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Coderivatives and Aubin property of efficient point and efficient solution set-valued maps in parametric vector optimization

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Abstract

The aim of this paper is computing the coderivatives of efficient point and efficient solution set-valued maps in a parametric vector optimization problem. By using a method different from the existing literature we establish an upper estimate and explicit expression for the coderivatives of an efficient point set-valued map where the independent variable can take values in the whole space. As an application, we give some characterizations on the Aubin property of an efficient point map and an explicit expression of the coderivative for an efficient solution map. We provide several examples illustrating the main results.

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1 Introduction

Consider the following parametric vector-valued optimization problem:

$$\mathrm{Min}_K \{f(p,x) \mid x \in C(p)\}, \tag{1}$$

where $f : \mathcal{R}^m \times \mathcal{R}^n \to \mathcal{R}^s$ is a vector-valued map, $C : \mathcal{R}^m \rightrightarrows \mathcal{R}^n$ is a set-valued map, K is a pointed closed convex cone of $\mathcal{R}^s$ that induces a partial ordering $\preceq_K$, $x \in \mathcal{R}^n$ is a decision variable, and $p \in \mathcal{R}^m$ is a parameter. The "Min_K" in (1) is understood as follows: $y \in \mathrm{Min}_K A$ if and only if $(y - K) \cap A = \{y\}$ for each $A \subseteq \mathcal{R}^s$; when $A = \emptyset$, $\mathrm{Min}_K A = \emptyset$. For convenience, we define $F : \mathcal{R}^m \rightrightarrows \mathcal{R}^s$ as

$$F(p) := (f \circ C)(p) = f\big(p, C(p)\big) = \big\{f(p,x) \mid x \in C(p)\big\} \tag{2}$$

and $\mathcal{F} : \mathcal{R}^m \rightrightarrows \mathcal{R}^s$ as

$$\mathcal{F}(p) := \mathrm{Min}_K F(p). \tag{3}$$

The latter is the efficient point set-valued map for (1). Clearly, the efficient solution map of (1) is given as

$$S(p) := \{x \in C(p) \mid 0 \in -f(p,x) + \mathcal{F}(p)\}. \tag{4}$$

The aim of this paper is discussion of the coderivatives and Aubin property of the efficient point and efficient solution maps in parametric vector optimization problems. The derivatives (especially, coderivatives) are very useful tools to discuss the Aubin property, which is an important concept in sensitivity analysis in optimization theory and applications. The key work of sensitivity analysis is analyzing the behavior of the efficient point set-valued maps $\mathcal{F}$ and the efficient solution set-valued maps S by using certain concepts of generalized derivatives for set-valued maps. Let us review the main work in this field. In primal spaces, by using the tangent derivatives for set-valued maps, which are generated by tangent cones to their graphs, sensitivity results are obtained for vector optimization problems with kinds of structure; see, for example, [2, 7, 8, 11, 14, 16, 22, 23] and references therein. In dual spaces, sensitivity results of scalar (single-objective) optimization problems are obtained by the coderivatives generated by normal cones to the graphs of set-valued maps; we refer the readers to [10, 15, 19, 20, 25] for just a few of them.

There are also some papers discussing the sensitivity of vector optimization problems. By using the results in [17], Huy et al. [6] studied the sensitivity properties of the parametric vector optimization problem via the Mordukhovich coderivatives with respect to the so-called generalized order optimality in the Asplund space setting. Chuong and Yao [4] and Chuong [3] gave some results on sensitivity analysis in parametric vector optimization problems by virtue of the Fréchet and Clarke coderivatives in the Banach space setting, respectively. Xue et al. [24] studied the upper estimates and equality formula for the coderivatives of parametric vector set-valued optimization problems in Banach spaces. The method to obtain the coderivatives of an efficient point map $\mathcal{F}$ in these papers is as follows: first, they gave some useful and effective expressions for the coderivatives of F by the coderivatives of $F + K$; then these effective characterizations are applied to the efficient point map $\mathcal{F}$; finally, by the domination property they showed that the coderivatives of the efficient point map $\mathcal{F}$ is equivalent to that of F. We can easily see that this method is technical and complex and that the drawback of this method is that these results cannot be used to establish the Aubin property of $\mathcal{F}$ and the coderivatives of solution map S since the independent variable of the coderivatives set-valued map of $\mathcal{F}$ is restricted to a part of the space; it also does not provide all the characters of $\mathcal{F}$, especially for the independent variable being 0, which is important in showing the Aubin property of $\mathcal{F}$.

