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A study on Copson operator and its associated sequence space II

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Dedicated to Prof. Maryam Mirzakhani who, in spite of a short lifetime, left a long-standing impact on mathematics.

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Abstract

In this paper, we investigate some properties of the domains $c(C^n)$, $c_0(C^n)$, and $\ell_p(C^n)$ ($0 < p < 1$) of the Copson matrix of order n , where c , c_0 , and ℓ_p are the spaces of all convergent, convergent to zero, and p -summable real sequences, respectively. Moreover, we compute the Köthe duals of these spaces and the lower bound of well-known operators on these sequence spaces. The domain $\ell_p(C^n)$ of Copson matrix C^n of order n in the sequence space ℓ_p , the norm of operators on this space, and the norm of Copson operator on several matrix domains have been investigated recently in (Roopaei in *J. Inequal. Appl.* 2020:120, 2020), and the present study is a complement of our previous research.

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1 Introduction

Let ω denote the set of all real-valued sequences. Any linear subspace of ω is called a sequence space. For $0 < p < 1$, the complete p -normed space ℓ_p is the set of all real sequences $x = (x_k)_{k=0}^{\infty} \in \omega$ such that

$$\|x\|_{\ell_p} = \sum_{k=0}^{\infty} |x_k|^p < \infty.$$

By c and c_0 , we denote the spaces of all convergent and convergent to zero real sequences, respectively. These spaces are Banach spaces with the norm $\|x\|_{\infty} = \sup_k |x_k|$. Here and in the rest of the paper, the supremum is taken over all $k \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$. Also, we use the notion $\mathbb{N} = \{1, 2, 3, \dots\}$.

One can consider an infinite matrix as a linear operator from a sequence space to another one. Given any two arbitrary sequence spaces X , Y and an infinite matrix $T = (t_{ij})$, we define a matrix transformation from X into Y as $Tx = ((Tx)_i) = (\sum_{j=0}^{\infty} t_{ij}x_j)$ provided that the series is convergent for each $i \in \mathbb{N}_0$. By (X, Y) , we denote the family of all infinite matrices from X into Y .

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The domain X_T of an infinite matrix T in a sequence space X is defined as

$$X_T = \{x \in \omega : Tx \in X\}, \quad (1.1)$$

which is also a sequence space. By using matrix domains of special triangle matrices in classical spaces, many authors have introduced and studied new Banach spaces. For the relevant literature, we refer to the papers [1, 2, 4, 9, 13, 16, 22, 23, 26–29] and textbooks [3, 20, 21].

The Köthe duals (α -, β -, γ -duals) of a sequence space X are defined by

$$\begin{aligned} X^\alpha &= \left\{ a = (a_k) \in \omega : \sum_{k=1}^{\infty} |a_k x_k| < \infty \text{ for all } x = (x_k) \in X \right\}, \\ X^\beta &= \left\{ a = (a_k) \in \omega : \left(\sum_{k=1}^n a_k x_k \right) \in c \text{ for all } x = (x_k) \in X \right\}, \\ X^\gamma &= \left\{ a = (a_k) \in \omega : \left(\sum_{k=1}^n a_k x_k \right) \in \ell_\infty \text{ for all } x = (x_k) \in X \right\}, \end{aligned}$$

respectively.

Copson matrix. The Copson matrix is an upper-triangular matrix which is defined by

$$c_{j,k} = \begin{cases} \frac{1}{k+1}, & 0 \leq j \leq k, \\ 0, & \text{otherwise,} \end{cases}$$

for all $j, k \in \mathbb{N}$. That is,

$$C = \begin{pmatrix} 1 & 1/2 & 1/3 & \dots \\ 0 & 1/2 & 1/3 & \dots \\ 0 & 0 & 1/3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and has the ℓ_p -norm $\|C\|_{\ell_p} = p$. This matrix is the transpose of the well-known Cesàro matrix.

Copson matrix of order n . Consider the Hausdorff matrix $H^\mu = (h_{j,k})_{j,k=0}^\infty$, with entries of the form:

$$h_{j,k} = \begin{cases} \int_0^1 \binom{j}{k} \theta^k (1-\theta)^{j-k} d\mu(\theta), & j \geq k, \\ 0, & j < k, \end{cases}$$

where μ is a probability measure on $[0, 1]$. The Hausdorff matrix contains the famous classes of matrices. For positive integer n , these classes are as follow:

- The choice $d\mu(\theta) = n(1-\theta)^{n-1} d\theta$ gives the Cesàro matrix of order n ,
- The choice $d\mu(\theta) = n\theta^{n-1} d\theta$ gives the Gamma matrix of order n ,
- The choice $d\mu(\theta) = \frac{|\log \theta|^{n-1}}{\Gamma(n)} d\theta$ gives the Hölder matrix of order n ,
- The choice $d\mu(\theta) = \text{point evaluation at } \theta = r$ ($0 < r < 1$) gives the Euler matrix of order r .

We use the notation $hau(p)$ as the set of all sequences whose H^μ -transforms are in the space ℓ_p , that is,

$$hau(p) = \left\{ x = (x_j) \in \omega : \sum_{j=0}^{\infty} \left| \sum_{k=0}^j \int_0^1 \binom{j}{k} \theta^k (1-\theta)^{j-k} d\mu(\theta) x_k \right|^p < \infty \right\},$$

where μ is a fixed probability measure on $[0, 1]$.

Hardy's formula [11, Theorem 216] states that the Hausdorff matrix is a bounded operator on ℓ_p if and only if $\int_0^1 \theta^{\frac{1}{p}-1} d\mu(\theta) < \infty$ and

$$\|H^\mu\|_{\ell_p} = \int_0^1 \theta^{\frac{1}{p}-1} d\mu(\theta) \quad (1 < p < \infty). \quad (1.2)$$

Hausdorff operator has the following norm property.

Theorem 1.1 ([7, Theorem 9]) *Let $p \geq 1$ and H^μ , H^φ , and H^ν be Hausdorff matrices such that $H^\mu = H^\varphi H^\nu$. Then H^μ is bounded on ℓ_p if and only if both H^φ and H^ν are bounded on ℓ_p . Moreover, we have*

$$\|H^\mu\|_{\ell_p} = \|H^\varphi\|_{\ell_p} \|H^\nu\|_{\ell_p}.$$

The following theorem is an analog of Hardy's formula.

