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On a variant of Čebyšev's inequality of the Mercer type

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Dedicated to Professor Shoshana Abramovich on the occasion of her 80th birthday.

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Abstract
We consider the discrete Jensen–Mercer inequality and Čebyšev's inequality of the Mercer type. We establish bounds for Čebyšev's functional of the Mercer type and bounds for the Jensen–Mercer functional in terms of the discrete Ostrowski inequality. Consequentially, we obtain new refinements of the considered inequalities.

MSC: 26D15

Keywords: Jensen–Mercer inequality; Čebyšev's inequality of the Mercer type

1 Introduction

Let $n \geq 2$ and let $\boldsymbol{w} = (w_1, \dots, w_n)$ be a real n-tuple such that

$$0 \leq W_k = \sum_{i=1}^{k} w_i \leq W_n, \quad k = 1, \dots, n, W_n > 0. \tag{1}$$

In [5] the following Čebyšev's inequality of the Mercer type:

$$\left(a + b - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)\left(c + d - \frac{1}{W_n}\sum_{i=1}^{n} w_i y_i\right) \leq ac + bd - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i y_i, \tag{2}$$

was proved for any real n-tuples $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$ monotone in the same direction and real numbers a, b, c, d such that

$$a \leq \min_{1\leq i\leq n} x_i, \qquad b \geq \max_{1\leq i\leq n} x_i, \qquad c \leq \min_{1\leq i\leq n} y_i, \qquad d \geq \max_{1\leq i\leq n} y_i. \tag{3}$$

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotonic in the opposite directions, inequality (2) is reversed.

Here, to be more precise, we cite that result with the slightly different notation.

In the same paper, the authors considered Čebyšev's functional (or Čebyšev's difference) of the Mercer type defined as the difference of the right- and left-hand sides of inequality (2). They established bounds in terms of the discrete Ostrowski inequality. Here we give more accurate bounds, which also provide refinements of inequality (2). In addition, using these results, we establish Ostrowski-like bounds for the Jensen–Mercer functional and, consequentially, a refinement of the Jensen–Mercer inequality.

2 Bounds for the Čebyšev's functional of the Mercer type

Let $m \geq 2$ and let $\boldsymbol{p} = (p_1, \dots, p_m)$ be a real m-tuple such that

$$0 \leq P_k = \sum_{i=1}^{k} p_i \leq P_m, \quad k = 1, \dots, m, P_m > 0. \tag{4}$$

Then $\overline{P}_k = \sum_{i=k}^{m} p_i \geq 0$, $k = 1, \dots, m$. Furthermore, from the summation by parts (sometimes called the Abel transformation) it follows that the identity

$$\begin{aligned} &\sum_{i=1}^{m} p_i \sum_{i=1}^{m} p_i \xi_i \zeta_i - \sum_{i=1}^{m} p_i \xi_i \sum_{i=1}^{m} p_i \zeta_i \\ &\quad = \sum_{i=1}^{m-1} \left(\sum_{j=1}^{i-1} \overline{P}_{i+1} P_j \Delta\xi_i \Delta\zeta_j + \sum_{j=i}^{m-1} P_i \overline{P}_{j+1} \Delta\xi_i \Delta\zeta_j \right) \end{aligned} \tag{5}$$

holds for any two real m-tuples $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m)$ and $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_m)$, where $\Delta\xi_i = \xi_{i+1} - \xi_i$, $\Delta\zeta_i = \zeta_{i+1} - \zeta_i$, $i = 1, \dots, m-1$ (see [7, 8]).

Here, and in the rest of the paper, we assume $\sum_{j=k}^{l} x_j = 0$ when $k > l$.

Lemma 1 *Let $n \geq 2$ and let $\boldsymbol{w}$ be a real n-tuple such that (1) is fulfilled. Then for any real n-tuples $\boldsymbol{x}$, $\boldsymbol{y}$ and real numbers a, b, c, d satisfying (3), the identity*

