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A note on maximal singular integrals with rough kernels

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Abstract

In this note we study the maximal singular integral operators associated with a homogeneous mapping with rough kernels as well as the corresponding maximal operators. The boundedness and continuity on the Lebesgue spaces, Triebel–Lizorkin spaces, and Besov spaces are established for the above operators with rough kernels in $H^1(S^{n-1})$, which complement some recent developments related to rough maximal singular integrals.

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1 Introduction

During the last several years, a considerable amount of attention has been given to investigate the boundedness for various kinds of integral operators on Triebel–Lizorkin spaces. For examples, see [1, 4, 5] for singular integrals, [15, 17, 26, 27] for Marcinkiewicz integrals, [16, 27] for Littlewood–Paley functions, [14, 18, 20] for maximal functions, and [21, 22] for maximal singular integrals. The main purpose of this paper is to prove the boundedness and continuity of the maximal singular integral and maximal operators related to homogeneous mappings on Triebel–Lizorkin spaces when their kernels are given by function Ω in the Hardy space $H^1(S^{n-1})$. Let $n, d \geq 2$ and $m = (m_1, \dots, m_d) \in \mathbb{R}^d$. We say that $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^d$ is a (nonisotropic) homogeneous mapping of degree m if

$$\Phi(ty) = \delta_t(\Phi(y))$$

holds for all $t > 0$ and $y \in \mathbb{R}^d$. Here, $\{\delta_t\}_{t>0}$ is a family of dilations on \mathbb{R}^d defined by

$$\delta_t(x_1, \dots, x_d) = (t^{m_1}x_1, \dots, t^{m_d}x_d).$$

Before stating our main results, let us recall some pertinent notations, definitions, and background. Let S^{n-1} be the unit sphere in \mathbb{R}^n equipped with the induced Lebesgue mea-

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sure $d\sigma$. Let Ω be integrable over S^{n-1} and satisfy

$$\int_{S^{n-1}} \Omega(u) d\sigma(u) = 0. \tag{1.1}$$

For a suitable mapping $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^d$, we define the singular integral operator $T_{\Omega,\Phi}$ by

$$T_{\Omega,\Phi}f(x) := \text{p.v.} \int_{\mathbb{R}^n} f(x - \Phi(y)) \frac{\Omega(y/|y|)}{|y|^n} dy, \tag{1.2}$$

where $f \in S(\mathbb{R}^d)$ (the Schwartz class on \mathbb{R}^d). When $n = d$ and $\Phi(y) = y$, the operator $T_{\Omega,\Phi}$ reduces to the classical Calderón–Zygmund singular integral operator, which is denoted by T_Ω . In their fundamental work on singular integrals, Calderón and Zygmund [3] first proved that T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if $\Omega \in L \log L(S^{n-1})$. The same conclusion was obtained independently by Coifman and Weiss [7] and Connett [8] under the less restrictive condition that $\Omega \in H^1(S^{n-1})$. Here, $H^1(S^{n-1})$ denotes the Hardy space on the unit sphere and contains $L \log L(S^{n-1})$ as a proper subspace. The above results were later extended to singular Radon transforms by many authors (see [2, 6, 9, 10]). In particular, Cheng [6] proved the following result.

Theorem A ([6]) *Let $\Phi = (\Phi_1, \dots, \Phi_d)$ be a homogeneous mapping of degree $m = (m_1, \dots, m_d)$ with each $m_i \neq 0$. Assume that $\Omega \in H^1(S^{n-1})$ satisfies (1.1) and $\Phi|_{S^{n-1}}$ is real-analytic. Then, for $1 < p < \infty$, there exists a positive constant C_p such that*

$$\|T_{\Omega,\Phi}f\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}.$$

In this paper, we study the maximal singular integrals and maximal operators related to homogeneous mappings. Let Ω, Φ be given as in (1.2). The maximal singular integral operator $T_{\Omega,\Phi}^*$ and the maximal operator $M_{\Omega,\Phi}$ are defined by

$$T_{\Omega,\Phi}^*f(x) := \sup_{\epsilon > 0} \left| \int_{|y| > \epsilon} f(x - \Phi(y)) \frac{\Omega(y/|y|)}{|y|^n} dy \right|,$$

$$M_{\Omega,\Phi}f(x) = \sup_{t > 0} \frac{1}{t^n} \int_{|y| < t} |f(x - \Phi(y))| |\Omega(y/|y|)h(|y|)| dy.$$

For the sake of simplification, we denote $T_{\Omega,\Phi}^* = T_\Omega^*$ and $M_{\Omega,\Phi} = M_\Omega$ when $n = d$ and $\Phi(y) = y$. Particularly, when $\Phi = \mathcal{P}$ is a real-valued polynomial mapping from \mathbb{R}^n to \mathbb{R}^d , we denote $T_{\Omega,\Phi}^* = T_{\Omega,\mathcal{P}}^*$ and $M_{\Omega,\Phi} = M_{\Omega,\mathcal{P}}$. In 1997, Fan and Pan [10] proved that $T_{\Omega,\mathcal{P}}^*$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, provided that $\Omega \in H^1(S^{n-1})$. It follows from a theorem of Stein and Wainger that $M_{\Omega,\mathcal{P}}$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$ if $\Omega \in L^1(S^{n-1})$. Other relevant results on the L^p bounds for $M_{\Omega,\mathcal{P}}$ can be found in [2, 24].

Based on the above, a natural question is the following:

Question A *Are the operators $T_{\Omega,\Phi}^*$ and $M_{\Omega,\Phi}$ bounded on $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$ if Ω and Φ are given as in Theorem A?*

This question can be addressed by the following theorem.

Theorem 1.1 *Let $\Phi = (\Phi_1, \dots, \Phi_d)$ be a homogeneous mapping of degree $m = (m_1, \dots, m_d)$ with each $m_i \neq 0$ and $\Phi|_{S^{n-1}}$ real-analytic. Assume that $\Omega \in H^1(S^{n-1})$ satisfies (1.1). Then, for $1 < p < \infty$, there exists a positive constant C_p such that*

$$\begin{aligned} \|T_{\Omega, \Phi}^* f\|_{L^p(\mathbb{R}^d)} &\leq C_p \|\Omega\|_{H^1(S^{n-1})} \|f\|_{L^p(\mathbb{R}^d)}, \\ \|M_{\Omega, \Phi} f\|_{L^p(\mathbb{R}^d)} &\leq C_p \|\Omega\|_{H^1(S^{n-1})} \|f\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

Here, the above constant $C > 0$ is independent of Ω .

Remark 1.1 We point out the following remarks, which are useful in comprehending Theorem 1.1.

(i) Theorem 1.1 is expectant. Actually, Theorem 1.1 can be proved by applying the arguments similar to those used in deriving [6, Theorem 1] and [10, Lemma 6.3]. However, our main arguments in proving Theorem 1.1 are different from those of [6, Theorem 1].

(ii) Theorem 1.1 is new. In a very recent paper [19], the authors have established the $L^p(1 < p < \infty)$ bounds for the singular integral operator and maximal singular integral operator related to homogeneous mappings when their integral kernels are given by the unit sphere kernel in $H^1(S^{n-1})$ and the weak size radial kernel, which contradicts the main result of [13].