Theorem 1.2 ([8, Theorem 7.18]) *Fix p , $0 < p < 1$, and let $H^{\mu t}$ be the transposed Hausdorff matrix. Then*

$$\|H^{\mu t} x\|_{\ell_p} \geq \left(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta) \right) \|x\|_{\ell_p} \quad (1.3)$$

for every sequence x of nonnegative terms. The constant is best possible, and there is equality only when $x = 0$ or $p = 1$ or $H = I$.

Theorem 1.3 ([8, Corollary 7.27]) *If $H^{\mu t}$ and $H^{\nu t}$ are two transposed Hausdorff matrices, then the lower bound (on ℓ_p , $0 < p < 1$) of their product is the product of their lower bounds.*

In order to define and know the Copson matrix details, we need the following theorem also known as *Hellinger–Toeplitz* theorem.

Theorem 1.4 ([8, Proposition 7.2]) *Suppose that $1 < p, q < \infty$. A matrix A maps ℓ_p into ℓ_q if and only if the transposed matrix, A^t , maps ℓ_{q^*} into ℓ_{p^*} . We then have*

$$\|A\|_{\ell_p, \ell_q} = \|A^t\|_{\ell_{q^*}, \ell_{p^*}},$$

where p^* is the conjugate of p , i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$.

For a nonnegative real number n , and by choosing $d\mu(\theta) = n(1-\theta)^{n-1} d\theta$ in the definition of Hausdorff matrix, we gain the Cesàro matrix of order n , which, according to Hardy's

formula, has the ℓ_p -norm

$$\frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)}.$$

Now, the Copson matrix of order n , $C^n = (c_{j,k}^n)$, which is defined as the transpose of Cesàro matrix of order n , has the entries

$$c_{j,k}^n = \begin{cases} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}}, & j \leq k, \\ 0, & \text{otherwise,} \end{cases} \quad (1.4)$$

and, according to Hellinger–Toeplitz theorem, the ℓ_p -norm

$$\|C^n\|_{\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p)}{\Gamma(n+1/p)}.$$

Note that $C^0 = I$, where I is the identity matrix and $C^1 = C$ is the well-known Copson matrix. For more examples,

$$C^2 = \begin{pmatrix} 1 & 2/3 & 3/6 & \dots \\ 0 & 1/3 & 2/6 & \dots \\ 0 & 0 & 1/6 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad C^3 = \begin{pmatrix} 1 & 3/4 & 6/10 & \dots \\ 0 & 1/4 & 3/10 & \dots \\ 0 & 0 & 1/10 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In this study, after introducing the domains of the Copson matrix of order n in the spaces c_0 , c , and ℓ_p , we study some properties of the Copson spaces. We also compute α -, β -, γ -duals of these spaces and determine Schauder basis. We seek lower bounds of the form

$$\|Tx\|_{\ell_p} > L\|x\|_{\ell_p},$$

valid for every $x \in \ell_p$ with $x_0 > x_1 > \dots > 0$. Here T is a matrix with nonnegative entries, assumed to map ℓ_p into itself, and L is a constant not depending on x . The lower bound of T is the greatest possible value of L , which we denote by $L(T)$.

Throughout this paper, we use the notations $L(\cdot)$ for the lower bound of operators on ℓ_p and $L(\cdot)_{X,Y}$ for the lower bound of operators from the sequence space X into the sequence space Y .

Motivation. Many mathematicians have and still publish numerous articles about the Cesàro matrix, Cesàro matrix domain, and Cesàro function spaces [8, 9, 14, 19, 29], while the importance of the Copson operator and its associated matrix domains have been ignored under the shadow of its transpose Cesàro matrix. Recently, the author have investigated the sequence space $\ell_p(C^n)$ for $1 \leq p < \infty$, as well as found the norm of well-known operators on this matrix domain. In this research, as a complement of [24], the matrix domains $c_0(C^n)$, $c(C^n)$, and $\ell_p(C^n)$, $0 < p < 1$, are investigated, while the lower bound of well-known operators on the Copson sequence space and the lower bound of the Copson operator on some matrix domains are computed as well, which has never been done before.

2 Copson Banach spaces $c_0(C^n)$, $c(C^n)$, and $\ell_p(C^n)$

In this section, the sequence spaces $c_0(C^n)$, $c(C^n)$, and $\ell_p(C^n)$ are introduced and the inclusion relations as well as dual spaces of these new spaces are determined.

Lemma 2.1 *The Copson matrix of order n , C^n , is invertible and its inverse, $C^{-n} = (c_{j,k}^{-n})$, is defined by*

$$c_{j,k}^{-n} = \begin{cases} (-1)^{(k-j)} \binom{n}{k-j} \binom{n+j}{j}, & j \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof Let us recall the forward difference matrix of order n , $\Delta^n = (\delta_{j,k}^n)$, which is a lower-triangular matrix with entries

$$\delta_{j,k}^n = \begin{cases} (-1)^{(k-j)} \binom{n}{k-j}, & j \leq k \leq j+n, \\ 0, & \text{otherwise.} \end{cases}$$

This matrix has the inverse $\Delta^{-n} = (\delta_{j,k}^{-n})$ with the following entries:

$$\delta_{j,k}^{-n} = \begin{cases} \binom{n+k-j-1}{k-j}, & j \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

From the relation (1.4), one can see that the Copson matrix of order n and its inverse can be rewritten based on the forward difference operator and its inverse. For $j \leq k$, we have

$$c_{j,k}^n = \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} = \frac{\delta_{j,k}^{-n}}{\binom{n+k}{k}} \quad \text{and} \quad c_{j,k}^{-n} = \delta_{j,k}^n \binom{n+j}{j}.$$

Now, by a simple calculation, we deduce that

$$(C^{-n}C^n)_{i,k} = \frac{\binom{n+i}{i}}{\binom{n+k}{k}} \sum_j \delta_{i,j}^n \delta_{j,k}^{-n} = \frac{\binom{n+i}{i}}{\binom{n+k}{k}} (\Delta^n \Delta^{-n})_{i,k} = \frac{\binom{n+i}{i}}{\binom{n+k}{k}} I_{i,k},$$

which completes the proof. \square

Now, we introduce the sequence spaces $c_0(C^n)$, $c(C^n)$, and $\ell_p(C^n)$ as the set of all sequences whose C^n -transforms are in the spaces c_0 , c , and ℓ_p , respectively, that is,

$$c_0(C^n) = \left\{ x = (x_j) \in \omega : \lim_{j \rightarrow \infty} \sum_{k=j}^{\infty} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} x_k = 0 \right\},$$

$$c(C^n) = \left\{ x = (x_j) \in \omega : \lim_{j \rightarrow \infty} \sum_{k=j}^{\infty} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} x_k < \infty \right\},$$

and

$$\ell_p(C^n) = \left\{ x = (x_j) \in \omega : \sum_{j=0}^{\infty} \left| \sum_{k=j}^{\infty} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} x_k \right|^p < \infty \right\} \quad (0 < p < 1).$$