$$\begin{aligned} &ac + bd - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i y_i - \left(a + b - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \right) \left(c + d - \frac{1}{W_n} \sum_{i=1}^{n} w_i y_i \right) \\ &\quad = (x_1 - a)(d - y_n) + (b - x_n)(y_1 - c) \\ &\qquad + \frac{1}{W_n} \left[\sum_{i=1}^{n-1} W_i (x_1 - a) \Delta y_i + \sum_{i=1}^{n-1} \overline{W}_{i+1} (b - x_n) \Delta y_i \right. \\ &\qquad \left. + \sum_{i=1}^{n-1} W_i \Delta x_i (y_1 - c) + \sum_{i=1}^{n-1} \overline{W}_{i+1} \Delta x_i (d - y_n) \right] \\ &\qquad + \frac{1}{W_n^2} \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} W_i \overline{W}_{j+1} \Delta x_i \Delta y_j + \sum_{j=i}^{n-1} \overline{W}_{i+1} W_j \Delta x_i \Delta y_j \right) \end{aligned} \tag{6}$$

holds, where $\Delta x_i = x_{i+1} - x_i$, $\Delta y_i = y_{i+1} - y_i$, $i = 1, \dots, n-1$.

Proof For $m = n + 2$, we define m-tuples $\boldsymbol{p}$, $\boldsymbol{\xi}$, and $\boldsymbol{\zeta}$ as

$$\begin{aligned} &p_1 = 1, && p_2 = -\frac{w_1}{W_n}, && p_3 = -\frac{w_2}{W_n}, && \dots, && p_{m-1} = -\frac{w_n}{W_n}, && p_m = 1, \\ &\xi_1 = a, && \xi_2 = x_1, && \xi_3 = x_2, && \dots, && \xi_{m-1} = x_n, && \xi_m = b, \\ &\zeta_1 = c, && \zeta_2 = y_1, && \zeta_3 = y_2, && \dots, && \zeta_{m-1} = y_n, && \zeta_m = d. \end{aligned} \tag{7}$$

Since $\boldsymbol{w}$ satisfies (1) it follows that

$$0 \le P_k = \sum_{i=1}^{k} p_i \le P_m, \quad k = 1, 2, \dots, m, P_m = 1 > 0.$$

Hence, we can apply identity (5). Its left-hand side is

$$\begin{aligned}
&\sum_{i=1}^{m} p_i \sum_{i=1}^{m} p_i \xi_i \zeta_i - \sum_{i=1}^{m} p_i \xi_i \sum_{i=1}^{m} p_i \zeta_i \\
&\quad = ac + bd - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i y_i - \left(a + b - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i\right)\left(c + d - \frac{1}{W_n} \sum_{i=1}^{n} w_i y_i\right).
\end{aligned}$$

It can be easily seen that

$$\begin{aligned}
&P_1 = 1, \qquad P_m = 1, \qquad P_{m-1} = 0, \qquad P_i = \frac{\overline{W}_i}{W_n}, \quad \text{for } i = 2, \dots, m-2, \\
&\overline{P}_1 = 1, \qquad \overline{P}_m = 1, \qquad \overline{P}_2 = 0, \qquad \overline{P}_i = \frac{W_{i-2}}{W_n}, \quad \text{for } i = 3, \dots, m-1,
\end{aligned}$$

hence, on the right-hand side of (5) we have

$$\begin{aligned}
&\sum_{i=1}^{m-1} \left(\sum_{j=1}^{i-1} \overline{P}_{i+1} P_j \Delta\xi_i \Delta\zeta_j + \sum_{j=i}^{m-1} P_i \overline{P}_{j+1} \Delta\xi_i \Delta\zeta_j \right) \\
&\quad = \sum_{i=1}^{m-1} \left(\overline{P}_{i+1} P_1 \Delta\xi_i \Delta\zeta_1 + \sum_{j=2}^{i-1} \overline{P}_{i+1} P_j \Delta\xi_i \Delta\zeta_j \right. \\
&\qquad \left. + \sum_{j=i}^{m-2} P_i \overline{P}_{j+1} \Delta\xi_i \Delta\zeta_j + P_i \overline{P}_m \Delta\xi_i \Delta\zeta_{m-1} \right) \\
&\quad = \sum_{i=1}^{m-1} \left(\overline{P}_{i+1} \Delta\xi_i (y_1 - c) + \frac{1}{W_n} \sum_{j=2}^{i-1} \overline{P}_{i+1} \overline{W}_j \Delta\xi_i \Delta y_{j-1} \right. \\
&\qquad \left. + \frac{1}{W_n} \sum_{j=i}^{m-2} P_i W_{j-1} \Delta\xi_i \Delta y_{j-1} + P_i \Delta\xi_i (d - y_n) \right).
\end{aligned}$$