In this paper, we introduce a new way, different from those in the recent literature, to investigate the sensitivity and Aubin properties of the parametric vector-valued optimization problem via the coderivatives. We first establish an upper estimate and explicit expression for computing the coderivative of an efficient point set-valued map $\mathcal{F}$ where the independent variable can take values in the whole space and then employ this formula to study the Aubin property of $\mathcal{F}$. As a byproduct, we use this formula to establish an explicit expression of the coderivative for efficient solution map S only by the coderivative of the objective function and the constraint map.

The rest of the paper is organized as follows. In Sect. 2, we recall and discuss some basic constructions from variational analysis and generalized differentiation, broadly employed in the formulations and proofs of the main results. In Sect. 3, we first establish an upper estimate and explicit expression for the coderivative of the efficient point set-valued map $\mathcal{F}$. Then, by the above formula, we obtain the Aubin property of $\mathcal{F}$. In Sect. 4, we study the coderivative of the efficient solution map S for the parametric vector-valued optimization problem. Moreover, we provide examples to analyze and illustrate the obtained results.

2 Basic definitions and preliminaries

We use standard notation. For all spaces, the norms are always denoted by $\|\cdot\|$, and $\langle\cdot,\cdot\rangle$ means the canonical pairing. The closed ball with center x and radius η is denoted by $B_\eta(x)$. The symbol A^* denotes the adjoint operator of a linear continuous operator A. If $F:\mathcal{R}^n \rightrightarrows \mathcal{R}^s$ is a set-valued map, then by $\operatorname{dom} F = \{x \in \mathcal{R}^n \mid F(x) \neq \emptyset\}$ and $\operatorname{gph} F = \{(x,y) \in \mathcal{R}^n \times \mathcal{R}^s \mid y \in F(x)\}$ we denote the domain and graph of F, respectively. The notation $x_n \xrightarrow{S} x$ means that the sequence x_n is contained in the subset S and converges to x. For a set-valued map $F:\mathcal{R}^n \rightrightarrows \mathcal{R}^s$, the expression

$$\begin{aligned} &\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) \\ &\quad = \bigl\{y \in \mathcal{R}^s \mid \exists \text{ sequences } x_k \to \bar{x}, y_k \in F(x_k), \text{s.t.}, y_k \to y \text{ for all } k \in \mathbb{N}\bigr\}, \end{aligned}$$

signifies the sequential Painlevé–Kuratowski upper (outer) limit of F at $\bar{x}$; $\mathbb{N} = \{1,2,\ldots\}$. The origins of all spaces are denoted by 0.

Next, we recall the basic concepts and constructions of variational analysis and generalized differentiation for formulations and justifications of the main results of the paper. Most of the concepts and properties can be found in the recent monographs [17, 18].

Definition 2.1 Let $\Omega \subset \mathcal{R}^n$ be a nonempty subset.

(i) Let $\bar{x} \in \Omega$ and $\varepsilon \geq 0$. The Fréchet normal cone (or the *prenormal cone*) to Ω at $\bar{x} \in \Omega$ is defined by

$$\hat{N}(\bar{x},\Omega) = \left\{x^* \in \mathcal{R}^n \,\middle|\, \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq 0\right\}. \tag{5}$$

We put $\hat{N}(\bar{x},\Omega) = \emptyset$ if $\bar{x} \notin \Omega$.

(ii) The Mordukhovich normal cone (or basic normal cone) to $\Omega \subset \mathcal{R}^n$ at $\bar{x}$ is defined through the Painlevé–Kuratowski upper (outer) limit as

$$N(\bar{x},\Omega) = \operatorname*{Lim\,sup}_{x_k\to\bar{x}} \hat{N}(x_k,\Omega). \tag{6}$$

Definition 2.2 Consider a set-valued map $\Phi:\mathcal{R}^n \rightrightarrows \mathcal{R}^s$.