On the other hand, the boundedness properties of maximal singular integral operator and maximal operator in Triebel–Lizorkin spaces have also received some attention of many authors. Let $\mathcal{S}'(\mathbb{R}^d)$ be the tempered distribution class on \mathbb{R}^d . For $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty (p \neq \infty)$, the homogeneous Triebel–Lizorkin spaces $\dot{F}_\alpha^{p,q}(\mathbb{R}^d)$ are defined by

$$\dot{F}_\alpha^{p,q}(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} = \left\| \left(\sum_{i \in \mathbb{Z}} 2^{-i\alpha q} |\Psi_i * f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} < \infty \right\}, \tag{1.3}$$

where $\widehat{\Psi}_i(\xi) = \phi(2^i \xi)$ for $i \in \mathbb{Z}$ and $\phi \in C_c^\infty(\mathbb{R}^d)$ satisfies the conditions: $0 \leq \phi(x) \leq 1$; $\text{supp}(\phi) \subset \{x : 1/2 \leq |x| \leq 2\}$; $\phi(x) > c > 0$ if $3/5 \leq |x| \leq 5/3$. The inhomogeneous version of Triebel–Lizorkin spaces, which is denoted by $F_\alpha^{p,q}(\mathbb{R}^d)$, is obtained by adding the term $\|\Theta * f\|_{L^p(\mathbb{R}^d)}$ to the right-hand side of (1.3) with $\sum_{i \in \mathbb{Z}}$ replaced by $\sum_{i \geq 1}$, where $\Theta \in \mathcal{S}(\mathbb{R}^d)$, $\text{supp}(\widehat{\Theta}) \subset \{\xi : |\xi| \leq 2\}$, $\widehat{\Theta}(x) > c > 0$ if $|x| \leq 5/3$. It is well known that the following are valid (see [11, 25] for more details):

$$\begin{aligned} \dot{F}_0^{p,2}(\mathbb{R}^d) &= L^p(\mathbb{R}^d) \quad \text{for } 1 < p < \infty; \\ F_\alpha^{p,q}(\mathbb{R}^d) &\sim \dot{F}_\alpha^{p,q}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \quad \text{and} \\ \|f\|_{F_\alpha^{p,q}(\mathbb{R}^d)} &\sim \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} + \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for } \alpha > 0. \end{aligned} \tag{1.4}$$

Recently, Liu et al. [21, 22] have established the bounds for $T_{\Omega, \mathcal{P}}^*$ and $M_{\Omega, \mathcal{P}}$ on Triebel–Lizorkin spaces when $\Omega \in L \log L(S^{n-1})$ or $\Omega \in \mathcal{F}_\beta(S^{n-1})$ (the Grafakos–Stefanov function class (see [12])). It should be pointed out that the following relationships are valid:

$$\begin{aligned} L(\log L)^{\alpha_1}(S^{n-1}) &\subsetneq L(\log L)^{\alpha_2}(S^{n-1}) \quad \text{for } 0 < \alpha_2 < \alpha_1; \\ L(\log L)^\alpha(S^{n-1}) &\subsetneq H^1(S^{n-1}) \subsetneq L^1(S^{n-1}) \quad \text{for } \alpha \geq 1; \end{aligned}$$

$$\begin{aligned}
 &L(\log L)^\alpha(S^{n-1}) \not\subseteq H^1(S^{n-1}) \not\subseteq L(\log L)^\alpha(S^{n-1}) \quad \text{for } 0 < \alpha < 1; \\
 &\bigcap_{\beta > 1} \mathcal{F}_\beta(S^{n-1}) \not\subseteq H^1(S^{n-1}) \not\subseteq \bigcup_{\beta > 1} \mathcal{F}_\beta(S^{n-1}); \\
 &\bigcap_{\beta > 1} \mathcal{F}_\beta(S^{n-1}) \not\subseteq L \log L(S^{n-1}).
 \end{aligned}$$

As far as I know the bounds for $T_{\Omega, \mathcal{P}}^*$ and $M_{\Omega, \mathcal{P}}$ on the Triebel–Lizorkin spaces and Besov spaces are unknown under the condition that $\Omega \in H^1(S^{n-1})$, even in the special case $n = d$ and $\mathcal{P}(y) = y$.

A natural question, which arises from the above, is the following:

Question 1.1 *Are the operators T_Ω^* and M_Ω bounded on the Triebel–Lizorkin spaces if $\Omega \in H^1(S^{n-1})$?*

Question 1.1 is the main motivation for this work. In this paper we give an affirmative answer to the above question by treating more general operators. Our main result can be stated as follows.

Theorem 1.2 *Let $\Phi = (\Phi_1, \dots, \Phi_d)$ be a homogeneous mapping of degree $m = (m_1, \dots, m_d)$ with each $m_i \in \mathbb{N} \setminus \{0\}$ and $\Phi|_{S^{n-1}}$ real-analytic. Assume that $\Omega \in H^1(S^{n-1})$ satisfies (1.1). Then*

(i) *For any $\alpha \in (0, 1)$ and $1 < p, q < \infty$, there exists a constant $C > 0$ such that*

$$\|T_{\Omega, \Phi}^* f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \leq C \|\Omega\|_{H^1(S^{n-1})} \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)},$$

where $C > 0$ is independent of Ω .

(ii) *$T_{\Omega, \Phi}^*$ is continuous from $F_\alpha^{p,q}(\mathbb{R}^d)$ to $\dot{F}_\alpha^{p,q}(\mathbb{R}^d)$ for all $\alpha \in (0, 1)$ and $1 < p, q < \infty$.*

The same conclusions hold for $M_{\Omega, \Phi}$.

The rest of this section is to present the bounds and continuity for $T_{\Omega, \Phi}^*$ and $M_{\Omega, \Phi}$ on Besov spaces. For $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty (p \neq \infty)$, the homogeneous Besov spaces $\dot{B}_\alpha^{p,q}(\mathbb{R}^d)$ are defined by

$$\dot{B}_\alpha^{p,q}(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)} = \left(\sum_{i \in \mathbb{Z}} 2^{-i\alpha q} \|\Psi_i * f\|_{L^p(\mathbb{R}^d)}^q \right)^{1/q} < \infty \right\}, \tag{1.5}$$

where $\widehat{\Psi}_i$ is given as in (1.3). The inhomogeneous version of Besov spaces $B_\alpha^{p,q}(\mathbb{R}^d)$ is obtained by adding the term $\|\Theta * f\|_{L^p(\mathbb{R}^d)}$ to the right-hand side of (1.5) with $\sum_{i \in \mathbb{Z}}$ replaced by $\sum_{i \geq 1}$, where Θ is given as in the definition of $F_\alpha^{p,q}(\mathbb{R}^d)$. The following property is well known (see [11, 25] for more details):

$$\begin{aligned}
 &B_\alpha^{p,q}(\mathbb{R}^d) \sim \dot{B}_\alpha^{p,q}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \quad \text{and} \\
 &\|f\|_{B_\alpha^{p,q}(\mathbb{R}^d)} \sim \|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)} + \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for } \alpha > 0.
 \end{aligned} \tag{1.6}$$

Recently, Liu and Wu [20] established a criterion on the boundedness and continuity of a class of operators on Besov spaces, which is listed as follows.

Proposition 1.1 ([20]) *Let T be a sublinear operator and be bounded on $L^p(\mathbb{R}^d)$ for some $p \in (1, \infty)$. Assume that*

$$|\Delta_\zeta(Tf)(x)| \leq |T(\Delta_\zeta f)(x)| \tag{1.7}$$

for any $x, \zeta \in \mathbb{R}^d$, where $\Delta_\zeta f(x) = f(x + \zeta) - f(x)$ for all $x, \zeta \in \mathbb{R}^d$. Then T is bounded on $\dot{B}_\alpha^{p,q}(\mathbb{R}^d)$ for any $\alpha \in (0, 1)$ and $q \in (1, \infty)$. Specially, if T satisfies the following:

$$|Tf - Tg| \leq |T(f - g)| \tag{1.8}$$

for arbitrary function f, g defined on \mathbb{R}^d , then T is continuous from $B_\alpha^{p,q}(\mathbb{R}^d)$ to $\dot{B}_\alpha^{p,q}(\mathbb{R}^d)$ for any $\alpha \in (0, 1)$ and $q \in (1, \infty)$.

It is easy to check that both the operators $T_{\Omega, \Phi}^*$ and $M_{\Omega, \Phi}$ satisfy conditions (1.7) and (1.8). This together with Theorem 1.1 and Proposition 1.1 implies the following theorem.

