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A natural Frenet frame for null curves on the lightlike cone in Minkowski space $\mathbb{R}_2^4$

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Abstract

In this paper, we investigate the representation of curves on the lightlike cone $\mathbb{Q}_2^3$ in Minkowski space $\mathbb{R}_2^4$ by structure functions. In addition, with this representation, we classify all of the null curves on the lightlike cone $\mathbb{Q}_2^3$ in four types, and we obtain a natural Frenet frame for these null curves. Furthermore, for this natural Frenet frame, we calculate curvature functions of a null curve, especially the curvature function $\kappa_2 = 0$, and we show that any null curve on the lightlike cone is a helix. Finally, we find all curves with constant curvature functions.

MSC: 51B20; 53A35; 53B30

Keywords: Constant curvature function; Helix; Lightlike cone; Natural Frenet frame; Null curve

1 Introduction

The study of semi-Riemannian manifolds plays an important role in differential geometry and physics, especially in the theory of relativity. In a semi-Riemannian manifold, the induced metric on a lightlike submanifold is degenerate. In general relativity, lightlike submanifolds usually appear to be some smooth parts of the achronal boundaries, for example, event horizon of Kruskal and Kerr black holes and the compact Cauchy horizons in Taub-NUT spacetime [5, 14]. One of the simplest examples of lightlike submanifolds is the lightlike cone $\mathbb{Q}_q^n$ in Minkowski space $\mathbb{R}_q^n$.

In differential geometry, one of the most important and applicable tools to analyse a curve is orthonormal frame. For a regular curve in Euclidean space $\mathbb{R}^n$, we can use $1\text{st}, 2\text{nd}, \dots, n\text{th}$ derivative vectors to construct Frenet frame [8]. Abazari, Bohner, Sağer, and Yayli [2] studied the relationship between Frenet elements of the stationary acceleration curve in four-dimensional Euclidean space. Also, by Frenet elements, they have provided a necessary and sufficient condition for a curve on a timelike surface that is an acceleration curve.

Bonnor [4] introduced Cartan frame as the most useful frame, and he used this frame to study null curves. Bejancu [3] gave a method for consideration of a null curve in semi-Riemannian manifold. Ferrández, Giménez and Lucas [7] generalized the Cartan frame to Lorentzian space form. Abazari, Bohner, Sağer, and Sedaghatdoost [1] studied some properties for spacelike curves in lightlike cones of index 1. Liu [12] studied curves in the lightlike cone and gave an asymptotic frame field along the curve and defined cone curvature functions for this frame field. To study the behavior of a curve in two- and three-dimensional lightlike cone, Liu and Meng [13] defined structure functions for a spacelike curve, and by these structure functions, they obtained representation formulas of spacelike curves in the lightlike cones $\mathbb{Q}_1^2$ and $\mathbb{Q}_1^3$ of Lorentzian space $\mathbb{R}_1^3$ and $\mathbb{R}_1^4$, respectively. Also, Külahci, Bektaş and Ergüt [11] considered AW(k)-type curves in the 3-dimensional lightlike cone, and recently Külahci [10] considered spacelike normal curves on the lightlike cones $\mathbb{Q}_1^2$ and $\mathbb{Q}_1^3$. Since for a null curve that lies on the lightlike cone, any order derivative vectors are null vectors [15], for a null curve x on the lightlike cone $\mathbb{Q}_2^3$, there exists a natural Frenet frame $\{x, x', N, W\}$ [6, 15]. Sun and Pei [15] considered null curves on the lightlike cone and unit semi-Euclidean 3-sphere of Minkowski space $\mathbb{R}_2^4$, and they obtained some results on AW(k)-type curves and null Bertrand curves on the lightlike cone $\mathbb{Q}_2^3$.

In this paper, we obtain representation formulas for any curve on the lightlike cone $\mathbb{Q}_2^3$ by structure functions, and by this representation, we classify all null curves on the lightlike cone. Furthermore, for a null curve on the lightlike cone, we construct a natural Frenet frame and calculate its curvature functions. Also, we show that the structure functions and the curvature functions of a null curve on the lightlike cone $\mathbb{Q}_2^3$ satisfy a special second-order differential equation, and by this natural Frenet frame, we conclude that any null curve on the lightlike cone is a helix.

2 Preliminaries

Let $\mathbb{R}^n$ be n-dimensional Euclidean space. For two vectors $v = (v_1, \dots, v_n)$, $w = (w_1, \dots, w_n)$ and $q \in \mathbb{N} \cap [1, n]$, we define the bilinear form

$$\langle v, w\rangle_q := -\sum_{i=1}^{q} v_i w_i + \sum_{i=q+1}^{n} v_i w_i,$$

which is a semi-Riemannian manifold. The resulting semi-Riemannian space is called Minkowski n-space of index q. If $n = 4$, then this is the simplest example of a relativistic spacetime [14]. In Minkowski space $\mathbb{R}_q^n$, we say that a nonzero vector $v \in \mathbb{R}_q^n$ is *spacelike*, *null*, or *timelike* if $\langle v, v\rangle_q$ is *positive*, *zero*, or *negative*, respectively. Also, the vector 0 is spacelike. The norm of $v \in \mathbb{R}_q^n$ is defined by $\|v\| := \sqrt{|\langle v, v\rangle|}$. In Minkowski space $\mathbb{R}_q^n$, there exist three types of hypersurfaces, called $(n-1)$-pseudo-sphere or de Sitter $(n-1)$-space for

$$\mathbb{S}_q^{n-1} := \left\{v \in \mathbb{R}_q^n : \langle v, v\rangle_q = 1\right\},$$

anti-de Sitter $(n-1)$-space for

$$\mathbb{H}_q^{n-1} := \left\{v \in \mathbb{R}_q^n : \langle v, v\rangle_q = -1\right\},$$

and lightlike cone for

$$\mathbb{Q}_q^{n-1} := \left\{v \in \mathbb{R}_q^n : \langle v, v\rangle_q = 0\right\}.$$

The curve $x : I \to \mathbb{R}_q^n$ is called a spacelike curve (null curve or timelike curve) if for any $t_0 \in I$, the velocity vector $\frac{\mathrm{d}x}{\mathrm{d}t}\big|_{t=t_0}$ of the curve is spacelike (null or timelike). Thus, a spacelike or timelike curve can be parameterized by arc length in the sense that $|\langle \frac{dx}{dt}, \frac{dx}{dt}\rangle_q| = 1$. Therefore, the arc length parameter of $x : I \to \mathbb{R}^n_q$ is denoted by s, and we have

$$\left\langle \frac{dx}{ds}, \frac{dx}{ds} \right\rangle_q = \varepsilon,$$

with $\varepsilon = +1$ for a spacelike curve and $\varepsilon = -1$ for a timelike curve. For a null curve with any parameter t, we have $\varepsilon = 0$ [8]. For a null curve $x : I \to \mathbb{Q}^3_2 \subset \mathbb{R}^4_2$, there is a *natural Frenet frame* $\{x(t), \xi(t), N(t), W(t)\}$ satisfying

$$\begin{aligned}
&\langle x(t), x(t)\rangle = \langle x(t), \xi(t)\rangle = \langle x(t), N(t)\rangle = 0,\\
&\langle \xi(t), \xi(t)\rangle = \langle \xi(t), W(t)\rangle = 0,\\
&\langle N(t), N(t)\rangle = \langle W(t), W(t)\rangle = \langle N(t), W(t)\rangle = 0,\\
&\langle \xi(t), N(t)\rangle = \langle x(t), W(t)\rangle = 1,
\end{aligned} \tag{1}$$

