugate transpose, the unitarily invariant norm, and the operator norm, respectively. We say A is a contraction if $\|A\|_\infty \leq 1$. By convention, the $n \times n$ identity matrix is denoted by I_n. $\|A\|$ and $\|A\|_\infty = s_1(A)$ are unitarily invariant, i.e., $\|UAV\| = \|A\|$ for all unitary matrices U, V. For $A, B \in \mathbb{M}_n$, the weak majorization relation $s(A) \prec_w s(B)$ means

$$\sum_{j=1}^{k} s_j(A) \leq \sum_{j=1}^{k} s_j(B), \quad k = 1, 2, \dots, n.$$

For $A \in \mathbb{M}_n$, recall the Cartesian (or Toeplitz) decomposition (see, e.g., [2, p. 6] and [3, p. 7])

$$A = \operatorname{Re} A + i \operatorname{Im} A,$$

where

$$\operatorname{Re} A := \frac{1}{2}(A + A^*), \qquad \operatorname{Im} A := \frac{1}{2i}(A - A^*).$$

The Cartesian decomposition of a matrix is unique. There are many interesting properties for such a decomposition. A celebrated result due to Fan and Hoffman (see, e.g., [2, p.73]) states that

$$\lambda_j(\operatorname{Re} A) \le s_j(A), \quad j = 1,\dots,n. \tag{1}$$

The numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax \mid x \in \mathbb{C}^n, x^*x = 1\},$$

which is a compact convex set (see, e.g., [4, Chap. 1]). For $\alpha \in [0,\pi/2)$, a sector on the complex plane is

$$S_\alpha = \{z \in \mathbb{C} \mid \operatorname{Re} z \ge 0, |\operatorname{Im} z| \le (\operatorname{Re} z)\tan\alpha\}.$$

A sector matrix $A \in \mathbb{M}_n$ is a matrix whose numerical range is contained in S_α for some $\alpha \in [0,\pi/2)$. It is clear that if $A \in \mathbb{M}_n$ is a sector matrix, then $\operatorname{Re} A$ is positive semidefinite. The interested readers can refer to [5–10], and [4] for recent results on sector matrices. If $W(A)$ is contained in the first quadrant of the complex plane, then $\operatorname{Re} A$ and $\operatorname{Im} A$ are positive semidefinite. We call such a matrix A accretive-dissipative. Note that if A is accretive-dissipative, then $W(e^{-\frac{i\pi}{4}}A) \subseteq S_{\frac{\pi}{4}}$. Recently this class of matrices has been studied by researchers partly due to the fact that it contains the class of positive semidefinite matrices (see, e.g., [1, 11–17]).

Next we introduce a special class of functions. Let $\mathcal{C}$ be the class of all nonnegative increasing functions f on $[0,\infty)$ preserving the weak-log majorization, i.e., for two nonincreasing sequences of nonnegative real numbers $(x_1,x_2,\dots,x_n)$ and $(y_1,y_2,\dots,y_n)$, $\prod_{j=1}^k x_j \le \prod_{j=1}^k y_j$ for $k = 1,\dots,n$ implies $\prod_{j=1}^k f(x_j) \le \prod_{j=1}^k f(y_j)$ for $k = 1,\dots,n$. There are many other properties on this class of functions; see [12, 18]. A nonnegative function $f \in \mathcal{C}$ on the interval $[0,\infty)$ is said to be submultiplicative if $f(ab) \le f(a)f(b)$ whenever $a,b \in [0,\infty)$. Recently, some unitarily invariant norm inequalities for submultiplicative functions of accretive-dissipative matrices have been shown in [12] and [13].

Bourahli et al. [1, Lemma 3.4] showed that if $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix} \in \mathbb{M}_{2n}$ is a positive semidefinite contraction and s,t are positive real numbers such that $\frac{1}{s}+\frac{1}{t}=1$, then

$$\left\|f\left(|A_{12}|^2\right)\right\| \le \left\|f^s\left(A_{11}^{\frac{1}{2}}\right)\right\|^{\frac{1}{s}} \left\|f^t\left(A_{22}^{\frac{1}{2}}\right)\right\|^{\frac{1}{t}}, \tag{2}$$

where f is an increasing submultiplicative function on $[0,\infty)$ with $f(0) = 0$. Moreover, if $A \in \mathbb{M}_{2n}$ is just positive semidefinite (not necessarily contraction matrices), then they presented a result related to (2) in [1, Remark 3.5] as follows:

$$\left\|f\left(|A_{12}|^2\right)\right\| \le f\left(\left\|A_{11}^{\frac{1}{2}}\right\|_\infty \left\|A_{22}^{\frac{1}{2}}\right\|_\infty\right) \left\|f^s\left(A_{11}^{\frac{1}{2}}\right)\right\|^{\frac{1}{s}} \left\|f^t\left(A_{22}^{\frac{1}{2}}\right)\right\|^{\frac{1}{t}}. \tag{3}$$

2 Unitarily invariant norms for submultiplicative functions

In [12] and [13], some unitarily invariant norms for accretive-dissipative matrices involving a special class of functions have been shown. In this section, we present inequalities for sector block matrices involving the class of function.