Theorem 1.3 *Let Ω, Φ be given as in Theorem 1.1. Then:*

- (i) *For any $\alpha \in (0, 1)$ and $1 < p, q < \infty$, there exists a constant $C > 0$ such that*

$$\|T_{\Omega, \Phi}^* f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)} \leq C \|\Omega\|_{H^1(S^{n-1})} \|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)},$$

where $C > 0$ is independent of Ω .

- (ii) *$T_{\Omega, \Phi}^*$ is continuous from $B_\alpha^{p,q}(\mathbb{R}^d)$ to $\dot{B}_\alpha^{p,q}(\mathbb{R}^d)$ for all $\alpha \in (0, 1)$ and $1 < p, q < \infty$. The same conclusions hold for $M_{\Omega, \Phi}$.*

By (1.4), (1.6), and Theorems 1.1–1.3, we can get immediately the following result.

Theorem 1.4 *Let Ω, Φ be given as in Theorem 1.2. Then both the operators $T_{\Omega, \Phi}^*$ and $M_{\Omega, \Phi}$ are bounded and continuous on $F_\alpha^{p,q}(\mathbb{R}^d)$ and $B_\alpha^{p,q}(\mathbb{R}^d)$ for $\alpha \in (0, 1)$ and $1 < p, q < \infty$.*

Remark 1.2 Theorems 1.2–1.4 are new, even in the special case $n = d$ and $\Phi(y) = y$.

The paper is organized as follows. Sect. 2 contains the atomic decomposition of Hardy space and some estimates of oscillatory integrals, which play key roles in the estimates of Fourier transforms on some measures. In Sect. 3, we prove Theorem 1.1 after presenting a general criterion on the L^p bounds of the convolution operators (see Lemma 3.1). The proof of Theorem 1.2 is given in Sect. 4. It should be pointed out that the proof of the boundedness (resp., continuity) part in Theorem 1.2 is greatly motivated by the idea in [21] (resp., [22]).

Throughout the paper, the letter C or c , sometimes with certain parameters, stands for positive constants not necessarily the same one at each occurrence, but are independent of the essential variables. For notational convenience, we set $\exp(it) = e^{it}$ for any $t \in \mathbb{R}$.

2 Preliminary definitions and lemmas

We start with the definition of Hardy space on S^{n-1} and its atomic decomposition. Recall that the Poisson kernel on S^{n-1} is defined by $P_{rw}(\theta) = \frac{1-r^2}{|rw-\theta|^n}$ for $0 \leq r < 1$ and $\theta, w \in S^{n-1}$.

The Hardy space $H^1(S^{n-1})$ is the set of all $L^1(S^{n-1})$ functions which satisfy

$$\|\Omega\|_{H^1(S^{n-1})} = \int_{S^{n-1}} \sup_{0 \leq r < 1} \left| \int_{S^{n-1}} \Omega(\theta) P_{rw}(\theta) d\sigma(\theta) \right| d\sigma(w) < \infty.$$

Definition 2.1 ($H^1(S^{n-1})$ atoms) A function $b(\cdot)$ on S^{n-1} is a regular atom if there exist $\varepsilon \in S^{n-1}$ and $\varrho \in (0, 2]$ such that

- (i) $\text{supp}(b) \subset S^{n-1} \cap B(\varepsilon, \varrho)$, where $B(\varepsilon, \varrho) = \{y \in \mathbb{R}^n : |y - \varepsilon| < \varrho\}$;
- (ii) $\|b\|_{L^2(S^{n-1})} \leq \varrho^{(1-n)/2}$;
- (iii) $\int_{S^{n-1}} b(y) d\sigma(y) = 0$.

The following atomic decomposition of Hardy space can be obtained by the idea in [7, 8].

Lemma 2.1 ([8]) *If $\Omega \in H^1(S^{n-1})$ satisfies the cancelation condition (1.1), then there exist a sequence of complex numbers $\{c_j\}$ and a sequence of $(1, \infty)$ atoms $\{\Omega_j\}$ such that $\Omega = \sum_j c_j \Omega_j$ and $\|\Omega\|_{H^1(S^{n-1})} \approx \sum_j |c_j|$.*

In our proofs of the main results we shall encounter oscillatory integrals with generalized polynomials as their phase functions. Thus the following lemma of van der Corput type is needed.

Lemma 2.2 ([6]) *Let $l \in \mathbb{N} \setminus \{0\}$, $\mu_1, \dots, \mu_l \in \mathbb{R}$, and d_1, \dots, d_l be distinct nonzero real numbers. Let $\psi \in C^1([1/2, 1])$. Then*

$$\left| \int_{\delta}^{\tau} \exp(i(\mu_1 t^{d_1} + \dots + \mu_l t^{d_l})) \psi(t) dt \right| \leq C |\mu_1|^{-1/l} \left(|\psi(\tau)| + \int_{\delta}^{\tau} |\psi'(t)| dt \right)$$

holds for $1/2 \leq \delta < \tau \leq 1$. Here, $C > 0$ is independent of $\{\mu_j\}_{j=1}^l$.

Applying Lemma 2.2 and the arguments similar to those used in deriving [6, Lemma 2.5], we can get the following lemma.

Lemma 2.3 *Let $l \in \mathbb{N} \setminus \{0\}$, $s \in \mathbb{N} \setminus \{0\}$, and h_1, \dots, h_l be distinct nonzero real numbers and*

$$Q(t, u) = t^{h_1} \sum_{|\alpha| \leq s} a_{\alpha} u^{\alpha} + \sum_{j=2}^l t^{h_j} w_j(u),$$

where $t \in \mathbb{R}$, $u = (u_1, \dots, u_{n-1}) \in \mathbb{R}^{n-1}$, $\alpha \in \mathbb{N}^{n-1}$, $a_{\alpha} \in \mathbb{R}$, and $w_j(\cdot)$ are real-valued. Let $r > 0$ and $b(\cdot)$ be a measurable function on $[-r, r]^{n-1}$ that satisfies $\|b\|_{L^2([-r, r]^{n-1})} \leq r^{-(n-1)/2}$. Then there exists a positive constant C independent of $\{a_{\alpha}\}$, $\{w_j(\cdot)\}$, r such that

$$\int_{1/2}^1 \left| \int_{[-r, r]^{n-1}} \exp(iQ(t, u)) b(u) du \right| dt \leq C \left(r^s \sum_{|\alpha|=s} |a_{\alpha}| \right)^{-\gamma}, \tag{2.1}$$

where $\gamma = \frac{1}{2} \min\{1/h_1, 1/l, 1/(s+1)\}$.

Proof Without loss of generality, we may assume that $|a_{(s,0,\dots,0)}| \approx \sum_{|\alpha|=s} |a_{\alpha}|$. Let $\eta = (u_2, \dots, u_{n-1})$ and $R(u) = \sum_{|\alpha| \leq s} a_{\alpha} u^{\alpha}$. By a change of variable and Hölder's inequality, we

have

$$\begin{aligned}
 & \int_{1/2}^1 \left| \int_{[-r,r]^{n-1}} \exp(iQ(t,u)) b(u) du \right| dt \\
 &= r^{n-1} \int_{1/2}^1 \left| \int_{[-1,1]^{n-1}} \exp(iQ(t,ru)) b(ru) du \right| dt \\
 &\leq r^{n-1} \int_{[-1,1]^{n-2}} \left(\int_{1/2}^1 \left| \int_{-1}^1 \exp(iQ(t,ru_1,r\eta)) b(ru_1,r\eta) du_1 \right|^2 dt \right)^{1/2} d\eta \\
 &\leq r^{n-1} \int_{[-1,1]^{n-2}} \left(\int_{-1}^1 \int_{-1}^1 \left| \int_{1/2}^1 \exp(i(Q(t,ru_1,r\eta) - Q(t,r\nu_1,r\eta))) dt \right| \right. \\
 &\quad \left. \times |b(ru_1,r\eta) \overline{b(r\nu_1,r\eta)}| du_1 d\nu_1 \right)^{1/2} d\eta \\
 &\leq r^{n-1} \int_{[-1,1]^{n-2}} \left(\int_{-1}^1 \int_{-1}^1 \left| \int_{1/2}^1 \exp(i(Q(t,ru_1,r\eta) - Q(t,r\nu_1,r\eta))) dt \right|^2 du_1 d\nu_1 \right)^{1/4} \\
 &\quad \times \left(\int_{-1}^1 \int_{-1}^1 |b(ru_1,r\eta) \overline{b(r\nu_1,r\eta)}|^2 du_1 d\nu_1 \right)^{1/4} d\eta. \tag{2.2}
 \end{aligned}$$