With the notation of (1.1), the spaces $c_0(C^n)$, $c(C^n)$, and $\ell_p(C^n)$ can be redefined as follows:

$$c_0(C^n) = (c_0)_{C^n}, \quad c(C^n) = (c)_{C^n}, \quad \text{and} \quad \ell_p(C^n) = (\ell_p)_{C^n}.$$

Throughout the study, $y = (y_j)$ will be the C^n -transform of a sequence $x = (x_j)$, that is,

$$y_j = (C^n x)_j = \sum_{k=j}^{\infty} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} x_k \quad (2.1)$$

for all $j \in \mathbb{N}_0$. Also, the relation

$$x_k = \sum_{i=k}^{\infty} (-1)^{i-k} \binom{n+k}{k} \binom{n}{i-k} y_i$$

holds for all $k \in \mathbb{N}_0$.

Theorem 2.2 *The following statements hold:*

- The spaces $c_0(C^n)$ and $c(C^n)$ are Banach spaces with the norm

$$\|x\|_{c_0(C^n)} = \|x\|_{c(C^n)} = \sup_{j \in \mathbb{N}_0} \left| \sum_{k=j}^{\infty} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} x_k \right|.$$

- The space $\ell_p(C^n)$ ($0 < p < 1$) is a complete p -normed space with the p -norm

$$\|x\|_{\ell_p(C^n)} = \sum_{j=0}^{\infty} \left| \sum_{k=j}^{\infty} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} x_k \right|^p$$

Proof We omit the proof which is a routine verification. □

Theorem 2.3 *The following statements hold:*

- The spaces $c_0(C^n)$ and $c(C^n)$ are linearly norm-isomorphic to c_0 and c , respectively.
- The space $\ell_p(C^n)$ is linearly p -norm isomorphic to ℓ_p .

Proof The proof follows from the fact that the mapping $L : X(C^n) \rightarrow X$ defined by $x \mapsto Lx = y = C^n x$ is a norm-preserving linear bijection, where $X \in \{c_0, c, \ell_p\}$ and $y = (y_j)$ is given by (2.1). □

Theorem 2.4 *The inclusion $\ell_q(C^n) \subset \ell_p(C^n)$ strictly holds, where $0 < p < q < 1$.*

Proof Choose any $x \in \ell_q(C^n)$. Then, $C^n x \in \ell_q$. Since the inclusion $\ell_q \subset \ell_p$ holds for $0 < p < q < 1$, we have $C^n x \in \ell_p$. This implies that $x \in \ell_p(C^n)$. Hence, we conclude that the inclusion $\ell_q(C^n) \subset \ell_p(C^n)$ holds.

Now, we show that the inclusion is strict. Since the inclusion $\ell_q \subset \ell_p$ is strict, we can choose $y = (y_j) \in \ell_p \setminus \ell_q$. Define a sequence $x = (x_j)$ as

$$x_j = \sum_{k=j}^{\infty} (-1)^{k-j} \binom{n+j}{j} \binom{n}{k-j} y_k \quad (j \in \mathbb{N}_0).$$

Then, we have

$$(C^n x)_j = y_j$$

for every $j \in \mathbb{N}_0$, which means $C^n x = y$, and so $C^n x \in \ell_p \setminus \ell_q$. Hence, we conclude that $x \in \ell_p(C^n) \setminus \ell_q(C^n)$, and so the inclusion $\ell_q(C^n) \subset \ell_p(C^n)$ is strict. \square

Corollary 2.5 *The inclusion $\ell_p(C^\alpha) \subset \ell_p(C^\beta)$ holds, where $0 < p < 1$ and $\alpha > \beta \geq 0$.*

Proof According to [8, Theorem 20.13], for $0 < p < 1$ and $\alpha > \beta \geq 0$, we have

$$\|C^\beta x\|_{\ell_p} \leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1/p)}{\Gamma(\beta+1)\Gamma(\alpha+1/p)} \|C^\alpha x\|_{\ell_p}.$$

Hence there remains nothing to prove. \square

It is known from Theorem 2.3 of Jarrah and Malkowsky [15] that if T is triangular then the domain X_T of T in a normed sequence space X has a basis if and only if X has a basis. As a direct consequence of this fact, we have

Corollary 2.6 *Define the sequence $(b^{(k)}) = (b_j^{(k)})$ for each $k \in \mathbb{N}$ by*

$$(b^{(k)})_j = \begin{cases} (-1)^{k-j} \binom{n+j}{j} \binom{n}{k-j}, & k \geq j, \\ 0, & k < j \end{cases} \quad (j \in \mathbb{N}_0).$$

Then, the sequence $(b^{(k)})$ is a basis for the spaces $c_0(C^n)$ and $\ell_p(C^n)$, and every sequence $x \in c_0(C^n)$ or $x \in \ell_p(C^n)$ has a unique representation of the form $x = \sum_k (C^n x)_k b^{(k)}$.

The following lemma is essential to determine the dual spaces. Throughout the paper, \mathcal{N} is the collection of all finite subsets of \mathbb{N} .

Lemma 2.7 ([30]) *The following statements hold:*

(i) $T = (t_{j,k}) \in (c_0, \ell_1) = (c, \ell_1)$ if and only if

$$\sup_{K \in \mathcal{N}} \sum_{j=0}^{\infty} \left| \sum_{k \in K} t_{j,k} \right| < \infty.$$

(ii) $T = (t_{j,k}) \in (c_0, c)$ if and only if

$$\sup_j \sum_{k=0}^{\infty} |t_{j,k}| < \infty, \quad (2.2)$$

and

$$\lim_{j \rightarrow \infty} t_{j,k} \text{ exists for each } k \in \mathbb{N}. \quad (2.3)$$

(iii) $T = (t_{j,k}) \in (c, c)$ if and only if (2.2) and (2.3) hold and

$$\lim_{j \rightarrow \infty} \sum_{k=0}^{\infty} t_{j,k} \text{ exists.}$$

(iv) $T = (t_{j,k}) \in (c_0, \ell_{\infty}) = (c, \ell_{\infty})$ if and only if (2.2) holds.