where $\xi(t)$ is the tangent vector, $N(t)$ is the unique transversal vector to $\xi(t)$, $W(t)$ is the unique transversal vector to $x(t)$ [15, (2.1)]. The vectors in this natural Frenet frame satisfy the equations

$$\begin{aligned}
&x'(t) = \xi(t),\\
&x''(t) = h(t)\xi(t) + \kappa_1(t)x(t),\\
&N'(t) = -h(t)N(t) + \kappa_2(t)x(t) - W(t),\\
&W'(t) = -\kappa_2(t)\xi(t) - \kappa_1(t)N(t),
\end{aligned} \tag{2}$$

which we call the natural Frenet equations of the null curve x on the lightlike cone $\mathbb{Q}^3_2$, where

$$h(t) = \langle x''(t), N(t)\rangle, \qquad \kappa_1(t) = \langle x''(t), W(t)\rangle, \qquad \kappa_2(t) = \langle N'(t), W(t)\rangle, \tag{3}$$

and the functions h, κ_1, κ_2 are called the curvature functions of x [6]. Duggal and Jin [6, Proposition 4.1 b] showed that x has a parameter such that the curvature function $h = 0$ for the null curve in $\mathbb{R}^n_2$, and, in addition, Sun and Pei [15] proved the following.

Proposition 2.1 (See [15, Proposition 2.1]) *Let x be a curve on the lightlike cone $\mathbb{Q}^3_2$. Then the type of the natural Frenet equations is invariant with respect to transformations of the coordinate neighborhood and the screen vector bundle of x. Moreover, it is possible to find a new parameter for x such that the curvature function $h = 0$ in the Frenet equation (2) of all possible types, using the same screen bundle.*

Hence, we can get the natural Frenet equations of a null curve on the lightlike cone $\mathbb{Q}_2^3$ as

$$\begin{aligned} x'(t) &= \xi(t), \\ x''(t) &= \kappa_1(t)x(t), \\ N'(t) &= \kappa_2(t)x(t) - W(t), \\ W'(t) &= -\kappa_2(t)\xi(t) - \kappa_1(t)N(t). \end{aligned} \tag{4}$$

In this paper, the semi-Riemannian manifold is Minkowski space $\mathbb{R}_2^4$, and especially, we consider its lightlike cone $\mathbb{Q}_2^3$.

3 Representation formulas of cone curves in $\mathbb{R}_2^4$

H. Liu [13] has obtained representation formulas of the spacelike curves in the lightlike cones $\mathbb{Q}_1^2$ and $\mathbb{Q}_1^3$ and proved the following two theorems.

Theorem 3.1 (See [13]) *Let $x : I \to \mathbb{Q}_1^2 \subset \mathbb{R}_1^3$ be a spacelike curve in $\mathbb{Q}_1^2$ parameterized by arc length. Then x can be written as*

$$x = (x_1, x_2, x_3) = \frac{1}{2} f'^{-1}\left(f^2 - 1, 2f, f^2 + 1\right)$$

for some nonconstant function f, which is called the structure function of x. The structure function f and the cone curvature function κ of x satisfy

$$\kappa = \frac{1}{2}\left[\left(\log f'\right)'\right]^2 - \left[\left(\log f'\right)'\right]',$$

where $f' = \frac{\mathrm{d}f}{\mathrm{d}s}$.

Theorem 3.2 (See [13]) *Let $x : I \to \mathbb{Q}_1^3 \subset \mathbb{R}_1^4$ be a spacelike curve in $\mathbb{Q}_1^3$ parameterized by arc length. Then x can be written as*

$$x = (x_1, x_2, x_3, x_4) = \rho\left(2f, 2g, 1 - f^2 - g^2, 1 + f^2 + g^2\right)$$

for some functions f and g, which are called structure functions of x. Here, ρ satisfies

$$4\rho^2\left(f'^2 + g'^2\right) = 1.$$

The structure functions f, g, the cone curvature function κ, and the cone torsion function τ of x satisfy

$$\kappa = \frac{1}{2}\left[(\log\rho)'\right]^2 + (\log f)'' - \frac{1}{2}\theta'^2, \qquad \tau = \pm\left(\theta'(\log\rho)' + \theta''\right),$$

where

$$\theta' = \left(1 + \frac{f'^2}{g'^2}\right)^{-1}\left(\frac{f'}{g'}\right)'.$$

First, in the following lemma, we give a result for spacelike and timelike curves in the lightlike cone Q_2^3. By this virtue, we will be able to give a representation formula for such curves.

Lemma 3.3 *Let $x = (x_1, x_2, x_3, x_4) : I \to \mathbb{Q}_2^3 \subset \mathbb{R}_2^4$ be a spacelike or timelike curve parameterized by arclength in the lightlike cone $\mathbb{Q}_2^3$. Then the functions $x_1 - x_3$, $x_1 + x_3$, $x_4 - x_2$, $x_4 + x_2$ are not zero.*

Proof Since $\langle x, x\rangle_2 = 0$, we get

$$-x_1^2 - x_2^2 + x_3^2 + x_4^2 = 0, \tag{5}$$

and from $x_1^2 - x_3^2 = x_4^2 - x_2^2$, we have

$$(x_1 - x_3)(x_1 + x_3) = (x_4 - x_2)(x_4 + x_2). \tag{6}$$

If one of the functions $x_1 - x_3$, $x_1 + x_3$, $x_4 - x_2$, $x_4 + x_2$ is zero, then

$$x_3 = \mu x_1, \qquad x_4 = \lambda x_2,$$

where $\mu, \lambda \in \{-1, 1\}$. Thus

$$x = (x_1, x_2, \mu x_1, \lambda x_2),$$

and hence,

$$x' = \bigl(x_1', x_2', \mu x_1', \lambda x_2'\bigr).$$

Since $\mu, \lambda \in \{-1, 1\}$, we obtain

$$\bigl\langle x', x'\bigr\rangle_2 = -{x_1'}^2 - {x_2'}^2 + \bigl(\mu x_1'\bigr)^2 + \bigl(\lambda x_2'\bigr)^2 = 0,$$

which is a contradiction. □

Theorem 3.4 *Let $x = (x_1, x_2, x_3, x_4) : I \to \mathbb{Q}_2^3 \subset \mathbb{R}_2^4$ be a spacelike or timelike curve parameterized by arclength in the lightlike cone $\mathbb{Q}_2^3$. Then we can write x as*

$$x = (f + \rho g, g - \rho f, f - \rho g, g + \rho f),$$

for some functions f and g. Here, ρ satisfies

$$4\bigl(fg' - f'g\bigr)\rho' = \varepsilon,$$

where $\varepsilon = +1$ for spacelike curves and $\varepsilon = -1$ for timelike curves.