Applying Lemma 2.2, we obtain

$$\left| \int_{1/2}^1 \exp(i(Q(t,ru_1,r\eta) - Q(t,r\nu_1,r\eta))) dt \right| \leq C |R(ru_1,r\eta) - R(r\nu_1,r\eta)|^{-2\gamma}, \tag{2.3}$$

where $\gamma = \frac{1}{2} \min\{1/h_1, 1/l, 1/(s + 1)\}$. Note that

$$R(ru_1,r\eta) = a_{(s,0,\dots,0)} r^s u_1^s + \text{lower powers in } u_1.$$

Then, by applying an inequality proved by Ricci and Stein (see [23, p. 182]), we get from (2.2) and (2.3) that

$$\begin{aligned}
 & \int_{-1}^1 \left| \int_{1/2}^1 \exp(i(Q(t,ru_1,r\eta) - Q(t,r\nu_1,r\eta))) dt \right|^2 du_1 \\
 &\leq \int_{-1}^1 |R(ru_1,r\eta) - R(r\nu_1,r\eta)|^{-4\gamma} du_1 \leq C \left(r^s \sum_{|\alpha|=s} |a_\alpha| \right)^{-4\gamma}, \tag{2.4}
 \end{aligned}$$

where $C > 0$ is independent of ν_1, η, r . On the other hand, by Hölder’s inequality and a change of variable, we have

$$\begin{aligned}
 & \int_{[-1,1]^{n-2}} \left(\int_{-1}^1 \int_{-1}^1 |b(ru_1,r\eta) \overline{b(r\nu_1,r\eta)}|^2 du_1 d\nu_1 \right)^{1/4} d\eta \\
 &\leq \int_{[-1,1]^{n-2}} \left(\int_{-1}^1 |b(ru_1,r\eta)|^2 du_1 \right)^{1/2} d\eta \\
 &\leq C \left(\int_{[-1,1]^{n-1}} |b(ru)|^2 du \right)^{1/2} \leq Cr^{-(n-1)/2} \|b\|_{L^2([-r,r]^{n-1})} \leq Cr^{-(n-1)}, \tag{2.5}
 \end{aligned}$$

where $C > 0$ is independent of r . (2.5) together with (2.2) and (2.4) yields (2.1). □

Lemma 2.4 ([6]) *For $j \in \{1, 2\}$, let U_j be a domain in \mathbb{R}^{m_j} and K_j be a compact subset of U_j . Let $h(\cdot, \cdot)$ be a real-analytic function on $U_1 \times U_2$ such that $h(\cdot, z)$ is a nonzero function for every $z \in U_2$. Then there exists a positive number $\delta = \delta(h, K_1, K_2)$ such that*

$$\sup_{z \in K_2} \int_{K_1} |h(w, z)|^{-\delta} dw < \infty.$$

3 Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following lemma, which is the main tool of proving Theorem 1.1.

Lemma 3.1 ([19]) *Let $A > 0$, $\Lambda \in \mathbb{N} \setminus \{0\}$ and $\{\sigma_{k,s} : 0 \leq s \leq \Lambda$ and $k \in \mathbb{Z}\}$ be a family of uniformly bounded Borel measures on \mathbb{R}^n with $\sigma_{k,0}(\xi) = 0$ for every $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$. For $1 \leq s \leq \Lambda$, let $v_s, \eta_s > 1$, $\delta_s, \beta_s > 0$, $\{a_{k,s}\}$ be a sequence of real positive numbers, $\ell_s \in \mathbb{N} \setminus \{0\}$ and $L_s : \mathbb{R}^n \rightarrow \mathbb{R}^{\ell_s}$ be a linear transformation. Suppose that*

- (i) $\max\{|\widehat{\sigma_{k,s}}(\xi)|, |\widehat{\sigma_{k,s}}(\xi)|\} \leq CA \min\{1, |a_{k,s}^{v_s} L_s(\xi)|^{-\delta_s/v_s}\}$;
- (ii) $\max\{|\widehat{\sigma_{k,s}}(\xi) - \widehat{\sigma_{k,s-1}}(\xi)|, |\widehat{\sigma_{k,s}}(\xi) - \widehat{\sigma_{k,s-1}}(\xi)|\} \leq CA |a_{k,s}^{v_s} L_s(\xi)|^{\beta_s/v_s}$;
- (iii) *for every $q > 1$, there exists $C_q > 0$ independent of $\{L_s\}_{s=1}^\Lambda$ such that*

$$\left\| \sup_{k \in \mathbb{Z}} |\sigma_{k,0}| * f \right\|_{L^q(\mathbb{R}^n)} \leq C_q A \|f\|_{L^q(\mathbb{R}^n)};$$

(iv) *for any $1 \leq s \leq \Lambda$, $\{a_{k,s}\}$ satisfies one of the following conditions:*

- (a) $\inf_{k \in \mathbb{Z}} \frac{a_{k+1,s}}{a_{k,s}} \geq \eta_s$, (b) $\inf_{k \in \mathbb{Z}} \frac{a_{k,s}}{a_{k+1,s}} \geq \eta_s$.

Then, for any $1 < p < \infty$, there exists $C_p > 0$ such that

$$\max \left\{ \left\| \sup_{k \in \mathbb{Z}} |\sigma_{k,\Lambda}| * f \right\|_{L^p(\mathbb{R}^n)}, \left\| \sup_{k \in \mathbb{Z}} \left| \sum_{j=k}^\infty \sigma_{j,\Lambda} * f \right\|_{L^p(\mathbb{R}^n)} \right\} \leq C_p A \|f\|_{L^p(\mathbb{R}^n)}.$$

Here, the constant C_p is independent of $\{L_s\}_{s=1}^\Lambda$ and $\{v_s\}_{s=1}^\Lambda$, but depends on $\Lambda, p, \{\delta_s\}_{s=1}^\Lambda, \{\beta_s\}_{s=1}^\Lambda$, and $\{\eta_s\}_{s=1}^\Lambda$.

We now proceed with the proof of Theorem 1.1. For notational convenience, we denote by \mathcal{V}_{n-1} the set of polynomials in $n - 1$ variables with real coefficients and set $[x] := \max\{k \in \mathbb{Z} : k \leq x\}$ for any $x \in \mathbb{R}$. For $s \in \mathbb{N}$, let $\mathcal{V}_{n-1,s}$ denote the subset of \mathcal{V}_{n-1} which contains homogeneous polynomials of degree s .

Proof of Theorem 1.1 By Lemma 2.1 and Minkowski’s inequality, it suffices to show that

$$\max \left\{ \|T_{\Omega, \Phi}^* f\|_{L^p(\mathbb{R}^d)}, \|M_{\Omega, \Phi}\|_{L^p(\mathbb{R}^d)} \right\} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}, \tag{3.1}$$

where Ω is an H^1 atom on S^{n-1} satisfying conditions (i)–(iii) of Definition 2.1.