Lemma 2.8 *The following statements hold:*

(i) [10, Theorem 5.1.0, with $p_k = p$ for all k] $T = (t_{j,k}) \in (\ell_p, \ell_1)$ if and only if

$$\sup_{N \in \mathcal{N}} \sup_{k \in \mathbb{N}} \left| \sum_{j \in N} t_{j,k} \right|^p < \infty \quad (0 < p \leq 1); \quad (2.4)$$

$$\sup_{N \in \mathcal{N}} \sum_{k=0}^{\infty} \left| \sum_{j \in N} t_{j,k} \right|^{p^*} < \infty \quad (1 < p < \infty). \quad (2.5)$$

(ii) [18, Theorem 1(i), with $p_k = p$ for all k] $T = (t_{j,k}) \in (\ell_p, \ell_{\infty})$ if and only if

$$\sup_{j,k \in \mathbb{N}_0} |t_{j,k}|^p < \infty, \quad (0 < p \leq 1). \quad (2.6)$$

(iii) [18, Corollary of Theorem 1, with $p_k = p$ for all k] $T = (t_{j,k}) \in (\ell_p, c)$ if and only if (2.6) holds and

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{j \rightarrow \infty} t_{j,k} = \alpha_k \quad \text{for each } k \in \mathbb{N}. \quad (2.7)$$

Theorem 2.9 *The α -duals of the spaces $c_0(C^n)$ and $c(C^n)$ are as follows:*

$$(c_0(C^n))^{\alpha} = (c(C^n))^{\alpha} = \left\{ b = (b_j) \in \omega : \sup_{K \in \mathcal{N}} \sum_{j=0}^{\infty} \left| \sum_{k \in K} (-1)^{k-j} \binom{n+j}{j} \binom{n}{k-j} b_j \right| < \infty \right\}.$$

Proof Let $b = (b_j) \in \omega$. Consider the matrix $A = (a_{j,k})$ defined by

$$a_{j,k} = \begin{cases} (-1)^{k-j} \binom{n+j}{j} \binom{n}{k-j} b_j, & j \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

Given any $x = (x_j) \in X(C^n)$, we have $b_j x_j = (Ay)_j$ for all $j \in \mathbb{N}$, where $X \in \{c_0, c\}$. This implies that $b x \in \ell_1$ with $x \in X(C^n)$ if and only if $Ay \in \ell_1$ with $y \in X$. Hence, we conclude that $b \in (X(C^n))^{\alpha}$ if and only if $A \in (X, \ell_1)$. This completes the proof by part (i) of Lemma 2.7. \square

Theorem 2.10 *Let define the following sets:*

$$A_1 = \left\{ b = (b_k) \in \omega : \sup_j \left\{ \sum_{k=0}^j \left| \sum_{i=0}^k (-1)^{k-i} \binom{n+i}{i} \binom{n}{k-i} b_i \right| + \sum_{k=j+1}^{\infty} \left| \sum_{i=0}^j (-1)^{j-i} \binom{n+i}{i} \binom{n}{j-i} b_i \right| \right\} < \infty \right\},$$

$$A_2 = \left\{ b = (b_k) \in \omega : \lim_{j \rightarrow \infty} \left(\sum_{k=0}^j \left| \sum_{i=0}^k (-1)^{k-i} \binom{n+i}{i} \binom{n}{k-i} b_i \right| + \sum_{k=j+1}^{\infty} \left| \sum_{i=0}^j (-1)^{j-i} \binom{n+i}{i} \binom{n}{j-i} b_i \right| \right) < \infty \right\},$$

and

$$A_3 = \left\{ b = (b_k) \in \omega : \sup_j \left| \sum_{i=0}^j (-1)^{j-i} \binom{n+i}{i} \binom{n}{j-i} b_i \right|^p < \infty \right\}.$$

Then, we have $(c_0(C^n))^\beta = A_1$, $(c(C^n))^\beta = A_1 \cap A_2$, and $(\ell_p(C^n))^\beta = A_3$ ($0 < p < 1$).

Proof Note that $b = (b_k) \in (c_0(C^n))^\beta$ if and only if the series $\sum_{k=0}^{\infty} b_k x_k$ is convergent for all $x = (x_k) \in c_0(C^n)$. The equality

$$\begin{aligned} \sum_{k=0}^j b_k x_k &= \sum_{k=0}^j b_k \left(\sum_{i=k}^{\infty} (-1)^{i-k} \binom{n+k}{k} \binom{n}{i-k} y_i \right) \\ &= \sum_{k=0}^j \left(\sum_{i=0}^k (-1)^{k-i} \binom{n+i}{i} \binom{n}{k-i} b_i \right) y_k \\ &\quad + \sum_{k=j+1}^{\infty} \left(\sum_{i=0}^j (-1)^{j-i} \binom{n+i}{i} \binom{n}{j-i} b_i \right) y_k \\ &= (By)_j, \end{aligned}$$

where $B = (b_{j,k})$ is defined by

$$b_{j,k} = \begin{cases} \sum_{i=0}^k (-1)^{k-i} \binom{n+i}{i} \binom{n}{k-i} b_i, & 0 \leq k \leq j, \\ \sum_{i=0}^j (-1)^{j-i} \binom{n+i}{i} \binom{n}{j-i} b_i, & k > j. \end{cases}$$

We deduce that $b = (b_k) \in (c_0(C^n))^\beta$ if and only if the matrix $B = (b_{j,k})$ is in the class (c_0, c) . Hence, we deduce from part (ii) of Lemma 2.7 that

$$\sup_j \left\{ \sum_{k=0}^j \left| \sum_{i=0}^k (-1)^{k-i} \binom{n+i}{i} \binom{n}{k-i} b_i \right| + \sum_{k=j+1}^{\infty} \left| \sum_{i=0}^j (-1)^{j-i} \binom{n+i}{i} \binom{n}{j-i} b_i \right| \right\} < \infty$$

and also

$$\lim_{j \rightarrow \infty} \sum_{i=0}^k (-1)^{k-i} \binom{n+i}{i} \binom{n}{k-i} b_i \quad \text{exists for each } k \in \mathbb{N},$$

which means $b = (b_k) \in A_1$, and so we have $(c_0(C^n))^\beta = A_1$. The other results can be proved similarly. \square