Proof Since $\langle x,x\rangle_2 = 0$, we get (5). Because the curve x is spacelike or timelike, from Lemma 3.3, we can define the nonzero smooth functions

$$\rho := \frac{x_1 - x_3}{x_4 + x_2} = \frac{x_4 - x_2}{x_1 + x_3}, \qquad f := \frac{x_1 + x_3}{2}, \qquad g := \frac{x_2 + x_4}{2},$$

where

$$x_1 = f + \rho g, \qquad x_2 = g - \rho f, \qquad x_3 = f - \rho g, \qquad x_4 = g + \rho f. \tag{7}$$

Let $\langle x',x'\rangle_2 = \varepsilon$, where $\varepsilon = +1$ for spacelike curves and $\varepsilon = -1$ for timelike curves. Then

$$-x_1'^2 - x_2'^2 + x_3'^2 + x_4'^2 = \varepsilon,$$

so that

$$x_1'^2 + x_2'^2 = x_3'^2 + x_4'^2 - \varepsilon,$$

and by (5), we have

$$\left(x_1^2 + x_2^2\right)\left(x_1'^2 + x_2'^2\right) = \left(x_3^2 + x_4^2\right)\left(x_3'^2 + x_4'^2\right) - \varepsilon\left(x_3^2 + x_4^2\right),$$

and thus

$$\begin{aligned} &\left(x_1x_1'\right)^2 + \left(x_1x_2'\right)^2 + \left(x_2x_1'\right)^2 + \left(x_2x_2'\right)^2 \\ &\quad = \left(x_3x_3'\right)^2 + \left(x_3x_4'\right)^2 + \left(x_4x_3'\right)^2 + \left(x_4x_4'\right)^2 - \varepsilon\left(x_3^2 + x_4^2\right). \end{aligned} \tag{8}$$

From $\langle x',x\rangle_2 = 0$, we have

$$x_1x_1' + x_2x_2' = x_3x_3' + x_4x_4',$$

and hence

$$\left(x_1x_1' + x_2x_2'\right)^2 = \left(x_3x_3' + x_4x_4'\right)^2,$$

so that

$$\left(x_1x_1'\right)^2 + \left(x_2x_2'\right)^2 + 2x_1x_1'x_2x_2' = \left(x_3x_3'\right)^2 + \left(x_4x_4'\right)^2 + 2x_3x_3'x_4x_4'. \tag{9}$$

From (8) and (9), we conclude

$$\left(x_1x_2' - x_2x_1'\right)^2 = \left(x_3x_4' - x_4x_3'\right)^2 - \varepsilon\left(x_3^2 + x_4^2\right). \tag{10}$$

By replacing (7) in (10), we have

$$\begin{aligned} \left((f + \rho g)(g - \rho f)' - (g - \rho f)(f + \rho g)'\right)^2 &= \left((f - \rho g)(g + \rho f)' - (g + \rho f)(f - \rho g)'\right)^2 \\ &\quad - \varepsilon\left((f - \rho g)^2 + (g + \rho f)^2\right), \end{aligned}$$

i.e.,

$$\begin{aligned}&\left(fg' + \rho gg' - \rho ff' - \rho' f^2 - \rho^2 f' g - \rho\rho' fg - f' g + \rho ff' - \rho gg' - \rho' g^2 + \rho^2 fg' + \rho\rho' fg\right)^2\\&\quad = \left(fg' - \rho gg' + \rho ff' + \rho' f^2 - \rho^2 f' g - \rho\rho' fg - f' g - \rho ff' + \rho gg' + \rho' g^2 + \rho^2 fg' + \rho\rho' fg\right)^2\\&\qquad - \varepsilon\left(f^2 - 2\rho fg + \rho^2 g^2 + g^2 + 2\rho fg + \rho^2 f^2\right),\end{aligned}$$

i.e.,

$$\begin{aligned}\left(fg' - \rho' f^2 - \rho^2 f' g - f' g - \rho' g^2 + \rho^2 fg'\right)^2 &= \left(fg' + \rho' f^2 - \rho^2 f' g - f' g + \rho' g^2 + \rho^2 fg'\right)^2\\&\quad - \varepsilon\left(f^2 + \rho^2 g^2 + g^2 + \rho^2 f^2\right),\end{aligned}$$

i.e.,

$$\left(\left(fg' - f' g\right)\left(1+\rho^2\right) - \left(f^2 + g^2\right)\rho'\right)^2 = \left(\left(fg' - f' g\right)\left(1+\rho^2\right) + \left(f^2 + g^2\right)\rho'\right)^2 - \varepsilon\left(f^2 + g^2\right)\left(1+\rho^2\right),$$

i.e.,

$$4\left(fg' - f' g\right)\left(1 + \rho^2\right)\left(f^2 + g^2\right)\rho' = \varepsilon\left(1 + \rho^2\right)\left(f^2 + g^2\right).$$

Since f and g are not zero, the proof is complete. □

Unlike in Theorem 3.4, for a null curve in the lightlike cone, some of the functions $x_1 - x_3$, $x_1 + x_3$, $x_4 - x_2$, $x_4 + x_2$ may be zero. Thus, in the following, we state these cases separately.

Proposition 3.5 *Let $x : I \to \mathbb{Q}^3_2 \subset \mathbb{R}^4_2$ be a null curve in the lightlike cone $\mathbb{Q}^3_2$.*

i. *If only one of the functions $x_1 - x_3$, $x_1 + x_3$, $x_4 - x_2$, $x_4 + x_2$ is not zero, then x is a null straight line.*
ii. *It is impossible that only one of the functions $x_1 - x_3$, $x_1 + x_3$, $x_4 - x_2$, $x_4 + x_2$ is zero and the others are not zero.*

Proof First, we prove i. Without loss of generality, we suppose

$$x_1 + x_3 = 0, \qquad x_1 - x_3 = 0, \qquad x_4 + x_2 = 0, \qquad x_4 - x_2 \neq 0.$$

From $x_1 + x_3 = x_1 - x_3 = 0$, we have $x_1 = x_3 = 0$, and $x_4 + x_2 = 0$ implies $x_4 = -x_2$. Hence

$$x = (0, x_2, 0, -x_2) = (0, 1, 0, -1)x_2,$$

which is a null straight line. Next, we prove ii. Let x be a null curve in the lightlike cone $\mathbb{Q}^3_2$. Without loss of generality, we suppose

$$x_1 + x_3 \neq 0, \qquad x_1 - x_3 \neq 0, \qquad x_4 + x_2 \neq 0, \qquad x_4 - x_2 = 0.$$

Thus $x_1^2 - x_3^2 \neq 0$, and $x_4^2 - x_2^2 = 0$, and this contradicts $\langle x, x\rangle_2 = 0$. □

In the following theorem, we classify all null curves in the lightlike cone $\mathbb{Q}^3_2$, which are not a straight line.

Theorem 3.6 *Let $x : I \to \mathbb{Q}^3_2 \subset \mathbb{R}^4_2$ be a null curve in the lightlike cone $\mathbb{Q}^3_2$ such that it is not a straight line. Then $x = (x_1, x_2, x_3, x_4)$ can be written as one of the following types.*

Type 1. $x_1 + x_3 = 0$ *or* $x_2 + x_4 = 0$. *In this case, we have three types of curves*:

Type 1.1. *If* $x_1 + x_3 = 0$ *and* $x_2 + x_4 = 0$, *then* $x = (f, g, -f, -g)$.

Type 1.2. *If* $x_1 + x_3 = 0$ *and* $x_2 + x_4 \neq 0$, *then* $x = (f, g, -f, g)$.

Type 1.3. *If* $x_1 + x_3 \neq 0$ *and* $x_2 + x_4 = 0$, *then* $x = (f, g, f, -g)$.

Here, the functions f and g are linearly independent.