In what follows, we assume that Ω is an H^1 atom on S^{n-1} satisfying conditions (i)–(iii) of Definition 2.1 with $\varrho \in (0, 2]$. Without loss of generality, we may assume the following:

- (i) λ is the number of distinct m_j ;

(ii) $\Phi = (\Phi_1, \dots, \Phi_d) = (\Phi^1, \dots, \Phi^\lambda)$, where $\Phi^s = (\Phi_{s,1}, \dots, \Phi_{s,a_s})$ with $\Phi_{s,j}(ty) = t^{m_{rs}} \Phi_{s,j}(y)$ for any $1 \leq s \leq \lambda$ and $1 \leq j \leq a_s$;

(iii) $\{\Phi_{s,1}, \dots, \Phi_{s,o_s}\}$ forms a basis for $\text{span}\{\Phi_{s,1}, \dots, \Phi_{s,a_s}\}$ for any $1 \leq s \leq \lambda$.

From assumption (ii) we see that $\sum_{s=1}^\lambda a_s = d$ and $\{r_1, \dots, r_\lambda\} \subset \{1, \dots, d\}$. We get from assumption (iii) that, for any $1 \leq s \leq \lambda$ and $1 \leq j \leq a_s$, there exists a sequence of numbers $\{b_{s,j,k}\}_{k=1}^{o_s}$ such that

$$\Phi_{s,j}(y) = b_{s,j,1} \Phi_{s,1}(y) + \dots + b_{s,j,o_s} \Phi_{s,o_s}(y).$$

Let $\xi = (\xi_1, \dots, \xi_d) = (\xi^1, \dots, \xi^\lambda)$ with $\xi^s = (\xi_{s,1}, \dots, \xi_{s,a_s})$ for $1 \leq s \leq \lambda$. For any $1 \leq s \leq \lambda$, let $\tilde{\Phi}^s = (\Phi_{s,1}, \dots, \Phi_{s,o_s})$ and $\tilde{\xi}^s = (\xi_{s,1}, \dots, \xi_{s,o_s})$. We define two sequences of linear transformations $\{H_{s,i}\}_{i=1}^{a_s} : \mathbb{R}^{o_s} \rightarrow \mathbb{R}$ and $\{R_{s,j}\}_{j=1}^{o_s} : \mathbb{R}^{a_s} \rightarrow \mathbb{R}$ as follows:

$$H_{s,i}(\tilde{\xi}^s) = b_{s,i,1} \xi_{s,1} + \dots + b_{s,i,o_s} \xi_{s,o_s}, \quad 1 \leq i \leq a_s;$$

$$R_{s,j}(\xi^s) = b_{s,1,j} \xi_{s,1} + \dots + b_{s,a_s,j} \xi_{s,a_s}, \quad 1 \leq j \leq o_s.$$

Define two sequences of linear transformations $\{H_s\}_{s=1}^\lambda$ and $\{R_s\}_{s=1}^\lambda$ by

$$H_s(\tilde{\xi}^s) = (H_{s,1}(\tilde{\xi}^s), \dots, H_{s,a_s}(\tilde{\xi}^s)), \quad R_s(\xi^s) = (R_{s,1}(\xi^s), \dots, R_{s,o_s}(\xi^s)).$$

It is not difficult to see that

$$\xi^s \cdot \Phi^s = \xi^s \cdot H_s(\tilde{\Phi}^s) = R_s(\xi^s) \cdot \tilde{\Phi}^s, \quad 1 \leq s \leq \lambda. \tag{3.2}$$

For any $1 \leq s \leq \lambda$ and $z \in S^{o_s-1}$, we have that $z \cdot \tilde{\Phi}^i(\cdot)$ is a nonzero real-analytic function since $\{\Phi_{s,1}, \dots, \Phi_{s,o_s}\}$ is linearly independent. Invoking Lemma 2.4, there exists $\delta_s > 0$ such that

$$\sup_{z \in S^{o_s-1}} \int_{S^{n-1}} |z \cdot \tilde{\Phi}^s(y)|^{-\delta_s} d\sigma(y) < \infty. \tag{3.3}$$

Let $\epsilon_s = \min\{1/s, \delta_s/2\}$. It follows from [9, (5.30)], for any $1 \leq s \leq \lambda$, that there exist an orthogonal $n \times n$ matrix U such that $\epsilon U = \mathbf{e} = (0, \dots, 0, 1) \in S^{n-1}$ and a polynomial $P_{s,j} \in \mathcal{V}_{n-1}$ such that $\deg(P_{s,j}) \leq \lceil \frac{n-1}{\epsilon_s} \rceil$ and

$$|\Phi_{s,j}(yU^{-1}) - P_{s,j}(\tilde{y})| \leq C \rho^{(n-1)/\epsilon_s} \tag{3.4}$$

for every $y \in B(\mathbf{e}, \rho) \cap S^{n-1}$ and $1 \leq j \leq a_s$, where $\tilde{y} = (y_1/|y|, \dots, y_{n-1}/|y|)$. For any $1 \leq s \leq \lambda$, let $\mathcal{P}_s = (P_{s,1}, \dots, P_{s,a_s})$ and $\deg(\mathcal{P}_s) = \max_{1 \leq j \leq a_s} \deg(P_{s,j})$. Then there are integers $0 \leq \Lambda_{s,1} < \dots < \Lambda_{s,M_s} \leq \deg(\mathcal{P}_s)$ and $Q_{s,j,\Lambda_{s,l}} \in \mathcal{V}_{n-1, \Lambda_{s,l}}$ for $1 \leq j \leq a_s$ and $1 \leq l \leq M_s$ such that

$$\mathcal{P}_s = \sum_{l=1}^{M_s} Q_{s,\Lambda_{s,l}}$$

where $Q_{s,\Lambda_{s,l}} = (Q_{s,1,\Lambda_{s,l}}, \dots, Q_{s,a_s,\Lambda_{s,l}})$ and $Q_{s,\Lambda_{s,l}} \neq (0, \dots, 0)$ for any $1 \leq l \leq M_s$. For any $1 \leq l \leq M_s$, let $\tilde{Q}_{s,\Lambda_{s,l}} = (Q_{s,1,\Lambda_{s,l}}, \dots, Q_{s,\theta_s,\Lambda_{s,l}})$ and

$$\tilde{P}_s = \sum_{l=1}^{M_s} \tilde{Q}_{s,\Lambda_{s,l}}.$$

It follows from (3.4) that

$$|\tilde{\Phi}^s(y) - \tilde{P}_s(y\tilde{U})| \leq C\varrho^{(n-1)/\epsilon_s} \tag{3.5}$$

for every $y \in B(\varepsilon, \varrho) \cap S^{n-1}$ and $1 \leq s \leq \lambda$. For $1 \leq s \leq \lambda$, $1 \leq l \leq M_s$, and $1 \leq j \leq a_s$, we set

$$Q_{s,j,\Lambda_{s,l}}(y) = \sum_{|\beta|=\Lambda_{s,l}} b_{s,j,l,\beta} y^\beta. \tag{3.6}$$

Let $\varpi(u) = \sum_{s=1}^u (M_s + 1)$ for $1 \leq u \leq \lambda$, $\varpi(0) = 0$, and define the mappings $\Theta_0, \dots, \Theta_{\varpi(\lambda)}$ by

$$\Theta_0(y) = (0, \dots, 0), \quad \Theta_{\varpi(\lambda)}(y) = \Phi(y) \tag{3.7}$$

and

$$\begin{aligned} &\Theta_{\varpi(u)+\theta}(y) \\ &= \left(\Phi^1(y), \dots, \Phi^u(y), |y|^{m_{r_{u+1}}} H_{u+1} \left(\sum_{l=1}^{\theta} \tilde{Q}_{u+1,\Lambda_{u+1,l}}(|y|^{-1}y\tilde{U}) \right), 0, \dots, 0 \right) \end{aligned} \tag{3.8}$$

for $0 \leq u \leq \lambda - 1$, $0 \leq \theta < \varpi(u + 1) - \varpi(u)$. In particular,

$$\Theta_{\varpi(u)-1}(y) = (\Phi^1(y), \dots, \Phi^{u-1}(y), |y|^{m_{r_u}} H_u(\tilde{P}_u(|y|^{-1}y\tilde{U})), 0, \dots, 0), \quad 1 \leq u \leq \lambda. \tag{3.9}$$