Theorem 2.11 *The γ -duals of the spaces $c_0(C^n)$, $c(C^n)$, and $\ell_p(C^n)$ ($0 < p < 1$) are as follows:*

$$\begin{aligned} c_0(C^n)^\gamma &= (c(C^n))^\gamma \\ &= \left\{ b = (b_k) \in \omega : \sup_j \left\{ \sum_{k=0}^j \left| \sum_{i=0}^k (-1)^{k-i} \binom{n+i}{i} \binom{n}{k-i} b_i \right| \right. \right. \\ &\quad \left. \left. + \sum_{k=j+1}^{\infty} \left| \sum_{i=0}^j (-1)^{j-i} \binom{n+i}{i} \binom{n}{j-i} b_i \right| \right\} < \infty \right\} \end{aligned}$$

and

$$(\ell_p(C^n))^\gamma = \left\{ b = (b_k) \in \omega : \sup_j \left| \sum_{i=0}^j (-1)^{j-i} \binom{n+i}{i} \binom{n}{j-i} b_i \right|^p < \infty \right\}.$$

Proof This follows by applying the same technique used in the proof of Theorem 2.10. \square

3 Lower bound of operators on the Copson matrix domain for ($0 < p < 1$)

In this section, we assume $0 < p < 1$ and intend to compute the lower bound of operators from ℓ_p into $\ell_p(C^n)$, from $\ell_p(C^n)$ into ℓ_p , and from $\ell_p(C^n)$ into itself. In so doing, we need the following lemma.

We emphasize again that we use the notations $L(\cdot)$ for the lower bound of operators on ℓ_p and $L(\cdot)_{X,Y}$ for the lower bound of operators from the sequence space X into the sequence space Y .

Lemma 3.1 ([25, Lemma 2.1]) *Let U is a bounded operator on ℓ_p and A_p and B_p be two matrix domains such that $A_p \simeq \ell_p$. Then, the following statements hold:*

- *If BT is a bounded operator on ℓ_p , then T is a bounded operator from ℓ_p into B_p and*

$$\|T\|_{\ell_p, B_p} = \|T\|_{\ell_p} \quad \text{and} \quad L(T)_{\ell_p, B_p} = L(BT).$$

- *If T has a factorization of the form $T = UA$, then T is a bounded operator from the matrix domain A_p into ℓ_p and*

$$\|T\|_{A_p, \ell_p} = \|U\|_{\ell_p} \quad \text{and} \quad L(T)_{A_p, \ell_p} = L(U).$$

- *If $BT = UA$, then T is a bounded operator from the matrix domain A_p into B_p and*

$$\|T\|_{A_p, B_p} = \|U\|_{\ell_p} \quad \text{and} \quad L(T)_{A_p, B_p} = L(U).$$

In particular, if $AT = UA$, then T is a bounded operator from the matrix domain A_p into itself and $\|T\|_{A_p} = \|U\|_{\ell_p}$ and $L(T)_{A_p} = L(U)$. Also, if T and A commute then $\|T\|_{A_p} = \|T\|_{\ell_p}$ and $L(T)_{A_p} = L(T)$.

3.1 Lower bound of operators from ℓ_p into $\ell_p(C^n)$

In this part of study we intend to compute the lower bound of transposed Hausdorff operators on the Copson matrix domain.

Theorem 3.2 *The transposed Hausdorff matrix $H^{\mu t}$ has a lower bound from ℓ_p into $\ell_p(C^n)$ and*

$$L(H^{\mu t})_{\ell_p, \ell_p(C^n)} = \frac{\Gamma(n+1)\Gamma(1/p)}{\Gamma(n+1/p)} \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta).$$

In particular, the transposed Cesàro, Gamma, Hölder, and Euler matrices of orders m have the following lower bounds:

$$L(C^{mt})_{\ell_p, \ell_p(C^n)} = \frac{\Gamma(n+1)\Gamma(m+1)(\Gamma(1/p))^2}{\Gamma(n+1/p^*)\Gamma(m+1/p)} \quad (n, m > 0),$$

$$L(\Gamma^{mt})_{\ell_p, \ell_p(C^n)} = \frac{mp^*\Gamma(n+1)\Gamma(1/p)}{(mp^*-1)\Gamma(n+1/p)} \quad (m > 1/p, n > 0),$$

$$L(H^{mt})_{\ell_p, \ell_p(C^n)} = \frac{p^m\Gamma(n+1)\Gamma(1/p)}{\Gamma(n+1/p)} \quad (n, m > 0),$$

$$L(E^{mt})_{\ell_p, \ell_p(C^n)} = \frac{m^{-1/p^*}\Gamma(n+1)\Gamma(1/p)}{\Gamma(n+1/p)} \quad (n > 0, 0 < m < 1).$$

Proof According to Lemma 3.1, Theorem 1.3, and relation (1.3), we have

$$\begin{aligned} L(H^{\mu t})_{\ell_p, \ell_p(C^n)} &= L(C^n H^{\mu t}) = L(C^n) L(H^{\mu t}) \\ &= \frac{\Gamma(n+1)\Gamma(1/p)}{\Gamma(n+1/p)} \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta). \end{aligned} \quad \square$$

3.2 Lower bound of operators from $\ell_p(C^n)$ into ℓ_p

In this section, we intend to find the lower bound of the transposed Hausdorff operators from $\ell_p(C^n)$ into ℓ_p .

Theorem 3.3 *The transposed Hausdorff operator $H^{\mu t}$ has the lower bound from $\ell_p(C^n)$ into ℓ_p and*

$$L(H^{\mu t})_{\ell_p(C^n), \ell_p} = \frac{\Gamma(n+1/p)}{\Gamma(n+1)\Gamma(1/p)} \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta). \quad (3.1)$$

In particular, the transposed Cesàro, Gamma, Hölder, and Euler matrices of orders m are bounded operators from $\ell_p(C^n)$ into ℓ_p and

$$L(C^{mt})_{\ell_p(C^n), \ell_p} = \frac{\Gamma(m+1)\Gamma(n+1/p)}{\Gamma(n+1)\Gamma(m+1/p)} \quad (n, m > 0),$$

$$L(\Gamma^{mt})_{\ell_p(C^n), \ell_p} = \frac{mp^*\Gamma(n+1/p)}{(mp^*-1)\Gamma(n+1)\Gamma(1/p)} \quad (m > 1/p, n > 0),$$

$$L(H^{mt})_{\ell_p(C^n), \ell_p} = \frac{p^m\Gamma(n+1/p)}{\Gamma(n+1)\Gamma(1/p)} \quad (n, m > 0),$$

$$L(E^{mt})_{\ell_p(C^n), \ell_p} = \frac{m^{-1/p^*}\Gamma(n+1/p)}{\Gamma(n+1)\Gamma(1/p)} \quad (n > 0, 0 < m < 1).$$