Type 2. *If the functions $x_1 + x_3$ and $x_2 + x_4$ are not zero, then*

$$x = (f + \rho g, g - \rho f, f - \rho g, g + \rho f),$$

where the smooth functions ρ, f, and g are

$$\rho = \frac{x_1 - x_3}{x_4 + x_2} = \frac{x_4 - x_2}{x_1 + x_3}, \qquad f = \frac{x_1 + x_3}{2}, \qquad g = \frac{x_2 + x_4}{2},$$

and satisfy $(fg' - f'g)\rho' = 0$. In this case, we have two types of curves:

Type 2.1. *If the functions f and g are linearly independent, then*

$$x = (f + \lambda g, g - \lambda f, f - \lambda g, g + \lambda f),$$

where λ is a real constant.

Type 2.2. *If the functions f and g are colinear, then*

$$x = (\lambda + \rho, 1 - \lambda\rho, \lambda - \rho, 1 + \lambda\rho)g,$$

where λ is a real constant.

Proof Type 1.1. If $x_1 + x_3 = x_4 + x_2 = 0$, then, by Proposition 3.5 i., $x_1 - x_3$ and $x_4 - x_2$ are not zero. Putting $f := x_1 = -x_3$ and $g := x_2 = -x_4$, we get

$$x = (f, g, -f, -g).$$

Type 1.2. If $x_1 + x_3 = 0$ and $x_4 + x_2 \neq 0$, then, from (6), we have $x_4 - x_2 = 0$. Since x is not a straight line, by Proposition 3.5, $x_1 - x_3 \neq 0$. Setting $f := x_1 = -x_3$ and $g := x_2 = x_4$, we get

$$x = (f, g, -f, g).$$

Type 1.3. If $x_1 + x_3 \neq 0$ and $x_4 + x_2 = 0$, then, from (6), we have $x_1 - x_3 = 0$. Since x is not a straight line, by Proposition 3.5 i., $x_4 - x_2 \neq 0$. Putting $f := x_1 = x_3$ and $g := x_2 = -x_4$, we obtain

$$x = (f, g, f, -g).$$

Types 1.1–1.3. In all Type 1 cases, if f and g are linearly dependent, then x is a straight line. Thus, the smooth functions f and g are linearly independent.

Type 2. In the proof of Theorem 3.4, from $x_1 + x_3 \neq 0$, $x_4 + x_2 \neq 0$, and $\langle x, x\rangle_2 = \varepsilon$, we conclude

$$x = (f + \rho g, g - \rho f, f - \rho g, g + \rho f), \qquad 4\big(fg' - f'g\big)\rho' = \varepsilon.$$

In a similar way, if we set $\varepsilon = 0$, then we can prove

$$x = (f + \rho g, g - \rho f, f - \rho g, g + \rho f), \qquad \big(fg' - f'g\big)\rho' = 0. \tag{11}$$

Thus $\rho' = 0$ or $fg' - f'g = 0$.

Type 2.1. If $\rho' = 0$ and $fg' - f'g \neq 0$, then $\rho \equiv \lambda$ is a real constant and the functions f and g in the first equation of (11) are linearly independent, and hence

$$x = (f + \rho g, g - \rho f, f - \rho g, g + \rho f).$$

Type 2.2. If $\rho' \neq 0$ and $fg' - f'g = 0$, then the functions f and g in the first equation of (11) are linearly dependent. Thus $f = \lambda g$, where λ is a real constant, and hence

$$x = (\lambda + \rho, 1 - \lambda\rho, \lambda - \rho, 1 + \lambda\rho)g.$$

If $\rho' = 0$ and $fg' - f'g = 0$, then x is a straight line, contradicting the assumption. □

In Table 1, we summarize all cases on the basis of (6).

Definition 3.7 For Types 1.1–1.3 and Type 2.1, the functions f and g in Theorem 3.6 are called the *structure functions* of the cone curve $x : I \to \mathbb{Q}^3_2 \subset \mathbb{R}^4_2$. A null curve of Type 2.2 by structure functions ρ and g writes as $x = (\lambda + \rho, 1 - \lambda\rho, \lambda - \rho, 1 + \lambda\rho)g$.

For a Frenet null curve in the lightlike cone $\mathbb{Q}^3_2$, there exists a nonunique natural Frenet frame $\{x, \xi, N, W\}$ that satisfies (1) [15]. In the following, we construct the vector fields N and W for one of the natural Frenet frames. Now we define the natural orthogonal vector field to a null vector field in $\mathbb{R}^4_2$ and prove some of its properties.

Table 1 All types of null curves in the lightlike cone $\mathbb{Q}^3_2$

Cases	$x_1 - x_3$	$x_1 + x_3$	$x_4 - x_2$	$x_4 + x_2$	The type of the curve
1	0	0	0	0	Impossible
2	0	0	0	$\neq 0$	Straight line
3	0	0	$\neq 0$	0	Straight line
4	0	0	$\neq 0$	$\neq 0$	Impossible
5	0	$\neq 0$	0	0	Straight line
6	0	$\neq 0$	0	$\neq 0$	Type 2
7	0	$\neq 0$	$\neq 0$	0	Type 1.3
8	0	$\neq 0$	$\neq 0$	$\neq 0$	Impossible
9	$\neq 0$	0	0	0	Straight line
10	$\neq 0$	0	0	$\neq 0$	Type 1.2
11	$\neq 0$	0	$\neq 0$	0	Type 1.1
12	$\neq 0$	0	$\neq 0$	$\neq 0$	Impossible
13	$\neq 0$	$\neq 0$	0	0	Impossible
14	$\neq 0$	$\neq 0$	0	$\neq 0$	Impossible
15	$\neq 0$	$\neq 0$	$\neq 0$	0	Impossible
16	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	Type 2

Definition 3.8 Let $V = (v_1, v_2, v_3, v_4)$ be a null vector in Minkowski space $\mathbb{R}^4_2$. Then the vectors $V^{\perp}_{13} := (v_2, -v_1, v_4, -v_3)$ and $V^{\perp}_{14} := (v_2, -v_1, -v_4, v_3)$ also are null vectors such that $\langle V, V^{\perp}_{1i}\rangle_2 = 0$, $i = 3, 4$. We call such null vectors natural orthogonal null vectors to V. If there is no confusion, instead of $V^{\perp}_{1i}$, $i = 3, 4$, we write $V^{\perp}$.

The following lemma is a direct consequence of Definition 3.8.

Lemma 3.9 *Let U, V be null vectors in Minkowski space $\mathbb{R}^4_2$. Then we have the following*:

i. $\langle U, V\rangle_2 = \langle U^{\perp}_{1i}, V^{\perp}_{1i}\rangle_2$, $i = 3, 4$.
ii. $\langle U, V^{\perp}_{1i}\rangle_2 = -\langle U^{\perp}_{1i}, V\rangle_2$, $i = 3, 4$.