For $0 \leq s \leq \varpi(\lambda)$, define two families of measures $\{\sigma_{k,s}\}$ and $\{|\sigma_{k,s}|\}$ by

$$\int_{\mathbb{R}^d} f(x) d\sigma_{k,s}(x) = \int_{2^{k-1} < |y| \leq 2^k} f(\Theta_s(y)) \frac{\Omega(y/|y|)}{|y|^n} dy.$$

The measure $|\sigma_{k,s}|$ is defined in the same way as $\sigma_{k,s}$, but with Ω replaced by $|\Omega|$. By (1.1) and (3.7) we have

$$\sigma_{k,0}(\xi) = 0 \quad \text{for } k \in \mathbb{Z}. \tag{3.10}$$

It is easy to check that

$$\max\{|\widehat{\sigma_{k,s}}(\xi)|, |\widehat{|\sigma_{k,s}|}(\xi)|\} \leq C\|\Omega\|_{L^1(S^{n-1})}. \tag{3.11}$$

For any $1 \leq s \leq \varpi(\lambda)$, by a change of variable, we have

$$\widehat{\sigma_{k,s}}(\xi) = \int_{2^{k-1}}^{2^k} \int_{S^{n-1}} \Omega(y') \exp(-2\pi i \xi \cdot \Theta_s(ty')) d\sigma(y') \frac{dt}{t}$$

$$= \int_{1/2}^1 \int_{S^{n-1}} \Omega(y') \exp(-2\pi i \xi \cdot \Theta_s(2^k ty')) d\sigma(y') \frac{dt}{t}. \tag{3.12}$$

Note that $2\epsilon_u \leq \delta_u$. By (3.3), (3.12), Hölder’s inequality and invoking Lemma 2.2, one finds that

$$\begin{aligned} |\widehat{\sigma_{k,\varpi(u)}}(\xi)| &\leq \int_{S^{n-1}} |\Omega(y')| \left| \int_{1/2}^1 \exp(-2\pi i \xi \cdot \Theta_{\varpi(u)}(2^k ty')) \frac{dt}{t} \right| d\sigma(y') \\ &\leq \int_{S^{n-1}} |\Omega(y')| \left| \int_{1/2}^1 \exp\left(-2\pi i \sum_{j=1}^u R_j(\xi^j) \cdot \tilde{\Phi}^j(y')(2^k t)^{mr_j}\right) \frac{dt}{t} \right| d\sigma(y') \\ &\leq C |2^{km_{ru}} R_u(\xi^u)|^{-\epsilon_u} \int_{S^{n-1}} |\Omega(y')| |(R_u(\xi^u))' \cdot \tilde{\Phi}^u(y')|^{-\epsilon_u} d\sigma(y') \\ &\leq C |2^{km_{ru}} R_u(\xi^u)|^{-\epsilon_u} \|\Omega\|_{L^2(S^{n-1})} \left(\int_{S^{n-1}} |(R_u(\xi^u))' \cdot \tilde{\Phi}^u(y')|^{-2\epsilon_u} d\sigma(y') \right)^{1/2} \\ &\leq C |2^{km_{ru}} R_u(\xi^u)|^{-\epsilon_u} \|\Omega\|_{L^2(S^{n-1})} \left(\int_{S^{n-1}} |(R_u(\xi^u))' \cdot \tilde{\Phi}^u(y')|^{-\delta_u} d\sigma(y') \right)^{1/2} \\ &\leq C |2^{km_{ru}} \varrho^{-(n-1)/(2\epsilon_u)} R_u(\xi^u)|^{-\epsilon_u} \end{aligned} \tag{3.13}$$

for any $1 \leq u \leq \lambda$, where $(R_u(\xi^u))' = \frac{R_u(\xi^u)}{|R_u(\xi^u)|}$. Similarly, we can get

$$|\widehat{\sigma_{k,\varpi(u)}}(\xi)| \leq C |2^{km_{ru}} \varrho^{-(n-1)/(2\epsilon_u)} R_u(\xi^u)|^{-\epsilon_u} \tag{3.14}$$

for $1 \leq u \leq \lambda$.

For $0 \leq u \leq \lambda - 1$ and $0 < \theta < \varpi(u + 1) - \varpi(u)$, we get from (3.8) and (3.12) that

$$\begin{aligned} |\widehat{\sigma_{k,\varpi(u)+\theta}}(\xi)| &= \left| \int_{1/2}^1 \int_{S^{n-1}} \Omega(y') \exp(-2\pi i \left(\sum_{\ell=1}^u \xi^\ell \cdot \Phi^\ell(yU^{-1})(2^k t)^{mr_\ell} \right. \right. \\ &\quad \left. \left. + \xi^{u+1} \cdot H_{u+1} \left(\sum_{l=1}^\theta \tilde{Q}_{u+1,\Lambda_{u+1,l}}(\tilde{y})(2^k t)^{mr_{u+1}} \right) \right) \frac{dt}{t} \right|. \end{aligned}$$

From (3.2) and (3.6) we have

$$\begin{aligned} \xi^{u+1} \cdot H_{u+1} \left(\sum_{l=1}^\theta \tilde{Q}_{u+1,\Lambda_{u+1,l}}(\tilde{y}) \right) &= R_{u+1}(\xi^{u+1}) \cdot \left(\sum_{l=1}^\theta \tilde{Q}_{u+1,\Lambda_{u+1,l}}(\tilde{y}) \right) \\ &= \sum_{j=1}^{o_{u+1}} R_{u+1,j}(\xi^{u+1}) \cdot \left(\sum_{l=1}^\theta Q_{u+1,j,\Lambda_{u+1,l}}(\tilde{y}) \right) \\ &= \sum_{l=1}^\theta \sum_{|\beta|=\Lambda_{u+1,l}} \left(\sum_{j=1}^{o_{u+1}} b_{u+1,j,l,\beta} R_{u+1,j}(\xi^{u+1}) \right) (\tilde{y})^\beta. \end{aligned}$$

Invoking Lemma 2.3, there exists a positive constant $\gamma_{u,\theta}$ such that, for $0 \leq u \leq \lambda - 1$ and $0 < \theta < \varpi(u + 1) - \varpi(u)$,

$$|\widehat{\sigma_{k,\varpi(u)+\theta}}(\xi)| \leq C |2^{km_{r_{u+1}}} \varrho^{\Lambda_{u+1,\theta}} L^{(\Lambda_{u+1,\theta})}(\xi)|^{-\gamma_{u,\theta}}, \tag{3.15}$$

where

$$L^{(\Lambda_{u+1,\theta})}(\xi) = \left(\sum_{j=1}^{o_{u+1}} b_{u+1,j,\theta,\beta} R_{u+1,j}(\xi^{u+1}) \right)_{|\beta|=\Lambda_{u+1,\theta}}.$$

Similarly, we get that, for $0 \leq u \leq \lambda - 1$ and $0 < \theta < \varpi(u + 1) - \varpi(u)$,

$$|\widehat{\sigma_{k,\varpi(u)+\theta}}(\xi)| \leq C |2^{km_{r_{u+1}}} \varrho^{\Lambda_{u+1,\theta}} L^{(\Lambda_{u+1,\theta})}(\xi)|^{-\gamma_{u,\theta}}. \tag{3.16}$$