Proof Following Bennett [8, p. 120], the Hausdorff operator H^μ has a factorization of the form $H^\mu = H^\varphi C^n$, where φ is a quotient measure. So $H^{\mu t} = H^{\varphi t} C^{nt}$, hence Lemma 3.1, Theorem 1.3, and relation (1.3) imply that

$$\begin{aligned} L(H^{\mu t})_{\ell_p(C^n), \ell_p} &= L(H^{\varphi t} C^n)_{\ell_p(C^n), \ell_p} = L(H^{\varphi t}) = L(H^{\mu t})/L(C^n) \\ &= \frac{\Gamma(n+1/p)}{\Gamma(n+1)\Gamma(1/p)} \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta). \end{aligned} \quad \square$$

Remark 3.4 As an example of factorization of Hausdorff operators we have $C^m = H^\varphi C^n$, where C^n is the Cesàro matrix of order n and the quotient measure, H^φ is a Hausdorff matrix associated with the measure

$$d\varphi(\theta) = \frac{\Gamma(m+1)}{\Gamma(n+1)\Gamma(m-n)} \theta^n (1-\theta)^{m-n-1} d\theta,$$

and by Hardy's formula has the ℓ_p -norm

$$\|H^\varphi\|_{\ell_p} = \frac{\Gamma(m+1)\Gamma(n+1/p^*)}{\Gamma(n+1)\Gamma(m+1/p^*)}.$$

3.3 Lower bound of operators on $\ell_p(C^n)$

In this part of the study, we try to find the lower bound of the transposed Hausdorff operators on the space $\ell_p(C^n)$.

Theorem 3.5 *The transposed Hausdorff operator $H^{\mu t}$ has a lower bound on $\ell_p(C^n)$ and*

$$L(H^{\mu t})_{\ell_p(C^n)} = L(H^{\mu t}) = \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta).$$

In particular, the transposed Cesàro, Gamma, Hölder, and Euler matrices of order m are bounded operators on $\ell_p(C^n)$ and

$$L(C^{mt})_{\ell_p(C^n)} = \frac{\Gamma(m+1)\Gamma(1/p)}{\Gamma(m+1/p)} \quad (m > 0),$$

$$L(\Gamma^{mt})_{\ell_p(C^n)} = \frac{mp^*}{mp^* - 1} \quad (mp > 1),$$

$$L(H^{mt})_{\ell_p(C^n)} = p^m \quad (m > 0),$$

$$L(E^{mt})_{\ell_p(C^n)} = m^{\frac{-1}{p^*}} \quad (0 < m < 1).$$

Proof Since Hausdorff matrices commute, we have $C^n H^{\mu t} = H^{\mu t} C^n$. Thus, part (iii) of Lemma 3.1 and relation (1.3) complete the proof. \square

Throughout the next two sections we assume that $1 \leq p < \infty$.

4 Lower bound of operators on the Copson matrix domain for $1 \leq p < \infty$

In this section, we intend to compute the lower bound of operators from ℓ_p into $\ell_p(C^n)$, from $\ell_p(C^n)$ into ℓ_p , and from $\ell_p(C^n)$ into itself.

Recall the Hilbert matrix $H = (h_{j,k})$, which was introduced by David Hilbert in 1894 to study a question in approximation theory:

$$h_{j,k} = \frac{1}{j+k+1} \quad (j, k = 0, 1, \dots).$$

We know that for $p \geq 1$, the Hilbert operator H is a bounded operator on ℓ_p with $\|H\|_{\ell_p} = \pi \csc(\pi/p)$ (see [12, Theorem 323]) and the lower bound $\zeta(p)^{1/p}$ (see [6, Theorem 5]).

For a positive integer n , we define the Hilbert matrix of order n , $H^n = (h_{j,k}^n)$, by

$$h_{j,k}^n = \frac{1}{j+k+n+1} \quad (j, k = 0, 1, \dots).$$

Note that for $n = 0$, $H^0 = H$ is the well-known Hilbert matrix. For more examples,

$$H^1 = \begin{pmatrix} 1/2 & 1/3 & 1/4 & \dots \\ 1/3 & 1/4 & 1/5 & \dots \\ 1/4 & 1/5 & 1/6 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad H^2 = \begin{pmatrix} 1/3 & 1/4 & 1/5 & \dots \\ 1/4 & 1/5 & 1/6 & \dots \\ 1/5 & 1/6 & 1/7 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For nonnegative integers n, j , and k , let us define the matrix $B^n = (b_{j,k}^n)$ by

$$b_{j,k}^n = \frac{(j+1) \cdots (j+n)}{(j+k+1) \cdots (j+k+n+1)} = \binom{n+j}{j} \beta(j+k+1, n+1), \quad (4.1)$$

where the β function is

$$\beta(m, n) = \int_0^1 z^{m-1} (1-z)^{n-1} dz \quad (m, n = 1, 2, \dots).$$

Consider that for $n = 0$, $B^0 = H$, where H is the Hilbert matrix.

Lemma 4.1 ([23, Remark 2.4]) *Let H and H^n be the Hilbert and Hilbert matrix of order n , respectively. We have the following identities:*

- $H = C^n B^n$,
- $H^n = B^n C^n$,
- $HC^n = C^n H^n$,
- B^n is a bounded operator on ℓ_p and $\|B^n\|_{\ell_p} = \frac{\Gamma(n+1/p)\Gamma(1/p^*)}{\Gamma(n+1)}$.

We say that $Q = (q_{n,k})$ is a quasisummability matrix if it is an upper-triangular matrix, i.e., $q_{n,k} = 0$ for $n < k$, and $\sum_{n=0}^k q_{n,k} = 1$ for all k . The product of two quasisummability matrices is also a quasisummability matrix and all these matrices have the lower bound 1 on ℓ_p , according to the following theorem.