Calculating separately summands for $i = 1$ and $i = m - 1$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{m-1} \left(\overline{P}_{i+1} \Delta\xi_i (y_1 - c) + \frac{1}{W_n} \sum_{j=2}^{i-1} \overline{P}_{i+1} \overline{W}_j \Delta\xi_i \Delta y_{j-1} \right. \\
&\qquad \left. + \frac{1}{W_n} \sum_{j=i}^{m-2} P_i W_{j-1} \Delta\xi_i \Delta y_{j-1} + P_i \Delta\xi_i (d - y_n) \right) \\
&\quad = \frac{1}{W_n} \sum_{j=2}^{m-2} W_{j-1} (x_1 - a) \Delta y_{j-1} + (x_1 - a)(d - y_n)
\end{aligned}$$

$$+\frac{1}{W_n}\sum_{i=2}^{m-2}W_{i-1}\Delta x_{i-1}(y_1-c)+\frac{1}{W_n^2}\sum_{i=2}^{m-2}\sum_{j=2}^{i-1}W_{i-1}\overline{W}_j\Delta x_{i-1}\Delta y_{j-1}$$
$$+\frac{1}{W_n^2}\sum_{i=2}^{m-2}\sum_{j=i}^{m-2}\overline{W}_iW_{j-1}\Delta x_{i-1}\Delta y_{j-1}+\frac{1}{W_n}\sum_{i=2}^{m-2}\overline{W}_i\Delta x_{i-1}(d-y_n)$$
$$+(b-x_n)(y_1-c)+\frac{1}{W_n}\sum_{j=2}^{m-2}\overline{W}_j(b-x_n)\Delta y_{j-1}.$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{m-1}\left(\sum_{j=1}^{i-1}\overline{P}_{i+1}P_j\Delta\xi_i\Delta\zeta_j+\sum_{j=i}^{m-1}P_i\overline{P}_{j+1}\Delta\xi_i\Delta\zeta_j\right)\\
&=(x_1-a)(d-y_n)+(b-x_n)(y_1-c)\\
&\quad+\frac{1}{W_n}\left[\sum_{j=2}^{n}W_{j-1}(x_1-a)\Delta y_{j-1}+\sum_{j=2}^{n}\overline{W}_j(b-x_n)\Delta y_{j-1}\right.\\
&\quad\left.+\sum_{i=2}^{n}W_{i-1}\Delta x_{i-1}(y_1-c)+\sum_{i=2}^{n}\overline{W}_i\Delta x_{i-1}(d-y_n)\right]\\
&\quad+\frac{1}{W_n^2}\sum_{i=2}^{n}\left(\sum_{j=2}^{i-1}W_{i-1}\overline{W}_j\Delta x_{i-1}\Delta y_{j-1}+\sum_{j=i}^{n}\overline{W}_iW_{j-1}\Delta x_{i-1}\Delta y_{j-1}\right),
\end{aligned}$$

which is equal to the right-hand side of (6). □

Using identity (6) and imposing stricter conditions than (3), we obtain refinements of inequality (2) which are more accurate than those previously established in [5].

Theorem 1 *Let $n \geq 2$ and let $\boldsymbol{w}$ be a real n-tuple such that (1) is fulfilled. Let $\boldsymbol{x}$, $\boldsymbol{y}$ be real n-tuples monotonic in the same direction. Suppose that real numbers a, b, c, d and nonnegative real numbers r, s satisfy*

$$\min_{1\le i\le n}x_i-a\ge r,\qquad b-\max_{1\le i\le n}x_i\ge r,\qquad |\Delta x_i|\ge r,\quad i=1,\dots,n-1,\tag{8}$$
$$\min_{1\le i\le n}y_i-c\ge s,\qquad d-\max_{1\le i\le n}y_i\ge s,\qquad |\Delta y_i|\ge s,\quad i=1,\dots,n-1.\tag{9}$$

Then

$$\begin{aligned}
&ac+bd-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_iy_i-\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i\right)\left(c+d-\frac{1}{W_n}\sum_{i=1}^{n}w_iy_i\right)\\
&\ge rs\left(2n+\frac{1}{W_n^2}\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1}W_i\overline{W}_{j+1}+\sum_{j=i}^{n-1}\overline{W}_{i+1}W_j\right)\right)\ge 0.
\end{aligned}\tag{10}$$

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotonic in the opposite directions, then the inequalities in (10) are reversed and the term rs appears with the negative sign.