On the other hand, by a change of variable, we get from (3.2), (3.5), and (3.8) that

$$\begin{aligned} & \max \left\{ \left| \widehat{\sigma_{k,\varpi(u)}}(\xi) - \widehat{\sigma_{k,\varpi(u)-1}}(\xi) \right|, \left| \widehat{\sigma_{k,\varpi(u)}}(\xi) - \widehat{\sigma_{k,\varpi(u)-1}}(\xi) \right| \right\} \\ &= \int_{2^{k-1} < |y| \leq 2^k} \left| \exp(-2\pi i \xi \cdot \Theta_{\varpi(u)}(y)) - \exp(-2\pi i \xi \cdot \Theta_{\varpi(u)-1}(y)) \right| \frac{|\Omega(y/|y|)|}{|y|^n} dy \\ &\leq C 2^{km_{r_u}} \int_{S^{n-1}} |\Omega(y)| |\xi^u \cdot \Phi^u(y) - \xi^u \cdot H_u(\mathcal{P}_u(\tilde{y}\tilde{U}))| d\sigma(y) \\ &\leq C 2^{km_{r_u}} \int_{S^{n-1}} |\Omega(y)| |R_u(\xi^u) \cdot (\tilde{\Phi}^u(y) - \tilde{P}_u(\tilde{y}\tilde{U}))| d\sigma(y) \\ &\leq C \|\Omega\|_{L^1(S^{n-1})} 2^{km_{r_u}} |\varrho^{(n-1)\epsilon_u} R_u(\xi^u)| \end{aligned} \tag{3.17}$$

for $1 \leq u \leq \lambda$. For $1 \leq u \leq \lambda - 1$ and $1 \leq \theta < \varpi(u) - \varpi(u - 1)$, we have

$$\begin{aligned} & \max \left\{ \left| \widehat{\sigma_{k,\varpi(u)+\theta}}(\xi) - \widehat{\sigma_{k,\varpi(u)+\theta-1}}(\xi) \right|, \left| \widehat{\sigma_{k,\varpi(u)+\theta}}(\xi) - \widehat{\sigma_{k,\varpi(u)+\theta-1}}(\xi) \right| \right\} \\ &= \int_{2^{k-1} < |y| \leq 2^k} \left| \exp(-2\pi i \xi \cdot \Theta_{\varpi(u)+\theta}(y)) - \exp(-2\pi i \xi \cdot \Theta_{\varpi(u)+\theta-1}(y)) \right| \frac{|\Omega(y/|y|)|}{|y|^n} dy \\ &\leq C \int_{2^{k-1} < |y| \leq 2^k} \left| 2^{km_{r_{u+1}}} \xi^{u+1} \cdot H_{u+1}(\tilde{\mathcal{Q}}_{u+1,\Lambda_{u+1,\theta}}(|y|^{-1}y\tilde{U})) \right| \frac{|\Omega(y/|y|)|}{|y|^n} dy \\ &\leq C \|\Omega\|_{L^1(S^{n-1})} 2^{km_{r_{u+1}}} |\varrho^{\Lambda_{u+1,\theta}} L^{(\Lambda_{u+1,\theta})}(\xi)|. \end{aligned} \tag{3.18}$$

Define the sequence of linear transformations $\{L_s\}_{s=1}^{\varpi(\lambda)}$ by

$$L_s(\xi) := \begin{cases} \varrho^{\Lambda_{u+1,\theta}} L^{(\Lambda_{u+1,\theta})}(\xi), & s = \varpi(u) + \theta, 0 \leq u \leq \lambda - 1, 0 < \theta < \varpi(u + 1) - \varpi(u), \\ \varrho^{(n-1)/(2\epsilon_u)} R_u(\xi^u), & s = \varpi(u), 1 \leq u \leq \lambda. \end{cases}$$

Also, we define $N_1, \dots, N_{\varpi(\lambda)}$ and $\eta_1, \dots, \eta_{\varpi(\lambda)}$ by

$$\begin{aligned} N_s &:= \begin{cases} \gamma_{u,\theta}, & s = \varpi(u) + \theta, 0 \leq u \leq \lambda - 1, 0 < \theta < \varpi(u + 1) - \varpi(u), \\ \epsilon_u, & s = \varpi(u), 1 \leq u \leq \lambda; \end{cases} \\ \eta_s &:= \begin{cases} m_{r_{u+1}}, & s = \varpi(u) + \theta, 0 \leq u \leq \lambda - 1, 0 < \theta < \varpi(u + 1) - \varpi(u), \\ m_{r_u}, & s = \varpi(u), 1 \leq u \leq \lambda. \end{cases} \end{aligned}$$

It follows from (3.11) and (3.13)–(3.18) that, for any $1 \leq s \leq \varpi(\lambda)$,

$$\max \left\{ \left| \widehat{\sigma_{k,s}}(\xi) - \widehat{\sigma_{k,s-1}}(\xi) \right|, \left| \widehat{\sigma_{k,s}}(\xi) - \widehat{\sigma_{k,s-1}}(\xi) \right| \right\} \leq C |2^{k\eta_s} L_s(\xi)|; \tag{3.19}$$

$$\max\{|\widehat{\sigma_{k,s}}(\xi)|, |\widehat{\sigma_{k,s}}(\xi)|\} \leq C \min\{1, |2^{k\eta_s} L_s(\xi)|^{-N_s}\}. \tag{3.20}$$

On the other hand, we get from (3.7) that

$$\sup_{k \in \mathbb{Z}} |\sigma_{k,0}| * f(x) \leq C|f(x)|. \tag{3.21}$$

From (3.21) we have

$$\left\| \sup_{k \in \mathbb{Z}} |\sigma_{k,0}| * f \right\|_{L^q(\mathbb{R}^d)} \leq C \|f\|_{L^q(\mathbb{R}^d)}, \quad 1 < q \leq \infty. \tag{3.22}$$

Invoking Lemma 3.1, we get from (3.10), (3.11), (3.19), (3.20), and (3.22) that

$$\left\| \sup_{k \in \mathbb{Z}} |\sigma_{k,\varpi}(\lambda)| * f \right\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}, \quad 1 < p < \infty, \tag{3.23}$$

$$\left\| \sup_{k \in \mathbb{Z}} \left| \sum_{j=k}^{\infty} \sigma_{j,\varpi}(\lambda) * f \right| \right\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}, \quad 1 < p < \infty. \tag{3.24}$$

For any $\epsilon > 0$, there exists an integer k such that $2^{k-1} \leq \epsilon < 2^k$. It follows that

$$T_{\Omega, \Phi}^* f \leq \sup_{k \in \mathbb{Z}} |\sigma_{k,\varpi}(\lambda)| * |f| + \sup_{k \in \mathbb{Z}} \left| \sum_{j=k}^{\infty} \sigma_{j,\varpi}(\lambda) * f \right|. \tag{3.25}$$

On the other hand, one can easily check that

$$M_{\Omega, \Phi} f(x) \leq C \sup_{k \in \mathbb{Z}} |\sigma_{k,\varpi}(\lambda)| * |f|(x) \quad \text{for } x \in \mathbb{R}^d. \tag{3.26}$$

Then (3.26) together with (3.23)–(3.25) yields (3.1) and completes the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

This section is devoted to proving Theorem 1.2. Before presenting the proof of Theorem 1.2, let us recall some useful lemmas. In what follows, we set $\mathfrak{R}_d = \{\xi \in \mathbb{R}^d; 1/2 < |\xi| \leq 1\}$.