Lemma 2.1 ([19, p. 280]) *Let A, X, B be $m \times p$, $p \times q$, $q \times n$ matrices, respectively. Then*

$$s_i(AXB) \le s_1(A)s_j(X)s_1(B), \quad i \le \min\{m,p,q,n\}.$$

Lemma 2.2 ([8, Theorem 2.1]) *Let $A \in \mathbb{M}_n$ be $n \times n$ such that $W(A) \subseteq S_\alpha$ for some $\alpha \in [0,\pi/2)$. Then there exist an invertible matrix X and a unitary and diagonal matrix $Z = \operatorname{diag}(e^{i\theta_1},\dots,e^{i\theta_n})$ with all $|\theta_j| \le \alpha$ such that $A = XZX^*$. Moreover, such a matrix Z is unique up to permutation.*

Lemma 2.3 ([8, Corollary 2.3 (ii)]) *Let $A \in \mathbb{M}_n$ be such that $W(A) \subseteq S_\alpha$ for some $\alpha \in [0,\pi/2)$, and let $A = XZX^*$ be a sectoral decomposition of A, where X is invertible and Z is unitary and diagonal. Then*

$$RR^* \le \sec(\alpha)\bigl(R(\operatorname{Re} Z)R^*\bigr) = \sec(\alpha)\bigl(\operatorname{Re}\bigl(RZR^*\bigr)\bigr)$$

for every matrix $R \in \mathbb{M}_n$.

We are ready to present our main result of this section.

Theorem 2.4 *Let $f \in \mathcal{C}$ be an increasing submultiplicative function on $[0,\infty)$ and $A \in \mathbb{M}_{2n}$ be a contraction matrix partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{4}$$

with $W(A) \subseteq S_\alpha$ for some $\alpha \in [0,\pi/2)$. Then, for all $r,s,t > 0$ with $\frac{1}{s} + \frac{1}{t} = 1$ and all unitarily invariant norms,

$$\bigl\|f\bigl(|A_{12}|^{2r}\bigr)\bigr\| \le \bigl\|f^s\bigl(\bigl(\sec^2(\alpha)|A_{11}|\bigr)^{r/2}\bigr)\bigr\|^{1/s}\bigl\|f^t\bigl(\bigl(\sec^2(\alpha)|A_{22}|\bigr)^{r/2}\bigr)\bigr\|^{1/t} \tag{5}$$

and

$$\bigl\|f\bigl(|A_{21}|^{2r}\bigr)\bigr\| \le \bigl\|f^s\bigl(\bigl(\sec^2(\alpha)|A_{11}|\bigr)^{r/2}\bigr)\bigr\|^{1/s}\bigl\|f^t\bigl(\bigl(\sec^2(\alpha)|A_{22}|\bigr)^{r/2}\bigr)\bigr\|^{1/t}. \tag{6}$$

Proof Note that A is a sector matrix with $W(A) \subseteq S_\alpha$. By Lemma 2.2, we have $A = XZX^*$, where X is invertible and Z is unitary and diagonal. We partition X as $\binom{X_1}{X_2}$, $X_1, X_2 \in \mathbb{M}_{n\times 2n}$. Thus, $\operatorname{Re} A_{11} = X_1(\operatorname{Re} Z)X_1^*$, $\operatorname{Re} A_{22} = X_2(\operatorname{Re} Z)X_2^*$, and $A_{12} = X_1ZX_2^*$. Consider the Cartesian decomposition

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = R + iS = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} + i\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

where R is positive semidefinite and S is Hermitian. Since A is a contraction matrix,

$$AA^* = R^2 + S^2 + i(SR - RS) \le I \tag{7}$$

and

$$A^*A = R^2 + S^2 + i(RS - SR) \le I. \tag{8}$$

Adding (7) and (8), we get

$$2\left(R^2 + S^2\right) \le 2I.$$

Thus, R and S are also contraction matrices, which implies that both $\operatorname{Re} A_{11} = R_{11}$ and $\operatorname{Re} A_{22} = R_{22}$ are positive semidefinite contractions.