Proof Under the given assumptions, using identity (6), we obtain

$$\begin{aligned} & ac+bd-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i y_i-\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)\left(c+d-\frac{1}{W_n}\sum_{i=1}^{n} w_i y_i\right) \\ & \geq 2rs+\frac{2rs}{W_n}\left(\sum_{i=1}^{n-1} W_i+\sum_{i=1}^{n-1}\overline{W}_{i+1}\right)+\frac{rs}{W_n^2}\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1} W_i\overline{W}_{j+1}+\sum_{j=i}^{n-1}\overline{W}_{i+1}W_j\right). \end{aligned}$$

Since

$$\sum_{i=1}^{n-1} W_i+\sum_{i=1}^{n-1}\overline{W}_{i+1}=\sum_{i=1}^{n-1} W_n=(n-1)W_n,$$

we obtain the first inequality in (10). Since r, s are nonnegative real numbers and obviously

$$2n+\frac{1}{W_n^2}\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1} W_i\overline{W}_{j+1}+\sum_{j=i}^{n-1}\overline{W}_{i+1}W_j\right)\geq 0,$$

the second inequality in (10) immediately follows. □

Using identity (6) and the triangle inequality, we can establish bounds for the Čebyšev's functional (or Čebyšev's difference) of the Mercer type in terms of the discrete Ostrowski inequality.

Throughout the rest of the paper, let $[a,b]$ and $[c,d]$ be intervals in $\mathbb{R}$, where $a<b$, $c<d$.

Theorem 2 *Let $n\geq 2$ and let $\mathbf{w}$ be a real n-tuple such that conditions (1) are fulfilled. Then for any real n-tuples $\mathbf{x}\in[a,b]^n$, $\mathbf{y}\in[c,d]^n$ the following inequalities hold*:

$$\begin{aligned} & \left|ac+bd-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i y_i-\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)\left(c+d-\frac{1}{W_n}\sum_{i=1}^{n} w_i y_i\right)\right| \\ & \leq (d-c)\left[\frac{1}{W_n}\left(\sum_{i=1}^{n} W_i|x_1-a|+\sum_{i=1}^{n}\overline{W}_i|b-x_n|\right)+\sum_{i=1}^{n-1}|\Delta x_i|\right. \\ & \quad\left.+\frac{1}{W_n^2}\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1} W_i\overline{W}_{j+1}+\sum_{j=i}^{n-1}\overline{W}_{i+1}W_j\right)|\Delta x_i|\right] \\ & \leq (b-a)(d-c)\left[2n+\frac{1}{W_n^2}\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1} W_i\overline{W}_{j+1}+\sum_{j=i}^{n-1}\overline{W}_{i+1}W_j\right)\right]. \end{aligned} \tag{11}$$

Proof Using identity (6) and the triangle inequality, we have

$$\begin{aligned} & \left|ac+bd-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i y_i-\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)\left(c+d-\frac{1}{W_n}\sum_{i=1}^{n} w_i y_i\right)\right| \\ & \leq |x_1-a||d-y_n|+|b-x_n||y_1-c| \\ & \quad+\frac{1}{W_n}\left[\sum_{i=1}^{n-1} W_i|x_1-a||\Delta y_i|+\sum_{i=1}^{n-1}\overline{W}_{i+1}|b-x_n||\Delta y_i|\right. \end{aligned}$$

$$+\sum_{i=1}^{n-1} W_i|\Delta x_i||y_1-c| + \sum_{i=1}^{n-1}\overline{W}_{i+1}|\Delta x_i||d-y_n|\Bigg]$$
$$+\frac{1}{W_n^2}\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1} W_i\overline{W}_{j+1}|\Delta x_i||\Delta y_j| + \sum_{j=i}^{n-1}\overline{W}_{i+1}W_j|\Delta x_i||\Delta y_j|\right),$$

because W_i and $\overline{W}_i$ are nonnegative for all $i=1,\dots,n$. Since $|y_1-c|$, $|d-y_n|$, $|\Delta y_i|$ for all $i=1,\dots,n$, are less or equal to $d-c$, and $|x_1-a|$, $|b-x_n|$, $|\Delta x_i|$ for all $i=1,\dots,n$, are less or equal to $b-a$, we obtain inequalities (11). □