Theorem 3.10 *Let $x = (x_1, x_2, x_3, x_4) : I \to \mathbb{Q}^3_2 \subset \mathbb{R}^4_2$ be a null curve in the lightlike cone $\mathbb{Q}^3_2$. Then the vector fields N and W of the natural Frenet frame $\{x, \xi, N, W\}$ for curves of the various types are as follows*:

Type 1.1. *If $x = (f, g, -f, -g)$ and $\Theta = fg' - f'g$, then*

$$N = \frac{1}{2\Theta}(g, -f, g, -f) = \frac{1}{2\Theta}x^{\perp}_{14},$$
$$W = \frac{-1}{2\Theta}\left(g', -f', g', -f'\right) = \frac{-1}{2\Theta}x'^{\perp}_{14}.$$

Type 1.2. *If $x = (f, g, -f, g)$ and $\Theta = fg' - f'g$, then*

$$N = \frac{1}{2\Theta}(g, -f, g, f) = \frac{1}{2\Theta}x^{\perp}_{13},$$
$$W = \frac{-1}{2\Theta}\left(g', -f', g', f'\right) = \frac{-1}{2\Theta}x'^{\perp}_{13}.$$

Type 1.3. *If $x = (f, g, f, -g)$ and $\Theta = fg' - f'g$, then*

$$N = \frac{1}{2\Theta}(g, -f, -g, -f) = \frac{1}{2\Theta}x^{\perp}_{13},$$
$$W = \frac{-1}{2\Theta}\left(g', -f', -g', -f'\right) = \frac{-1}{2\Theta}x'^{\perp}_{13}.$$

Type 2.1. *If $x = (f + \lambda g, g - \lambda f, f - \lambda g, g + \lambda f)$, then*

$$N = \frac{1}{2(1+\lambda^2)\Theta}(g - \lambda f, -f - \lambda g, -g - \lambda f, f - \lambda g) = \frac{1}{2(1+\lambda^2)\Theta}x^{\perp}_{14},$$
$$W = \frac{-1}{2(1+\lambda^2)\Theta}\left(g' - \lambda f', -f' - \lambda g', -g' - \lambda f', f' - \lambda g'\right) = \frac{-1}{2(1+\lambda^2)\Theta}x'^{\perp}_{14},$$

where $\Theta = fg' - f'g$.

Type 2.2. *If $x = Ag$, where*

$$A := (\lambda + \rho, 1 - \lambda\rho, \lambda - \rho, 1 + \lambda\rho),$$

then

$$N = \frac{-1}{2(1+\lambda^2)\rho' g}(1 - \lambda\rho, -\lambda - \rho, 1 + \lambda\rho, -\lambda + \rho) = \frac{-1}{2(1+\lambda^2)\rho' g^2}x^{\perp}_{13},$$

$$W = \frac{1}{2(1+\lambda^2)\rho' g^2}\left(g' A_{13}^{\perp} + g\rho'(-\lambda,-1,\lambda,1)\right) = \frac{1}{2(1+\lambda^2)\rho' g^2}x'^{\perp}_{13}.$$

Proof Since the vector fields N and W are defined by natural orthogonal vectors to x and x', we have

$$\langle x, N\rangle_2 = \frac{1}{2\Theta}\langle x, x^{\perp}\rangle_2 = 0, \qquad \langle x', W\rangle_2 = \frac{-1}{2\Theta}\langle x', x'^{\perp}\rangle_2 = 0.$$

Moreover, since $\langle x, x'\rangle_2 = 0$, by Lemma 3.9 i., for any type of null curves, we have

$$\langle x, \xi\rangle_2 = \langle N, W\rangle_2 = 0.$$

In order to obtain the vector fields N and W, it is thus sufficient to calculate $\langle x^{\perp}, x'\rangle_2$ in all types of null curves.

For a null curve of Type 1.1, we have

$$\begin{aligned}\langle x_{14}^{\perp}, x'\rangle_2 &= \left\langle (g,-f,g,-f), \left(f', g', -f', -g'\right)\right\rangle_2\\ &= -f'g + fg' - f'g + fg'\\ &= 2\left(fg' - f'g\right) = 2\Theta.\end{aligned}$$

From Theorem 3.6, f and g are not proportional, so $\Theta \neq 0$, and by Lemma 3.9 ii., we have

$$\begin{aligned}N &= \frac{1}{2\Theta}(g,-f,g,-f) = \frac{1}{2\Theta}x_{14}^{\perp},\\ W &= \frac{-1}{2\Theta}\left(g',-f',g',-f'\right) = \frac{-1}{2\Theta}x'^{\perp}_{14}.\end{aligned}$$

Similarly, the calculations for null curves of Type 1.2 and Type 1.3 are valid.

For a curve of Type 2.1, by direct calculations, we have

$$\begin{aligned}\langle x_{14}^{\perp}, x'\rangle_2 &= \left\langle (g-\lambda f, -f-\lambda g, -g-\lambda f, f-\lambda g), \left(f'+\lambda g', g'-\lambda f', f'-\lambda g', g'+\lambda f'\right)\right\rangle\\ &= (\lambda f - g)\left(f'+\lambda g'\right) + (f+\lambda g)\left(g'-\lambda f'\right)\\ &\quad - (g+\lambda f)\left(f'-\lambda g'\right) + (f-\lambda g)\left(g'+\lambda f'\right)\\ &= \lambda ff' + \lambda^2 fg' - f'g - \lambda gg' + fg' - \lambda ff' + \lambda gg' - \lambda^2 f'g\\ &\quad - f'g + \lambda gg' - \lambda ff' + \lambda^2 fg' + fg' + \lambda ff' - \lambda gg' - \lambda^2 f'g\\ &= 2\left(1+\lambda^2\right)\left(fg' - f'g\right) = 2\left(1+\lambda^2\right)\Theta.\end{aligned}$$

This equality for a null curve of Type 2.1, by Lemma 3.9 ii., yields

$$\begin{aligned}N &= \frac{1}{2(1+\lambda^2)\Theta}(g-\lambda f, -f-\lambda g, -g-\lambda f, f-\lambda g) = \frac{1}{2(1+\lambda^2)\Theta}x_{14}^{\perp},\\ W &= \frac{-1}{2(1+\lambda^2)\Theta}\left(g'-\lambda f', -f'-\lambda g', -g'-\lambda f', f'-\lambda g'\right) = \frac{-1}{2(1+\lambda^2)\Theta}x'^{\perp}_{14}.\end{aligned}$$

For a null curve of Type 2.2, we have $x = Ag$, where

$$A = (\lambda+\rho, 1-\lambda\rho, \lambda-\rho, 1+\lambda\rho).$$

Moreover,

$$\langle A_{13}^{\perp}, A' \rangle_2 = -2(1+\lambda^2)\rho' g,$$

and by Lemma 3.9, we have

$$N = \frac{-1}{2(1+\lambda^2)\rho' g}(1-\lambda\rho, -\lambda-\rho, 1+\lambda\rho, -\lambda+\rho) = \frac{-1}{2(1+\lambda^2)\rho' g^2} x_{13}^{\perp},$$
$$W = \frac{1}{2(1+\lambda^2)\rho' g^2}\big(g' A_{13}^{\perp} + g\rho'(-\lambda, -1, \lambda, 1)\big) = \frac{1}{2(1+\lambda^2)\rho' g^2} x'^{\perp}_{13}.$$

This completes the proof. □

Theorem 3.11 *Let $x = (x_1, x_2, x_3, x_4) : I \to \mathbb{Q}_2^3 \subset \mathbb{R}_2^4$ be a null curve in the lightlike cone $\mathbb{Q}_2^3$ with natural Frenet frame $\{x, \xi, N, W\}$. Then the curvature functions of the curve x in the natural Frenet equations (2) for a null curve of Types* 1.1, 1.2, 1.3 *or Type* 2.1 *are*

$$h = \frac{fg'' - f''g}{fg' - f'g}, \qquad \kappa_1 = \frac{f''g' - f'g''}{fg' - f'g}, \qquad \kappa_2 = 0,$$

and for a null curve of Type 2.2 *are*

$$h = \frac{\rho''}{\rho'} + \frac{2g'}{g}, \qquad \kappa_1 = \frac{g''\rho' - g'\rho''}{g\rho'} - 2\left(\frac{g'}{g}\right)^2, \qquad \kappa_2 = 0.$$