Theorem 4.2 ([7, Theorem 2]) *Let p be fixed, $1 < p < \infty$, and let T be a quasisummability matrix. If $x \in \ell_p$ satisfies $x_0 > x_1 > \dots > 0$, then*

$$\|Tx\|_{\ell_q} \geq \|x\|_{\ell_p}.$$

4.1 Lower bound of operators from ℓ_p into $\ell_p(C^n)$

In this part of study we compute the lower bound of some well-known operators like Hilbert and transposed Hausdorff operators on the domain of Copson matrix.

Theorem 4.3 *The matrix B^n defined in relation (4.1) has a lower bound from ℓ_p into $\ell_p(C^n)$ and $L(B^n)_{\ell_p, \ell_p(C^n)} = \zeta(p)^{1/p}$.*

Proof According to Lemmas 3.1 and 4.1, we have

$$L(B^n)_{\ell_p, \ell_p(C^n)} = L(C^n B^n) = L(H) = \zeta(p)^{1/p}. \quad \square$$

Lemma 4.4 ([24, Theorem 3.7]) *Let α and n be two nonnegative integers such that $\alpha > n \geq 0$. The Copson matrix of order α has a factorization of the form $C^\alpha = C^n S^{\alpha, n} = S^{\alpha, n} C^n$, where C^n is the Copson matrix of order n while $S^{\alpha, n} = (s_{j,k}^{\alpha, n})$ is a bounded operator on ℓ_p with the entries*

$$s_{j,k}^{\alpha, n} = \frac{\binom{n+j}{j} \binom{\alpha-n+k-j-1}{k-j}}{\binom{\alpha+k}{k}} \quad (j, k = 0, 1, \dots) \quad (4.2)$$

and ℓ_p -norm

$$\|S^{\alpha, n}\|_{\ell_p} = \frac{\Gamma(\alpha+1)\Gamma(n+1/p)}{\Gamma(n+1)\Gamma(\alpha+1/p)}.$$

Corollary 4.5 *Let α, n be two nonnegative integers that $\alpha > n \geq 0$. The matrix $S^{\alpha, n}$ defined in relation (4.2) has a lower bound from ℓ_p into $\ell_p(C^n)$ and*

$$L(S^{\alpha, n})_{\ell_p, \ell_p(C^n)} = 1.$$

Proof According to Lemmas 3.1 and 4.4 and Theorem 4.2, we have

$$L(S^{\alpha, n})_{\ell_p, \ell_p(C^n)} = L(C^n S^{\alpha, n}) = L(C^\alpha) = 1. \quad \square$$

Corollary 4.6 *For every quasisummability matrix Q , we have*

$$L(Q)_{\ell_p, \ell_p(C^n)} = 1.$$

In particular, for every Hausdorff matrix H^μ , we have $L(H^{\mu t})_{\ell_p, \ell_p(C^n)} = 1$.

4.2 Lower bound of operators from $\ell_p(C^n)$ into ℓ_p

In this part of study we compute the lower bound of transposed Hausdorff operators on the domain of Copson matrix.

Theorem 4.7 *The transposed Hausdorff matrix $H^{\mu t}$ has a lower bound from $\ell_p(C^n)$ into ℓ_p and $L(H^{\mu t})_{\ell_p(C^n), \ell_p} = 1$.*

Proof Since the factor $H^{\varphi t}$ in the factorization $H^{\mu t} = C^n H^{\varphi t}$ is a quasisummability matrix, the proof is similar to that of Theorem 2.1, so

$$L(H^{\mu t})_{\ell_p(C^n), \ell_p} = L(H^{\varphi t}) = 1. \quad \square$$

4.3 Lower bound of operators on $\ell_p(C^n)$

In this part of study we compute the lower bound of Hilbert and transposed Hausdorff operators on the domain of Copson matrix.

Theorem 4.8 *The Hilbert matrix of order n , H^n , has a lower bound on $\ell_p(C^n)$ and $L(H^n)_{\ell_p(C^n)} = \zeta(p)^{1/p}$.*

Proof According to Lemmas 3.1 and 4.1, we have $L(H^n)_{\ell_p(C^n)} = L(H) = \zeta(p)^{1/p}$. \square

Theorem 4.9 *The transposed Hausdorff operator H^{ut} has a lower bound on $\ell_p(C^n)$ and $L(H^{ut})_{\ell_p(C^n)} = 1$.*

Proof Since every two transposed Hausdorff matrices commute hence the proof is the direct result of Lemma 3.1 and Theorem 4.2. \square

5 Lower bound of Copson operator on some sequence spaces

In this section, we investigate the problem of finding the lower bound of Copson operator on some sequence spaces. Through out this section we assume that $1 \leq p < \infty$.

5.1 Lower bound of Copson operator on the difference sequence spaces

In this part of study, we investigate the lower bound of the Copson matrix of order n on the difference sequence spaces. In so doing we need the following preliminaries.

Let $n \in \mathbb{N}$ and $\Delta^{nF} = (\delta_{j,k}^{nF})$ be the forward difference operator of order n with entries

$$\delta_{j,k}^{nF} = \begin{cases} (-1)^{k-j} \binom{n}{k-j}, & j \leq k \leq n+j, \\ 0, & \text{otherwise.} \end{cases}$$

We define the sequence space $\ell_p(\Delta^{nF})$ as the set $\{x = (x_k) : \Delta^{nF}x \in \ell_p\}$ or

$$\ell_p(\Delta^{nF}) = \left\{ x = (x_k) : \sum_{j=0}^{\infty} \left| \sum_{k=0}^n (-1)^k \binom{n}{k} x_{k+j} \right|^p < \infty \right\},$$

with seminorm, $\|\cdot\|_{\ell_p(\Delta^{nF})}$, which is defined by

$$\|x\|_{\ell_p(\Delta^{nF})} = \left(\sum_{j=0}^{\infty} \left| \sum_{k=0}^n (-1)^k \binom{n}{k} x_{k+j} \right|^p \right)^{\frac{1}{p}}.$$

Note that this function will not be a norm, since if $x = (1, 1, 1, \dots)$ then $\|x\|_{\ell_p(\Delta^{nF})} = 0$ while $x \neq 0$. The definition of the backward difference sequence space $\ell_p(\Delta^{nB})$ is similar to $\ell_p(\Delta^{nF})$, except that $\|\cdot\|_{\ell_p(\Delta^{nB})}$ is a norm.