Lemma 4.1 ([26]) *Let $\alpha \in (0, 1)$, $p \in (1, \infty)$, $q \in (1, \infty]$, and $r \in [1, \min\{p, q\}]$. Then*

$$\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \sim \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_d} |\Delta_{2^{-l}\zeta} f|^r d\zeta \right)^{q/r} \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)}.$$

Lemma 4.2 ([21]) *Let $v \geq 1$, $\Lambda \in \mathbb{N} \setminus \{0\}$, and $\{\sigma_k^s : k \in \mathbb{Z}, 1 \leq s \leq \Lambda\}$ be a family of Borel measures on \mathbb{R}^d with $\sigma_k^0 = 0$ for all $k \in \mathbb{Z}$. Let $|\sigma_k^s|$ be the total variation of σ_k^s . For $1 \leq s \leq \Lambda$, let $\beta_s, \gamma_s > 0$, $M_s \in \mathbb{N} \setminus \{0\}$, and $L_s : \mathbb{R}^d \rightarrow \mathbb{R}^{M_s}$ be linear transformations. Suppose that there exist $C, A > 0$ independent of v such that, for $1 \leq s \leq \Lambda$, $k \in \mathbb{Z}$, and $\xi \in \mathbb{R}^d$, the following conditions are satisfied:*

- (a) $\max\{|\widehat{\sigma_k^s}(\xi) - \widehat{\sigma_k^{s-1}}(\xi)|, |\widehat{\sigma_k^s}(\xi) - |\widehat{\sigma_k^{s-1}}(\xi)||\} \leq CA|2^{kv\gamma_s} L_s(\xi)|^{1/v}$;
- (b) $\max\{|\widehat{\sigma_k^s}(\xi)|, |\widehat{\sigma_k^s}(\xi)|\} \leq CA \min\{1, |2^{kv\gamma_s} L_s(\xi)|^{-\beta_s/v}\}$;

- (c) *There exists $\vartheta \in \mathbb{R}^d$ such that $\sup_{k \in \mathbb{Z}} \|\sigma_k^0 * f(x)\| \leq CA|f(x + \vartheta)|$ for any $k \in \mathbb{Z}$;*
- (i) *Suppose that there exist $p_0, q_0 > 1$ satisfying $(p_0, q_0) \neq (2, 2)$, $1 < r_0 < \min\{p_0, q_0\}$, and $2 \leq u < \infty$ such that the following condition (d) holds for $1 \leq s \leq \Lambda$:*
- (d)

$$\begin{aligned} & \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \left(\sum_{k \in \mathbb{Z}} \|\sigma_k^s * g_{l,\zeta,k}\|^u \right)^{1/u} \right\|_{L^{p_0}(\mathfrak{R}_d)}^{q_0} \right)^{1/q_0} \right\|_{L^{p_0}(\mathbb{R}^d)} \\ & \leq CA \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \left(\sum_{k \in \mathbb{Z}} |g_{l,\zeta,k}|^u \right)^{1/u} \right\|_{L^{r_0}(\mathfrak{R}_d)}^{q_0} \right)^{1/q_0} \right\|_{L^{p_0}(\mathbb{R}^d)}. \end{aligned}$$

Let P_1P_2 be the line segment from P_1 to P_2 with $P_1 = (\frac{1}{2}, \frac{1}{2})$ and $P_2 = (\frac{1}{p_0}, \frac{1}{q_0})$. Then, for $1 \leq s \leq \Lambda$, $\alpha \in (0, 1)$, and $(\frac{1}{p}, \frac{1}{q}) \in P_1P_2 \setminus \{(\frac{1}{p_0}, \frac{1}{q_0})\}$, there exists $C > 0$ independent of A and v such that

$$\left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_d} \sup_{k \in \mathbb{Z}} \|\sigma_k^s * |\Delta_{2^{-l}\zeta} f|\right| d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \leq CA \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)}.$$

- (ii) *Suppose also that the following condition (e) holds for $1 \leq s \leq \Lambda$:*
- (e)

$$\left\| \left(\sum_{j \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} \|\sigma_k^s * g_{k,j}\|^2 \right)^{q_0/2} \right)^{1/q_0} \right\|_{L^{p_0}(\mathbb{R}^d)} \leq CA \left\| \left(\sum_{j \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} |g_{k,j}|^2 \right)^{q_0/2} \right)^{1/q_0} \right\|_{L^{p_0}(\mathbb{R}^d)}.$$

Then, for $\alpha \in (0, 1)$ and $(\frac{1}{p}, \frac{1}{q}) \in P_1P_2 \setminus \{(\frac{1}{p_0}, \frac{1}{q_0}), (\frac{1}{2}, \frac{1}{2})\}$, there exists a constant $C > 0$ independent of A and v such that

$$\left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_d} \sup_{k \in \mathbb{Z}} \left| \sum_{j=k}^\infty \sigma_j^\Lambda * \Delta_{2^{-l}\zeta} f \right| d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \leq CA \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)}.$$

We now give the proof of Theorem 1.2.

Proof of Theorem 1.2 We first prove the boundedness part. One can easily check that $T_{\Omega,\Phi}^*$ and $M_{\Omega,\Phi}$ satisfy condition (1.7). This together with Lemma 4.1 yields that

$$\begin{aligned} \|T_{\Omega,\Phi}^* f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} & \leq C \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_d} |\Delta_{2^{-l}\zeta} (T_{\Omega,\Phi}^* f)| d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \\ & \leq C \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_d} |T_{\Omega,\Phi}^* (\Delta_{2^{-l}\zeta} f)| d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)}, \end{aligned} \tag{4.1}$$

$$\|M_{\Omega,\Phi} f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \leq C \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_d} |M_{\Omega,\Gamma} (\Delta_{2^{-l}\zeta} f)| d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \tag{4.2}$$

for $\alpha \in (0, 1)$ and $1 < p, q < \infty$. By Lemma 2.1 and Minkowski's inequality, it suffices to show that

$$\left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_d} |T_{\Omega,\Phi}^* (\Delta_{2^{-l}\zeta} f)| d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)}, \tag{4.3}$$

$$\left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_d} |M_{\Omega, \Phi}(\Delta_{2^{-l}\zeta} f)| d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)}, \tag{4.4}$$

where Ω is an H^1 atom on S^{n-1} satisfying conditions (i)–(iii) of Definition 2.1. Without loss of generality, we may assume that Ω, Φ are given as in the proof of Theorem 1.1. We also let $\{\sigma_{k,s}\}_{s=0}^{\varpi(\lambda)}$ and $\{|\sigma_{k,s}|\}_{s=1}^{\varpi(\lambda)}$ be defined as in the proof of Theorem 1.1. By [22, Lemmas 2.4–2.5], we obtain that, for any $1 \leq s \leq \varpi(\lambda)$ and $1 < p, q, r < \infty$,

$$\left\| \left(\sum_{j \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} |\sigma_{k,s} * g_{k,j}|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \leq C \left\| \left(\sum_{j \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} |g_{k,j}|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)}; \tag{4.5}$$

$$\begin{aligned} & \left\| \left(\sum_{j \in \mathbb{Z}} \left\| \left(\sum_{k \in \mathbb{Z}} |\sigma_{k,s}| * g_{j,\zeta,k}|^2 \right)^{1/2} \right\|_{L^r(\mathfrak{R}_d)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \\ & \leq C \left\| \left(\sum_{j \in \mathbb{Z}} \left\| \left(\sum_{k \in \mathbb{Z}} |g_{j,\zeta,k}|^2 \right)^{1/2} \right\|_{L^r(\mathfrak{R}_d)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)}. \end{aligned} \tag{4.6}$$

By (3.10), (3.19)–(3.21), (4.5), (4.6) and applying Lemma 4.2, we have that, for $\alpha \in (0, 1)$ and $1 < p, q < \infty$,

$$\left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_d} \sup_{k \in \mathbb{Z}} |\sigma_{k,\varpi(\lambda)}| * |\Delta_{2^{-l}\zeta} f| d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)}, \tag{4.7}$$

$$\left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_d} \sup_{k \in \mathbb{Z}} \left| \sum_{j=k}^\infty \sigma_{j,\varpi(\lambda)} * \Delta_{2^{-l}\zeta} f \right| d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)}. \tag{4.8}$$

Then (4.1) follows from (3.25), (4.7), and (4.8). Inequality (3.26) together with (4.7) yields (4.2). One can easily check that both the operators $T_{\Omega, \Phi}^*$ and $M_{\Omega, \Phi}$ satisfy conditions (1.7) and (1.8). This together with (1.4), (4.1), (4.2), Theorem 1.1, and the arguments similar to those used to derive the continuity part in [22, Theorem 1.1]. We omit the details. \square

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Authors’ contributions

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