Proof By using (3) and direct calculations, we can obtain the curvature functions. For example, the curvature functions for a null curve of Type 2.1 are

$$\begin{aligned} h &= \langle x'', N \rangle_2 \\ &= \left\langle x'', \frac{1}{2(1+\lambda^2)\Theta} x^{\perp} \right\rangle_2 \\ &= \frac{1}{2(1+\lambda^2)\Theta} 2(1+\lambda^2)(fg'' - f''g) \\ &= \frac{fg'' - f''g}{fg' - f'g} \end{aligned}$$

and

$$\begin{aligned} \kappa_1 &= \langle x'', W \rangle_2 \\ &= \left\langle x'', \frac{-1}{2(1+\lambda^2)\Theta} x'^{\perp} \right\rangle_2 \\ &= \frac{1}{2(1+\lambda^2)\Theta} 2(1+\lambda^2)(f''g' - f'g'') \\ &= \frac{f''g' - f'g''}{fg' - f'g}. \end{aligned}$$

For a null curve of Type 2.2, let $x = Ag$. Then

$$x'' = B(g\rho'' + 2g'\rho') + Ag'',$$

where $B := \frac{1}{\rho'}A'$, ${A'_{13}}^{\perp} = B_{13}^{\perp}\rho'$, and $\langle B, A_{13}^{\perp}\rangle_2 = -2(1+\lambda^2)$. Therefore,

$$\begin{aligned}
h &= \langle x'', N\rangle_2 \\
&= \left\langle B\left(\rho'' g + 2\rho' g'\right) + Ag'', \frac{-1}{2(1+\lambda^2)\rho' g} A_{13}^{\perp}\right\rangle_2 \\
&= \frac{\rho''}{\rho'} + \frac{2g'}{g}
\end{aligned}$$

and

$$\begin{aligned}
\kappa_1 &= \langle x'', W\rangle_2 \\
&= \left\langle B\left(\rho'' g + 2\rho' g'\right) + Ag'', \frac{1}{2(1+\lambda^2)\rho' g^2}\left(g' A_{13}^{\perp} + g{A'_{13}}^{\perp}\right)\right\rangle_2 \\
&= \left\langle B\left(\rho'' g + 2\rho' g'\right) + Ag'', \frac{g'}{2(1+\lambda^2)\rho' g^2} A_{13}^{\perp}\right\rangle_2 \\
&\quad + \left\langle B\left(\rho'' g + 2\rho' g'\right) + Ag'', \frac{\rho' g}{2(1+\lambda^2)\rho' g} B_{13}^{\perp}\right\rangle_2 \\
&= -\left(\frac{\rho'' g'}{\rho' g} + \frac{2g'^2}{g^2}\right) + \frac{g''}{g} \\
&= \frac{g''\rho' - \rho'' g'}{\rho' g} - 2\left(\frac{g'}{g}\right)^2.
\end{aligned}$$

Since $\langle x, x'\rangle_2 = \langle x', x'\rangle_2 = 0$, by Lemma 3.9 i., $\langle x^{\perp}, {x'}^{\perp}\rangle_2 = \langle {x'}^{\perp}, {x'}^{\perp}\rangle_2 = 0$, so for any type of null curve, we have $\kappa_2 = \langle N', W\rangle = 0$. □

If we use the parameter $\tilde{s}$ presented in Proposition 2.1 for the natural Frenet frame obtained in Theorem 3.10, together with Theorem 3.11 that gives $\kappa_2 = 0$ for any type of null curve, the natural Frenet equations (4) are

$$\begin{aligned}
&x'(\tilde{s}) = \xi(\tilde{s}), \\
&x''(\tilde{s}) = \kappa(\tilde{s})x(\tilde{s}), \\
&N'(\tilde{s}) = -W(\tilde{s}), \\
&W'(\tilde{s}) = -\kappa(\tilde{s})N(\tilde{s}),
\end{aligned} \tag{12}$$

where κ_1 is now denoted by κ.

Lemma 3.12 *Let $x : I \to \mathbb{Q}_2^3 \subset \mathbb{R}_2^4$ be a null curve parameterized by $\tilde{s}$ such that its tangent vector field has a constant angle with a constant vector β. Then the curve x is a straight line or β is in the plane generated by* $\mathrm{Span}\{x(\tilde{s}), x'(\tilde{s})\}$.

Proof For any straight line, the statement is true. Thus, we assume that the curvature function $\kappa(\tilde{s}) \neq 0$. Let β be a constant vector such that $\langle x'(\tilde{s}), \beta\rangle_2 = l$ for any $\tilde{s} \in I$. Thus, $\langle x(\tilde{s}), \beta\rangle_2 = l\tilde{s} + l_0$ and $\langle x''(\tilde{s}), \beta\rangle_2 = 0$. From (12), we conclude that $\kappa(\tilde{s})\langle x(\tilde{s}), \beta\rangle_2 = 0$, and from $\kappa(\tilde{s}) \neq 0$, we get $l = l_0 = 0$ so that

$$\beta = r_1(\tilde{s})x(\tilde{s}) + r_2(\tilde{s})x'(\tilde{s}).$$

Hence $r_1(\tilde{s}) = \langle W(\tilde{s}), \beta\rangle_2$ and $r_2(\tilde{s}) = \langle N(\tilde{s}), \beta\rangle_2$. □

Theorem 3.13 *Any null curve on the lightlike cone $\mathbb{Q}_2^3$ is a helix.*

Proof For a straight line, the tangent vector has constant angle with any constant vector. Let β be a constant vector such that $\langle x'(\tilde{s}), \beta\rangle_2 = l$. Then, by Lemma 3.12,

$$\beta = r_1(\tilde{s})x(\tilde{s}) + r_2(\tilde{s})x'(\tilde{s}). \tag{13}$$

By differentiation of (13), we have

$$r_1'(\tilde{s})x(\tilde{s}) + r_1(\tilde{s})x'(\tilde{s}) + r_2'(\tilde{s})x'(\tilde{s}) + r_2(\tilde{s})x''(\tilde{s}) = 0,$$

and by the second equation of (12), we get

$$\bigl(r_1'(\tilde{s}) + \kappa(\tilde{s})r_2(\tilde{s})\bigr)x(\tilde{s}) + \bigl(r_1(\tilde{s}) + r_2'(\tilde{s})\bigr)x'(\tilde{s}) = 0.$$

Since x and x' are linearly independent, we obtain

$$r_1'(\tilde{s}) + \kappa(\tilde{s})r_2(\tilde{s}) = 0, \qquad r_1(\tilde{s}) + r_2'(\tilde{s}) = 0,$$

and thus

$$r_2''(\tilde{s}) = -r_1'(\tilde{s}) = \kappa(\tilde{s})r_2(\tilde{s}).$$

Again by the second equation of (12), since $x_i(\tilde{s})$, $i = 1, \dots, 4$, and $r_2(\tilde{s})$ satisfy the same differential equation, we can choose $r_2(\tilde{s}) = x_i(\tilde{s})$, $i = 1, \dots, 4$. In this situation, for $r_2(\tilde{s}) := x_1(\tilde{s})$, the constant vector β is

$$\beta = \bigl(0, x_1x_2' - x_2x_1', x_1x_3' - x_3x_1', x_1x_4' - x_4x_1'\bigr).$$

The equations ${x_1'}^2 + {x_2'}^2 = {x_3'}^2 + {x_4'}^2$ and $x_1x_1' + x_2x_2' = x_3x_3' + x_4x_4'$ yield $\langle x'(\tilde{s}), \beta\rangle_2 = 0$. □

Kula and Yayli, in [9, Proposition 6.1], proved that, if $x : \mathbb{R} \to \mathbb{R}^4$ satisfies a second-order linear homogeneous differential equation, then the image of x lies in a two-dimensional subspace of $\mathbb{R}^4$. Thus, we can prove the following corollary.