For the special case $n = 1$, we use the notations Δ^B and Δ^F to indicate the backward and forward difference matrices of order 1, respectively. These matrices are defined by

$$\delta_{j,k}^B = \begin{cases} 1, & k = j, \\ -1, & k = j - 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \delta_{j,k}^F = \begin{cases} 1, & k = j, \\ -1, & k = j + 1, \\ 0, & \text{otherwise,} \end{cases}$$

and their associated sequence spaces $\ell_p(\Delta^B)$ and $\ell_p(\Delta^F)$ are

$$\ell_p(\Delta^B) = \left\{ x = (x_n) : \sum_{n=1}^{\infty} |x_n - x_{n-1}|^p < \infty \right\},$$

and

$$\ell_p(\Delta^F) = \left\{ x = (x_n) : \sum_{n=1}^{\infty} |x_n - x_{n+1}|^p < \infty \right\},$$

respectively. The domains $c_0(\Delta^F)$, $c(\Delta^F)$, and $\ell_\infty(\Delta^F)$ of the forward difference matrix Δ^F in the spaces c_0 , c , and ℓ_∞ were introduced by Kizmaz [17]. Moreover, the domain bv_p of the backward difference matrix Δ^B in the space ℓ_p has been recently investigated for $0 < p < 1$ by Altay and Başar [1], and for $1 \leq p \leq \infty$ by Başar and Altay [5].

Theorem 5.1 ([7, Theorem 1]) *Let $p \geq 1$, and let H^μ be a bounded Hausdorff matrix on ℓ_p . Then,*

$$\|H^\mu x\|_{\ell_p} \geq L \|x\|_{\ell_p} \quad (5.1)$$

for every decreasing sequence x of nonnegative terms, where

$$L^p = \sum_{k=0}^{\infty} \left(\int_0^1 (1-\theta)^k d\mu(\theta) \right)^p.$$

The constant in (5.1) is the best possible, and there is equality only when $x = 0$ or $p = 1$, or when $d\mu(\theta)$ is the point mass at 1.

For example, by choosing $d\mu(\theta) = n(1-\theta)^{n-1} d\theta$, the lower bound of the Cesàro matrix of order n is

$$L(C^{nt}) = \left\{ \sum_{k=0}^{\infty} \left(\frac{n}{n+k} \right)^p \right\}^{1/p}. \quad (5.2)$$

In particular, for $n = 1$, the well-known Cesàro operator has the lower bound $L(C^t) = \zeta(p)^{1/p}$.

Theorem 5.2 *The Copson matrix of order n , C^n , is a bounded operator from $\ell_p(\Delta^{nB})$ into $\ell_p(\Delta^{nF})$ and*

$$L(C^n)_{\ell_p(\Delta^{nB}), \ell_p(\Delta^{nF})} = \left\{ \sum_{k=0}^{\infty} \left(\frac{n}{n+k} \right)^p \right\}^{1/p}.$$

In particular, the Copson matrix is a bounded operator from $\ell_p(\Delta^B)$ into $\ell_p(\Delta^F)$ and $L(C)_{\ell_p(\Delta^B), \ell_p(\Delta^F)} = \zeta(p)^{1/p}$.

Proof Let $\Delta^{n_F} C^n = D^n$. It has proved by Theorem 4.1 in [24] that the matrix $D^n = (d_{ij}^n) = I_{ij} / \binom{n+j}{j}$ is a diagonal matrix, where I is the identity matrix. The facts that Δ^{n_B} is the transpose of Δ^{n_F} and $\Delta^{n_F} C^n$ is a diagonal matrix result in the identity $\Delta^{n_F} C^n = C^{nt} \Delta^{n_B}$. Now, by applying Lemma 3.1 and relation (5.2), we have

$$L(C^n)_{\ell_p(\Delta^{n_B}), \ell_p(\Delta^{n_F})} = L(C^{nt}) = \left\{ \sum_{k=0}^{\infty} \left(\frac{n}{n+k} \right)^p \right\}^{1/p},$$

which completes the proof. \square

5.2 Lower bound of Copson operator on the domain of Hilbert matrix

Let n be a nonnegative integer and $hil(n, p)$ be the sequence space associated with the Hilbert matrix of order n , H^n , which is

$$hil(n, p) = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \sum_{k=0}^{\infty} \frac{x_k}{j+k+n+1} \right|^p < \infty \right\},$$

and has the norm

$$\|x\|_{hil(n, p)} = \left(\sum_{j=0}^{\infty} \left| \sum_{k=0}^{\infty} \frac{x_k}{j+k+n+1} \right|^p \right)^{\frac{1}{p}}.$$

Note that, by letting $n = 0$ in the above definition, we obtain the domain of Hilbert matrix $hil(p)$.

Corollary 5.3 *The Copson operator of order n , C^n , is a bounded operator from $hil(n, p)$ into $hil(p)$ and*

$$L(C^n)_{hil(n, p), hil(p)} = 1.$$

Proof According to Lemma 4.1, we have $HC^n = C^n H^n$. Now, Lemma 3.1 and Theorem 4.2 complete the proof. \square

5.3 Lower bound of Copson operator on the domain of the transposed Hausdorff matrix

Let $hau^t(p)$ be the domain of the transposed Hausdorff matrix in the space ℓ_p , that is, $\{x \in \ell_p : H^{\mu t} x \in \ell_p\}$. Then

Corollary 5.4 *The Copson operator of order n , C^n , is a bounded operator from*

- ℓ_p into $hau^t(p)$ and $L(C^n)_{\ell_p, hau^t(p)} = 1$,
- $hau^t(p)$ into ℓ_p and $L(C^n)_{hau^t(p), \ell_p} = 1$,
- $hau^t(p)$ into itself and $L(C^n)_{hau^t(p)} = 1$.

Proof According to Lemma 3.1, $L(C^n)_{\ell_p, hau^t(p)} = L(H^{\mu t} C^n)$. Now, since the product of any two quasi-Hausdorff matrices is also a quasi-Hausdorff matrix, Theorem 4.2 completes the proof.

Let ν be the quotient measure in the factorization of the Copson matrix $C^n = H^\nu H^{\mu t}$, where H^ν is a quasi Hausdorff matrix. Now, by applying Lemma 3.1 and Theorem 4.2, $L(C^n)_{\text{haus}^t(p), \ell_p} = L(H^\nu) = 1$.

The fact that the Hausdorff matrices commute is also valid for their transposes $H^{\mu t} C^n = C^n H^{\mu t}$. Hence the proof is obvious by applying Lemma 3.1 and Theorem 4.2. \square

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