(i) The Fréchet coderivative $\hat{D}^*\Phi(\bar{x},\bar{y})$ at $(\bar{x},\bar{y})$ is defined through the Fréchet normal cone (5) to the graph as

$$\hat{D}^*\Phi(\bar{x},\bar{y})\bigl(y^*\bigr) = \bigl\{x^* \in \mathcal{R}^n \mid \bigl(x^*,-y^*\bigr) \in \hat{N}\bigl((\bar{x},\bar{y}),\operatorname{gph}\Phi\bigr)\bigr\}. \tag{7}$$

(ii) The normal (Mordukhovich) coderivative of Φ at $(\bar{x},\bar{y})$ is

$$D^*\Phi(\bar{x},\bar{y})\bigl(y^*\bigr) = \bigl\{x^* \in \mathcal{R}^n \mid \bigl(x^*,-y^*\bigr) \in N\bigl((\bar{x},\bar{y}),\operatorname{gph}\Phi\bigr)\bigr\}, \tag{8}$$

that is, $D^*\Phi(\bar{x},\bar{y})(y^*)$ is the collection of all x^* for which there are sequences $(x_k,y_k)\to(\bar{x},\bar{y})$ and $(x_k^*,y_k^*)\to(x^*,y^*)$ with $(x_k,y_k)\in\operatorname{gph}\Phi$ and $x_k^*\in\hat{D}^*\Phi(x_k,y_k)(y_k^*)$.

The symbol $D^*\Phi(\bar{x})$ is used when Φ is single-valued at $\bar{x}$ and $\bar{y}=\Phi(\bar{x})$.

We say that Ω is regular at $\bar{x}\in\Omega$ if $N(\bar{x},\Omega)=\hat{N}(\bar{x},\Omega)$ and that Φ is regular at $(\bar{x},\bar{y})$ if $D^*\Phi(\bar{x},\bar{y})=\hat{D}^*\Phi(\bar{x},\bar{y})$.

The following proposition gives a sufficient condition for the regularity of Φ and special representations of the coderivatives.

Proposition 2.1 *Let $\Phi:\mathcal{R}^n\to\mathcal{R}^s$ be Fréchet differentiable at $\bar{x}$. Then*

$$\hat{D}^*\Phi(\bar{x})(y^*)=\{(\nabla\Phi(\bar{x}))^*y^*\},\quad\forall y^*\in\mathcal{R}^s.$$

Moreover, if Φ is strictly differentiable at $\bar{x}$, that is, Φ is single-valued around $\bar{x}$ and

$$\lim_{x,x'\to\bar{x}}[\Phi(x)-\Phi(x')-\nabla\Phi(\bar{x})(x-x')]/\|x-x'\|=0,$$

then Φ is regular at $\bar{x}$, and we have

$$D^*\Phi(\bar{x})(y^*)=\{(\nabla\Phi(\bar{x}))^*y^*\},\quad\forall y^*\in\mathcal{R}^s.$$

Definition 2.3 Let $f:\mathcal{R}^n\to\mathcal{R}^s$ be a single-valued map, and let $\bar{x}\in\operatorname{dom}f$. The map f is said to be locally upper Lipschitzian at $\bar{x}$ if there are numbers $\eta>0$ and $L>0$ such that

$$\|f(x)-f(\bar{x})\|\le L\|x-\bar{x}\|\quad\text{for all } x\in B_\eta(\bar{x})\cap\operatorname{dom}f.$$

We say that a set-valued map $F:\mathcal{R}^n\rightrightarrows\mathcal{R}^s$ admits a local upper Lipschitzian selection at $(\bar{x},\bar{y})\in\operatorname{gph}F$ if there is a single-valued map $f:\operatorname{dom}F\to\mathcal{R}^s$ which is locally upper Lipschitzian at $\bar{x}$ satisfying $f(\bar{x})=\bar{y}$ and $f(x)\in F(x)$ for all $x\in\operatorname{dom}F$ in a neighborhood of $\bar{x}$. We say that F admits a locally upper Lipschitzian selection around $(\bar{x},\bar{y})\in\operatorname{gph}F$ if there is a neighborhood U of $(\bar{x},\bar{y})$ such that F admits a locally upper Lipschitzian selection at any $(x,y)\in\operatorname{gph}F\cap U$.