Corollary 3.14 *Let $x : I \to \mathbb{Q}_2^3 \subset \mathbb{R}_2^4$ be a null curve in the lightlike cone $\mathbb{Q}_2^3$. Then the image of x lies in a 2-dimensional subspace of $\mathbb{R}^4$.*

Proof Since x satisfies the second-order linear homogeneous differential equation in (12), by [9, Proposition 6.1], the proof is complete. □

4 Curves with special curvature functions

In this section, we classify all curves with constant curvature functions on the lightlike cone $\mathbb{Q}^3_2$. These curves are solutions of a second-order differential equation with constant coefficients. From now on, we assume that the parameter $\tilde{s}$ of the curve $x(\tilde{s})$ satisfies Proposition 2.1, i.e., $h = 0$. Also, since from Theorem 3.11, $\kappa_2 = 0$ for a null curve on the lightlike cone $\mathbb{Q}^3_2$ with respect to the natural Frenet frame obtained in Theorem 3.10, there exists only one curvature function κ. Also, by the second equation of (12), a null curve x on the lightlike cone $\mathbb{Q}^3_2$ satisfies the second-order differential equation

$$x''(\tilde{s}) - \kappa(\tilde{s})x(\tilde{s}) = 0. \tag{14}$$

Hence, with initial conditions $x(\tilde{s}_0) = (a_1, a_2, a_3, a_4) \in \mathbb{Q}^3_2$ and $x'(\tilde{s}_0) = (b_1, b_2, b_3, b_4) \in \mathbb{Q}^3_2$, the differential equation (14) has a unique solution. Thus, we have proved the following result.

Theorem 4.1 *Let $q_0 = (a_1, a_2, a_3, a_4) \in \mathbb{Q}^3_2 \subset \mathbb{R}^4_2$ be a point on the lightlike cone $\mathbb{Q}^3_2$, $X_0 = (b_1, b_2, b_3, b_4)$ be a null tangent vector at q_0, $\kappa : I \to R$ be a nonzero smooth function, and $\tilde{s}_0 \in I$. Then there is a unique null curve $x : I \to \mathbb{Q}^3_2 \subset \mathbb{R}^4_2$ parameterized by $\tilde{s}$ with the natural Frenet frame $\{x(\tilde{s}), x'(\tilde{s}), N(\tilde{s}), W(\tilde{s})\}$ such that $x(\tilde{s}_0) = q_0$, $x'(\tilde{s}_0) = X_0$, and its curvature function is $\kappa(\tilde{s})$.*

For constant curvature functions $\kappa(\tilde{s}) = \kappa_0$, the solutions of the differential equation (14) are

$$\begin{aligned} x(\tilde{s}) = \big(&a_1 \sinh(\sqrt{\kappa_0}\tilde{s}) + a_2 \cosh(\sqrt{\kappa_0}\tilde{s}), b_1 \sinh(\sqrt{\kappa_0}\tilde{s}) + b_2 \cosh(\sqrt{\kappa_0}\tilde{s}), \\ &c_1 \sinh(\sqrt{\kappa_0}\tilde{s}) + c_2 \cosh(\sqrt{\kappa_0}\tilde{s}), d_1 \sinh(\sqrt{\kappa_0}\tilde{s}) + d_2 \cosh(\sqrt{\kappa_0}\tilde{s})\big) \end{aligned}$$

for $\kappa_0 > 0$, and

$$\begin{aligned} x(\tilde{s}) = \big(&a_1 \sin(\sqrt{-\kappa_0}\tilde{s}) + a_2 \cos(\sqrt{-\kappa_0}\tilde{s}), b_1 \sin(\sqrt{-\kappa_0}\tilde{s}) + b_2 \cos(\sqrt{-\kappa_0}\tilde{s}), \\ &c_1 \sin(\sqrt{-\kappa_0}\tilde{s}) + c_2 \cos(\sqrt{-\kappa_0}\tilde{s}), d_1 \sin(\sqrt{-\kappa_0}\tilde{s}) + d_2 \cos(\sqrt{-\kappa_0}\tilde{s})\big) \end{aligned}$$

for $\kappa_0 < 0$, where it is possible to calculate the coefficients a_i, b_i, c_i, and d_i, $i = 1, 2$, from the initial conditions $x(\tilde{s}_0)$ and $x'(\tilde{s}_0)$.

The following example appears in [16, Example 5.1], where the authors have obtained a natural Frenet frame and its curvature functions for the given null curve in the way of Duggal and Bejancu [6]. In this example, we calculate our natural Frenet frame by Theorem 3.10.

Example 4.2 Let $x : I \to \mathbb{Q}^3_2$ be the null curve given by

$$x(s) = \sqrt{s}(\cos\theta - \sin\theta, \cos\theta + \sin\theta, \sqrt{2}\cos\theta, \sqrt{2}\sin\theta),$$

where $I \subset \mathbb{R}$ is an open set and $\theta = \frac{\sqrt{15}}{2} \ln s^2$. The structure functions of $x(s)$ are

$$f(s) = \frac{\sqrt{s}}{2}\big((1 + \sqrt{2})\cos\theta - \sin\theta\big), \qquad g(s) = \frac{\sqrt{s}}{2}\big(\cos\theta + (1 + \sqrt{2})\sin\theta\big).$$

The tangent vector $x'(s)$ is given by

$$x'(s) = \frac{1}{2\sqrt{s}}\big((1-\sqrt{15})\cos\theta - (1+\sqrt{15})\sin\theta, (1+\sqrt{15})\cos\theta + (1-\sqrt{15})\sin\theta,$$
$$\sqrt{2}(\cos\theta - \sqrt{15}\sin\theta), \sqrt{2}(\sin\theta + \sqrt{15}\cos\theta)\big).$$

Since $x_1(s)+x_3(s)\neq 0$, $x_2(s)+x_4(s)\neq 0$, and $\lambda := \rho(s) = 1-\sqrt{2}$, we get $\rho'(s)=0$, and

$$\begin{aligned}
\Theta(s) &= f(s)g'(s) - f'(s)g(s)\\
&= \frac{\sqrt{s}}{2}\big((1+\sqrt{2})\cos\theta - \sin\theta\big)\\
&\quad \times \left\{\frac{1}{4\sqrt{s}}\big(\cos\theta + (1+\sqrt{2})\sin\theta\big) + \frac{\sqrt{s}}{2}\frac{\sqrt{15}}{s}\big((1+\sqrt{2})\cos\theta - \sin\theta\big)\right\}\\
&\quad - \frac{\sqrt{s}}{2}\big(\cos\theta + (1+\sqrt{2})\sin\theta\big)\\
&\quad \times \left\{\frac{1}{4\sqrt{s}}\big((1+\sqrt{2})\cos\theta - \sin\theta\big) - \frac{\sqrt{s}}{2}\frac{\sqrt{15}}{s}\big(\cos\theta + (1+\sqrt{2})\sin\theta\big)\right\}\\
&= \frac{\sqrt{15}}{4}\big((1+\sqrt{2})\cos\theta - \sin\theta\big)^2 + \frac{\sqrt{15}}{4}\big(\cos\theta + (1+\sqrt{2})\sin\theta\big)^2\\
&= \frac{\sqrt{15}}{4}\big((1+\sqrt{2})^2\cos^2\theta + \sin^2\theta + \cos^2\theta + (1+\sqrt{2})^2\sin^2\theta\big)\\
&= \frac{\sqrt{15}}{4}\big((1+\sqrt{2})^2 + 1\big) = \frac{\sqrt{15}}{2}(2+\sqrt{2}).
\end{aligned}$$