Remark 1 If in Theorem 2 we add assumption that R, S are nonnegative real numbers such that

$$|x_1-a|\le R,\qquad |b-x_n|\le R,\qquad |\Delta x_i|\le R,\quad i=1,\dots,n-1, \tag{12}$$
$$|y_1-c|\le S,\qquad |d-y_n|\le S,\qquad |\Delta y_i|\le S,\quad i=1,\dots,n-1, \tag{13}$$

then we obtain refinements of the two inequalities proved in [5] under the same assumption. Namely, we have inequalities

$$\begin{aligned}
&\left|ac+bd-\frac{1}{W_n}\sum_{i=1}^n w_i x_i y_i - \left(a+b-\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right)\left(c+d-\frac{1}{W_n}\sum_{i=1}^n w_i y_i\right)\right| \\
&\le S\Bigg[\frac{1}{W_n}\left(\sum_{i=1}^n W_i|x_1-a| + \sum_{i=1}^n \overline{W}_i|b-x_n|\right) + \sum_{i=1}^{n-1}|\Delta x_i| \\
&\quad + \frac{1}{W_n^2}\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1}W_i\overline{W}_{j+1} + \sum_{j=i}^{n-1}\overline{W}_{i+1}W_j\right)|\Delta x_i|\Bigg] \\
&\le RS\left[2n + \frac{1}{W_n^2}\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1}W_i\overline{W}_{j+1} + \sum_{j=i}^{n-1}\overline{W}_{i+1}W_j\right)\right]
\end{aligned} \tag{14}$$

and, as a special case when $w_i=1$ $(i=1,\dots,n)$, we have inequalities

$$\begin{aligned}
&\left|ac+bd-\frac{1}{n}\sum_{i=1}^n x_i y_i - \left(a+b-\frac{1}{n}\sum_{i=1}^n x_i\right)\left(c+d-\frac{1}{n}\sum_{i=1}^n y_i\right)\right| \\
&\le S\left(\frac{n+1}{2}\left(|x_1-a| + \sum_{i=1}^{n-1}|\Delta x_i| + |b-x_n|\right) - \frac{1}{2n}\sum_{i=1}^{n-1} i(n-i)|\Delta x_i|\right) \\
&\le RS\frac{(n+1)(5n+7)}{12}.
\end{aligned}$$

3 Bounds for the Jensen–Mercer functional

Jensen–Mercer inequality

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) \le f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^n w_i f(x_i), \tag{15}$$

for a convex function $f:(\alpha,\beta)\to\mathbb{R}$, real n-tuple $\boldsymbol{x}\in[a,b]^n$, and positive real n-tuple $\boldsymbol{w}$, where $-\infty\le\alpha<a<b<\beta\le\infty$, was proved in [6]. In [1], it was proved that it remains valid when $\boldsymbol{x}$ is monotonic and $\boldsymbol{w}$ satisfies conditions (1).

Using our results from the previous section, we establish Ostrowski-like bounds for the Jensen–Mercer functional, i.e., the difference of the right- and left-hand sides of inequality (15).

Theorem 3 *Let $f:(\alpha,\beta)\to\mathbb{R}$ be a differentiable function and suppose that γ,δ are real numbers such that $\gamma\le f'(x)\le\delta$, for all $x\in(\alpha,\beta)$. Let $n\ge 2$ and suppose that n-tuple $\boldsymbol{x}\in[a,b]^n$, where $-\infty\le\alpha<a<b<\beta\le\infty$, satisfies conditions (12). Let $\boldsymbol{w}$ be a real n-tuple such that conditions (1) are fulfilled and $a+b-\frac{1}{W_n}\sum_{i=1}^n w_ix_i\in[a,b]$. Then*