Now

$$\begin{aligned}
s_\ell\left(|A_{12}|^r\right) &= s_\ell^r\left(|A_{12}|\right) = s_\ell^r(A_{12}) = s_\ell^r\left(X_1 Z X_2^*\right) \\
&\le s_1^r(X_1)s_\ell^r\left(ZX_2^*\right) \quad \text{(by Lemma 2.1)} \\
&= \lambda_1^{\frac{r}{2}}\left(X_1^*X_1\right)\lambda_\ell^{\frac{r}{2}}\left(X_2Z^*ZX_2^*\right) \\
&= \lambda_1^{\frac{r}{2}}\left(X_1X_1^*\right)\lambda_\ell^{\frac{r}{2}}\left(X_2X_2^*\right) \\
&\le \lambda_1^{\frac{r}{2}}\left(\sec(\alpha)X_1(\operatorname{Re} Z)X_1^*\right)\lambda_\ell^{\frac{r}{2}}\left(\sec(\alpha)X_2(\operatorname{Re} Z)X_2^*\right) \quad \text{(by Lemma 2.3)} \\
&= \lambda_1^{\frac{r}{2}}\left(\sec(\alpha)\operatorname{Re} A_{11}\right)\lambda_\ell^{\frac{r}{2}}\left(\sec(\alpha)\operatorname{Re} A_{22}\right) && (9) \\
&\le \sec^{\frac{r}{2}}(\alpha)\lambda_\ell^{\frac{r}{2}}\left(\sec(\alpha)\operatorname{Re} A_{22}\right) \quad \text{(since } \operatorname{Re} A_{11} \text{ is a contraction)} \\
&\le \sec^r(\alpha)s_\ell^{\frac{r}{2}}\left(|A_{22}|\right) \quad \text{(by (1))} && (10)
\end{aligned}$$

for $l = 1, 2, \dots, n$.

Since $\operatorname{Re} A_{22}$ is also a contraction, it follows from (1) and (9) that

$$s_\ell\left(|A_{12}|^r\right) \le \sec^r(\alpha)s_\ell^{\frac{r}{2}}\left(|A_{11}|\right) \tag{11}$$

for $l = 1, 2, \dots, n$.

Multiplying inequalities (10) and (11) by each other implies that

$$s_\ell\left(|A_{12}|^{2r}\right) \le \sec^{2r}(\alpha)s_\ell^{\frac{r}{2}}\left(|A_{11}|\right)s_\ell^{\frac{r}{2}}\left(|A_{22}|\right) \tag{12}$$

for $l = 1, 2, \dots, n$.

So,

$$\begin{aligned}
s_\ell\left(f\left(|A_{12}|^{2r}\right)\right) &= f\left(s_\ell\left(|A_{12}|^{2r}\right)\right) \\
&\le f\left(\sec^{2r}(\alpha)s_\ell^{\frac{r}{2}}\left(|A_{11}|\right)s_\ell^{\frac{r}{2}}\left(|A_{22}|\right)\right) \quad \text{(by (12))} \\
&= f\left(\sec^r(\alpha)s_\ell^{\frac{r}{2}}\left(|A_{11}|\right)\right)f\left(\sec^r(\alpha)s_\ell^{\frac{r}{2}}\left(|A_{22}|\right)\right) && (13) \\
&\quad \text{(since } f \text{ is submultiplicative)}
\end{aligned}$$

for $l = 1, 2, \dots, n$. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a decreasing sequence of nonnegative real numbers. The α-norm of a matrix $B \in \mathbb{M}_n$ is defined by

$$\|B\|_\alpha = \sum_{\ell=1}^{n} \alpha_\ell s_\ell(B).$$

The α-norms are unitarily invariant [4, p. 204].

Actually, inequality (13) means that

$$\prod_{\ell=1}^{k} \alpha_\ell s_\ell\big(f\big(|A_{12}|^{2r}\big)\big) \le \prod_{\ell=1}^{k} \alpha_\ell s_\ell\big(f\big(\big(\sec^2(\alpha)|A_{11}|\big)^{\frac{r}{2}}\big)\big) s_\ell\big(f\big(\big(\sec^2(\alpha)|A_{22}|\big)^{\frac{r}{2}}\big)\big)$$