From Theorem 3.6, the curve x is a curve of Type 2.1, and by Theorem 3.10, the unique transversal vector field to x is

$$N(s) = \frac{1}{2(1+\lambda^2)\Theta(s)}x_{14}^{\perp}(s) = \frac{\sqrt{s}}{4\sqrt{15}}(\sin\theta+\cos\theta, \sin\theta-\cos\theta, -\sqrt{2}\sin\theta, \sqrt{2}\cos\theta)$$

since

$$\begin{aligned}
2\big(1+\lambda^2\big)\Theta &= 2\big(1+(1-\sqrt{2})^2\big)\frac{\sqrt{15}}{2}(2+\sqrt{2}) = \sqrt{15}(4-2\sqrt{2})(2+\sqrt{2})\\
&= 2\sqrt{15}(2-\sqrt{2})(2+\sqrt{2}) = 4\sqrt{15},
\end{aligned}$$

and the unique transversal vector field to x' is

$$\begin{aligned}
W(s) &= -\frac{1}{2(1+\lambda^2)\Theta(s)}{x'_{14}}^{\perp}(s)\\
&= -\frac{1}{8\sqrt{15s}}\big((1-\sqrt{15})\sin\theta + (1+\sqrt{15})\cos\theta, (1+\sqrt{15})\sin\theta - (1-\sqrt{15})\cos\theta,\\
&\quad -\sqrt{2}(\sin\theta + \sqrt{15}\cos\theta), \sqrt{2}(\cos\theta - \sqrt{15}\sin\theta)\big).
\end{aligned}$$

For this natural Frenet frame, we have, by Theorem 3.11, one curvature function, namely

$$\kappa_1(s) = \frac{(2+\sqrt{2})\sqrt{15}}{s^2},$$

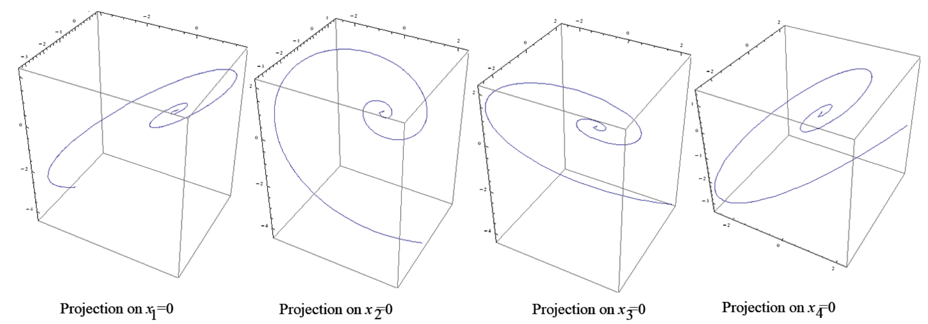

Figure 1 $x(s) = \sqrt{s}(\cos\theta - \sin\theta, \cos\theta + \sin\theta, \sqrt{2}\cos\theta, \sqrt{2}\sin\theta)$, $\theta = \frac{\sqrt{15}}{2}\ln s^2$

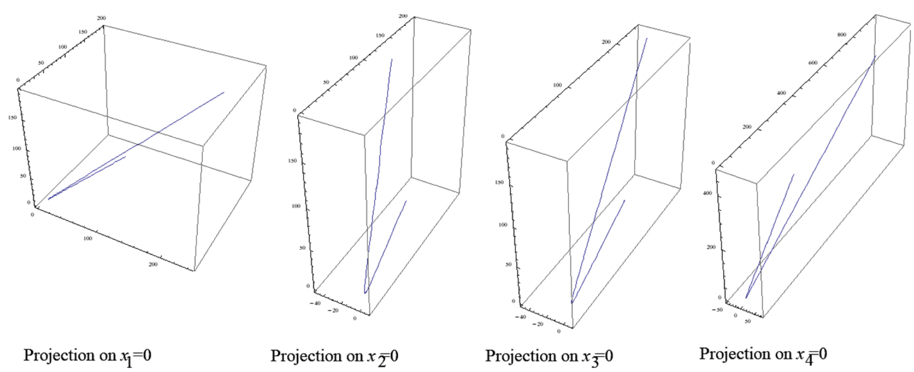

Figure 2 $x(s) = (\cosh s + \frac{5}{3}\sinh s, 17\cosh s + \frac{35}{3}\sinh s, 11\cosh s + \frac{25}{3}\sinh s, 13\cosh s + \frac{25}{3}\sinh s)$

and $h(s) = \kappa_2(s) = 0$. From Corollary 3.14, this curve lies in a 2-dimensional subspace of $\mathbb{R}^4$. By direct calculations, we can represent x as

$$x(s) = \left(\sqrt{s}\cos\theta - \frac{\sqrt{s}}{\sqrt{15}}\sin\theta\right)(1, 1, \sqrt{2}, 0) + \frac{\sqrt{s}}{\sqrt{15}}\sin\theta(1 - \sqrt{15}, 1 + \sqrt{15}, \sqrt{2}, \sqrt{30}),$$

where $\theta(s) = \frac{\sqrt{15}}{2}\ln s^2$. In Fig. 1, we draw the projections of the null curve x on $x_i = 0$, $i = 1, \dots, 4$.

In the following example, we give a null curve with constant curvature $\kappa = 1$.

Example 4.3 Let $x_0 = (1, 17, 11, 13)$ be a point of the lightlike cone $\mathbb{Q}^3_2$ and the vector $x'_0 = (\frac{5}{3}, \frac{35}{3}, \frac{25}{3}, \frac{25}{3})$ be a tangent vector at x_0. Then the null curve

$$x(s) = \left(\cosh s + \frac{5}{3}\sinh s, 17\cosh s + \frac{35}{3}\sinh s, 11\cosh s + \frac{25}{3}\sinh s, 13\cosh s + \frac{25}{3}\sinh s\right)$$

is a null curve of Type 2.1 that passes from x_0 with tangent vector x'_0, and its curvature function is constant $\kappa = 1$. For this curve, $h = -\frac{1}{3}$ and the structure functions are

$$f(s) = 6\cosh s + 5\sinh s, \qquad g(s) = 15\cosh s + 10\sinh s,$$

also $\Theta(s) = f(s)g'(s) - f'(s)g(s) = -15$. From Corollary 3.14, this curve lies in the 2-dimensional subspace spanned by the initial condition vectors x_0, x_0' as

$$x(s) = \cosh s(1, 17, 11, 13) + \sinh s\left(\frac{5}{3}, \frac{35}{3}, \frac{25}{3}, \frac{25}{3}\right).$$

In Fig. 2, we draw the projections of the null curve x on $x_i = 0$, $i = 1, \ldots, 4$.

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