Definition 2.4 ([1]) Let $S\subset\mathcal{R}^n$ be a nonempty subset. The paratingent cone to K at $z\in clK$ is the set

$$P(K,z)=\operatorname*{Lim\,sup}_{t\downarrow 0,z'\xrightarrow{S}z}\frac{K-z'}{t}.$$

The paratingent (or strict) derivative to a set-valued map $F:\mathcal{R}^n\rightrightarrows\mathcal{R}^s$ at $(\bar{x},\bar{y})\in\operatorname{cl}(\operatorname{gph}F)$ is the set-valued map $D_*F(\bar{x},\bar{y})$ whose graph is the paratingent cone to $\operatorname{gph}F$ at $(\bar{x},\bar{y})$. That is, $v\in D_*F(\bar{x},\bar{y})(u)$ if and only if there are sequences $t_k\to 0_+$, $(x_k,y_k)\to(\bar{x},\bar{y})$, and $(u_k,v_k)\to(u,v)$ with $(x_k,y_k)\in\operatorname{gph}F$ and $y_k+t_kv_k\in F(x_k+t_ku_k)$.

3 Sensitivity analysis of the efficient point map

In this section, we provide sensitivity analysis of the efficient point set-valued map $\mathcal{F}$ for the parametric vector-valued optimization problem. By upper estimates for the coderivatives of an efficient point set-valued map, we obtain the Aubin property of $\mathcal{F}$.

Recall that a map T is said to be compact at $\bar{x} \in \operatorname{dom} T$ if for any sequence $(x_k, y_k) \subset \operatorname{gph} T$ with $x_k \to \bar{x}$, there exists a subsequence y_{k_i} converging to some $\bar{y} \in T(\bar{x})$. We say that T is locally compact around $\bar{x}$ if there is a neighborhood U of $\bar{x}$ such that T is compact at any $x \in U \cap \operatorname{dom} T$.

The following property is very important in computing the coderivative of an efficient point set-valued map $\mathcal{F}$.

Proposition 3.1 *Let $\bar{x} \in C(\bar{p})$ and $\bar{y} = f(\bar{p}, \bar{x})$, and let the set-valued map $M : \mathcal{R}^m \times \mathcal{R}^s \rightrightarrows \mathcal{R}^n$ be defined by*

$$M(p, y) = \{x \in \mathcal{R}^n \mid y = f(p, x) \text{ and } x \in C(p)\}.$$

Suppose that f is locally Lipschitz at $(\bar{p}, \bar{x})$, $M(\bar{p}, \bar{y}) = \{\bar{x}\}$, and M is compact at $(\bar{p}, \bar{y})$. If

$$D_* C(\bar{p}, \bar{x})(0) = \{0\}, \tag{9}$$

then

$$D_* F(\bar{p}, \bar{y})(0) = \{0\}. \tag{10}$$

Proof Let $v \in D_* F(\bar{p}, \bar{y})(0)$. By the definition of strict derivative, there are sequences t_n, (p_n, y_n), and (u_n, v_n) with $t_n \downarrow 0$, $(p_n, y_n) \xrightarrow{\operatorname{gph} F} (\bar{p}, \bar{y})$, and $(u_n, v_n) \to (0, v)$ such that $y_n + t_n v_n \in F(p_n + t_n u_n)$. By the definition of F there exist $x_n \in C(p_n)$ and $x'_n \in C(p_n + t_n u_n)$ satisfying $y_n = f(p_n, x_n)$ and $y_n + t_n v_n = f(p_n + t_n u_n, x'_n)$. So

$$v_n = \frac{f(p_n + t_n u_n, x'_n) - f(p_n, x_n)}{t_n}.$$

For the above p_n, y_n, and $x_n \in M(p_n, y_n)$, since M is compact at $(\bar{p}, \bar{y})$, there exists a convergent subsequence, and we may assume without loss of generality that $x_n \to \hat{x} \in M(\bar{p}, \bar{y})$. The assumption $M(\bar{p}, \bar{y}) = \{\bar{x}\}$ implies that $x_n \to \bar{x}$. Similarly, since $x'_n \in M(p_n + t_n u_n, y_n + t_n v_n)$, we have $x'_n \to \bar{x}$. Then by [9, Theorem 1.3] the nonsingularity condition of C ensures the existence of a constant $L > 0$ such that