for $k = 1, 2, \dots, n$, which implies that

$$\sum_{\ell=1}^{k} \alpha_\ell s_\ell\big(f\big(|A_{12}|^{2r}\big)\big) \le \sum_{\ell=1}^{k} \alpha_\ell s_\ell\big(f\big(\big(\sec^2(\alpha)|A_{11}|\big)^{\frac{r}{2}}\big)\big) s_\ell\big(f\big(\big(\sec^2(\alpha)|A_{22}|\big)^{\frac{r}{2}}\big)\big)$$

for $k = 1, 2, \dots, n$. Thus,

$$\begin{aligned}
&\big\|f\big(|A_{12}|^{2r}\big)\big\|_\alpha \\
&\quad = \sum_{\ell=1}^{n} \alpha_\ell s_\ell\big(f\big(|A_{12}|^{2r}\big)\big) \\
&\quad \le \sum_{\ell=1}^{n} \alpha_\ell s_\ell\big(f\big(\big(\sec^2(\alpha)|A_{11}|\big)^{r/2}\big)\big) s_\ell\big(f\big(\big(\sec^2(\alpha)|A_{22}|\big)^{r/2}\big)\big) \\
&\quad = \sum_{\ell=1}^{n} \alpha_\ell^{1/s} s_\ell\big(f\big(\big(\sec^2(\alpha)|A_{11}|\big)^{r/2}\big)\big) \alpha_\ell^{1/t} s_\ell\big(f\big(\big(\sec^2(\alpha)|A_{22}|\big)^{r/2}\big)\big) \\
&\quad \le \left(\sum_{\ell=1}^{n} \alpha_\ell s_\ell^s\big(f\big(\big(\sec^2(\alpha)|A_{11}|\big)^{r/2}\big)\big)\right)^{1/s} \left(\sum_{\ell=1}^{m} \alpha_\ell s_\ell^t\big(f\big(\big(\sec^2(\alpha)|A_{22}|\big)^{r/2}\big)\big)\right)^{1/t} \\
&\quad \big(\text{by Hölder's inequality}\big) \\
&\quad = \left(\sum_{\ell=1}^{n} \alpha_\ell s_\ell\big(f^s\big(\big(\sec^2(\alpha)|A_{11}|\big)^{r/2}\big)\big)\right)^{1/s} \left(\sum_{\ell=1}^{m} \alpha_\ell s_\ell\big(f^t\big(\big(\sec^2(\alpha)|A_{22}|\big)^{r/2}\big)\big)\right)^{1/t} \\
&\quad = \big\|f^s\big(\big(\sec^2(\alpha)|A_{11}|\big)^{r/2}\big)\big\|_\alpha^{1/s} \big\|f^t\big(\big(\sec^2(\alpha)|A_{22}|\big)^{r/2}\big)\big\|_\alpha^{1/t}
\end{aligned}$$

for all decreasing sequences $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative real numbers. It follows from the above inequality that

$$\big\|f\big(|A_{12}|^{2r}\big)\big\| \le \big\|f^s\big(\big(\sec^2(\alpha)|A_{11}|\big)^{r/2}\big)\big\|^{1/s} \big\|f^t\big(\big(\sec^2(\alpha)|A_{22}|\big)^{r/2}\big)\big\|^{1/t}.$$

The inequality for A_{21} is similarly proved. □

Remark 1 In particular, when A is a positive semidefinite contraction ($\alpha = 0$) and $r = 1$, Theorem 2.4 gives

$$\left\| f\left(|A_{12}|^2\right)\right\| \leq \left\| f^s\left(A_{11}^{1/2}\right)\right\|^{1/s} \left\| f^t\left(A_{22}^{1/2}\right)\right\|^{1/t}, \tag{14}$$

which is due to Bourahli et al. [1, Lemma 3.4]. Thus, our result (5) is a generalization of (14).

Remark 2 If A is just a general sector matrix with $W(A) \subseteq S_\alpha$ for $\alpha \in [0, \frac{\pi}{2})$ (not a contraction matrix), then we have the following result: Let $f \in \mathcal{C}$ be an increasing submultiplicative function on $[0,\infty)$ and $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right) \in \mathbb{M}_{2n}$ be a sector matrix with $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \pi/2)$. Then, for all $r, s, t > 0$ with $\frac{1}{s} + \frac{1}{t} = 1$ and all unitarily invariant norms,

$$\left\| f\left(|A_{12}|^{2r}\right)\right\| = f\left(\sec^{2r}(\alpha)\|A_{11}\|_\infty^{\frac{r}{2}}\|A_{22}\|_\infty^{\frac{r}{2}}\right) \left\| f^s\left(|A_{11}|^{\frac{r}{2}}\right)\right\|^{\frac{1}{s}} \left\| f^t\left(|A_{22}|^{\frac{r}{2}}\right)\right\|^{\frac{1}{t}}. \tag{15}$$