$$\begin{aligned}
&\left|f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^n w_if(x_i)-f\left(a+b-\frac{1}{W_n}\sum_{i=1}^n w_ix_i\right)\right|\\
&\quad\le(\delta-\gamma)\left[\frac{1}{W_n}\left(\sum_{i=1}^n W_i|x_1-a|+\sum_{i=1}^n \overline{W}_i|b-x_n|\right)+\sum_{i=1}^{n-1}|\Delta x_i|\right.\\
&\qquad\left.+\frac{1}{W_n^2}\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1}W_i\overline{W}_{j+1}+\sum_{j=i}^{n-1}\overline{W}_{i+1}W_j\right)|\Delta x_i|\right]\\
&\quad\le R(\delta-\gamma)\left[2n+\frac{1}{W_n^2}\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1}W_i\overline{W}_{j+1}+\sum_{j=i}^{n-1}\overline{W}_{i+1}W_j\right)\right].
\end{aligned}\tag{16}$$

Proof By the mean-value theorem, for any $\zeta,\eta\in(\alpha,\beta)$, there exists some ξ between them such that $f(\zeta)-f(\eta)=f'(\xi)(\zeta-\eta)$. Hence, choosing $\zeta=x_i$ and $\eta=a+b-\frac{1}{W_n}\sum_{i=1}^n w_ix_i$, we obtain

$$f(x_i)-f\left(a+b-\frac{1}{W_n}\sum_{i=1}^n w_ix_i\right)=f'(\xi_i)\left(x_i-(a+b)+\frac{1}{W_n}\sum_{i=1}^n w_ix_i\right).\tag{17}$$

Multiplying (17) by $-\frac{w_i}{W_n}$, and then summing over i, we have

$$\begin{aligned}
&-\frac{1}{W_n}\sum_{i=1}^n w_if(x_i)+f\left(a+b-\frac{1}{W_n}\sum_{i=1}^n w_ix_i\right)\\
&\quad=-\frac{1}{W_n}\sum_{i=1}^n w_ix_if'(\xi_i)+\frac{1}{W_n}(a+b)\sum_{i=1}^n w_if'(\xi_i)-\frac{1}{W_n^2}\sum_{i=1}^n w_ix_i\sum_{i=1}^n w_if'(\xi_i).
\end{aligned}$$

Choosing $\zeta=a$, $\zeta=b$, respectively, and $\eta=a+b-\frac{1}{W_n}\sum_{i=1}^n w_ix_i$, we have

$$\begin{aligned}
&f(a)-f\left(a+b-\frac{1}{W_n}\sum_{i=1}^n w_ix_i\right)=f'(\xi_a)\left(\frac{1}{W_n}\sum_{i=1}^n w_ix_i-b\right),\\
&f(b)-f\left(a+b-\frac{1}{W_n}\sum_{i=1}^n w_ix_i\right)=f'(\xi_b)\left(\frac{1}{W_n}\sum_{i=1}^n w_ix_i-a\right).
\end{aligned}$$

Summing the above three equalities, we obtain

$$\begin{aligned} & f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i)-f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \\ & \quad = -\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i f'(\xi_i)+(a+b)\frac{1}{W_n}\sum_{i=1}^{n} w_i f'(\xi_i)-\frac{1}{W_n^2}\sum_{i=1}^{n} w_i x_i \sum_{i=1}^{n} w_i f'(\xi_i) \\ & \qquad + f'(\xi_a)\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i - b\right)+f'(\xi_b)\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i - a\right) \\ & \quad = af'(\xi_a)+bf'(\xi_b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i f'(\xi_i) \\ & \qquad -\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)\left(f'(\xi_a)+f'(\xi_b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i f'(\xi_i)\right). \end{aligned}$$

Since $\gamma \leq f'(x) \leq \delta$, for all $x \in (\alpha, \beta)$, it holds

$$\begin{aligned} & \left|f'(\xi_1)-f'(\xi_a)\right| \leq \delta-\gamma, \qquad \left|f'(\xi_b)-f'(\xi_n)\right| \leq \delta-\gamma, \\ & \left|\Delta f'(\xi_i)\right| \leq \delta-\gamma, \quad i=1,\ldots,n-1, \end{aligned}$$

and inequalities (16) immediately follow from Theorem 2 and Remark 1. □

Remark 2 An integral variant of identity (5) can be found in [9] and there is a way to obtain integral variants in terms of Riemann–Stieltjes integral of the Jensen–Mercer inequality from the Jensen–Steffensen inequality (see, for example, [2–4]). Hence, our discrete results can be extended to the continuous case.

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Authors' contributions
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