By (9), (10), and Lemma 2.1, we have

$$\begin{aligned} s_\ell\left(|A_{12}|^r\right) &= s_\ell^r\left(|A_{12}|\right) = s_\ell^r(A_{12}) = s_\ell^r\left(X_1 Z X_2^*\right) \\ &\leq \sec^r(\alpha)\|A_{11}\|_\infty^{\frac{r}{2}} s_\ell^{\frac{r}{2}}\left(|A_{22}|\right) \end{aligned} \tag{16}$$

and

$$s_\ell\left(|A_{12}|^r\right) \leq \sec^r(\alpha)\|A_{22}\|_\infty^{\frac{r}{2}} s_\ell^{\frac{r}{2}}\left(|A_{11}|\right) \tag{17}$$

for $l = 1, 2, \ldots, n$.

Multiplying inequalities (16) and (17) by each other implies that

$$s_\ell\left(|A_{12}|^{2r}\right) \leq \sec^{2r}(\alpha)\|A_{11}\|_\infty^{\frac{r}{2}}\|A_{22}\|_\infty^{\frac{r}{2}} s_\ell^{\frac{r}{2}}\left(|A_{11}|\right) s_\ell^{\frac{r}{2}}\left(|A_{22}|\right) \tag{18}$$

for $l = 1, 2, \ldots, n$.

So,

$$\begin{aligned} s_\ell\left(f\left(|A_{12}|^{2r}\right)\right) &= f\left(s_\ell\left(|A_{12}|^{2r}\right)\right) \\ &\leq f(\sec^{2r}(\alpha)\|A_{11}\|_\infty^{\frac{r}{2}}\|A_{22}\|_\infty^{\frac{r}{2}} s_\ell^{\frac{r}{2}}\left(|A_{11}|\right) s_\ell^{\frac{r}{2}}\left(|A_{22}|\right) \quad \text{(by (18))} \\ &= f\left(\sec^{2r}(\alpha)\|A_{11}\|_\infty^{\frac{r}{2}}\|A_{22}\|_\infty^{\frac{r}{2}}\right) f\left(s_\ell^{\frac{r}{2}}\left(|A_{11}|\right)\right) f\left(s_\ell^{\frac{r}{2}}\left(|A_{22}|\right)\right) \end{aligned} \tag{19}$$

for $l = 1, 2, \ldots, n$. Based on (19), we can obtain the desired result by a proof similar to that given for inequality (13). Therefore, when $\alpha = 0$ and $r = 1$, our result (15) is (2).

Theorem 2.5 *Let $f \in \mathcal{C}$ be an increasing submultiplicative function on $[0,\infty)$ and $A \in \mathbb{M}_{2n}$ be a contraction matrix partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

with $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \pi/2)$. Then, for all $r, s, t > 0$ with $\frac{1}{s} + \frac{1}{t} = 1$ and all unitarily invariant norms,

$$\begin{aligned} &\| f(|A_{12}|^{2r}) \| + \| f(|A_{21}|^{2r}) \| \\ &\quad \le 2 \| f^s((\sec^2(\alpha)|A_{11}|)^{r/2}) \|^{1/s} \| f^t((\sec^2(\alpha)|A_{22}|)^{r/2}) \|^{1/t}. \end{aligned} \tag{20}$$

Proof By Theorem 2.4, we can have the desired result. □

Remark 3 When $f(t) = t$, $r = 1$, and $\alpha = \frac{\pi}{4}$, result (20) becomes

$$\| |A_{12}|^2 \| + \| |A_{21}|^2 \| \le 2 \| f^s((\sec^2(\alpha)|A_{11}|)^{r/2}) \|^{1/s} \| f^t((\sec^2(\alpha)|A_{22}|)^{r/2}) \|^{1/t}.$$

Acknowledgements
The author wants to express her sincere thanks to the referee for his or her valuable remarks and suggestions, which made this paper more readable.

Funding
Zhou's work was supported by the High-level Talents Program of Natural Science Foundation of Hainan Province, China (2019RC193) and the Scientific Research Project of Colleges and Universities of Hainan Province, China (Hnky2020-31).

Availability of data and materials
Not applicable.

Competing interests
The author declares that they have no competing interests.

Authors' contributions
The author drafted and approved the final manuscript.

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Publisher's Note

Received: 27 July 2020 Accepted: 12 November 2020 Published online: 19 November 2020

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