$$\|x'_n - x_n\| \le L\|p_n + t_n u_n - p_n\| = L t_n \|u_n\|$$

for n large enough. So we have $\frac{x'_n - x_n}{t_n} \to 0$. Since f is locally Lipschitz at $(\bar{p}, \bar{x})$ and $p_n, p_n + t_n u_n \to \bar{p}$ as $n \to \infty$, there exists a constant $L > 0$ such that

$$\begin{aligned} \frac{\|f(p_n + t_n u_n, x'_n) - f(p_n, x_n)\|}{t_n} &\le \frac{L(\|t_n u_n\| + \|x'_n - x_n\|)}{t_n} \\ &= L\left(\|u_n\| + \frac{\|x'_n - x_n\|}{t_n}\right). \end{aligned}$$

Since u_n converges to 0, we have

$$v = \lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{f(p_n + t_n u_n, x'_n) - f(p_n, x_n)}{t_n} = 0,$$

which implies that (10) holds. This completes the proof. □

Remark 3.1 If C is a single-valued map that is continuous at $\bar{p}$, then the hypothetical conditions of M hold naturally, and the nonsingularity condition (9) is equivalent to the Lipschitzian property of C at $\bar{p}$. In this case, this proposition reduces to the classic result of composition of two Lipschitz maps.

The following example shows that our result is a generalization of the classic result on composition of two Lipschitz maps even in the single-valued case.

Example 3.1 Let $C : \mathcal{R} \to \mathcal{R}$ and $f : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be defined by

$$C(p) = \begin{cases} 0, & p = 0 \text{ or } p = \frac{1}{n}, n \in N, \\ \emptyset, & \text{else} \end{cases}$$

and

$$f(p,x) = p + x,$$

respectively. Then, we have

$$F(p) = \begin{cases} p, & p = 0 \text{ or } p = \frac{1}{n}, n \in N \\ \emptyset, & \text{else.} \end{cases}$$

Consider $(\bar{p},\bar{x}) = (0,0)$ and $\bar{y} = f(\bar{p},\bar{x}) = 0$. By direct computation we obtain that for any $p \in \mathcal{R}$,

$$D_*C(\bar{p},\bar{x})(p) = \{0\}$$

and

$$D_*F(\bar{p},\bar{x})(p) = \{p\}.$$

Although C is not continuous at $\bar{p}$, our result still works.

We give some sufficient conditions for the hypothetical conditions of M in Proposition 3.1.

Proposition 3.2 *Let $\bar{x} \in C(\bar{p})$ and $\bar{y} = f(\bar{p},\bar{x})$, and let the set-valued map $M : \mathcal{R}^m \times \mathcal{R}^s \rightrightarrows \mathcal{R}^n$ be defined by*

$$M(p,y) = \big\{x \in \mathcal{R}^n \mid y = f(p,x) \text{ and } x \in C(p)\big\}.$$

Consider the following conditions:

(i) *f is strictly differentiable at $(\bar{p},\bar{x})$, and $\nabla f(\bar{p},\bar{x})$ is invertible*;
(ii) *f^{-1} is locally single-valued around $\bar{y}$ and continuous at this point*;
(iii) *$M(\bar{p},\bar{y}) = \{\bar{x}\}$, and f^{-1} is compact at $\bar{y}$*;
(iv) *$M(\bar{p},\bar{y}) = \{\bar{x}\}$, and C is compact at $\bar{p}$*;
(v) *$M(\bar{p},\bar{y}) = \{\bar{x}\}$, and M is compact at $(\bar{p},\bar{y})$*;
(vi) *$M(\bar{p},\bar{y}) = \{\bar{x}\}$, and M is upper semicontinuous at $(\bar{p},\bar{y})$*;
(vii) *$M(\bar{p},\bar{y}) = \{\bar{x}\}$, and M is locally upper Lipschitz at $(\bar{p},\bar{y})$*;

Then we have

$$\begin{array}{ccccccccccc} \text{(i)} & \Rightarrow & \text{(ii)} & \Rightarrow & \text{(iii)} & \Rightarrow & \text{(v)} & \Leftarrow & \text{(vi)} & \Leftarrow & (vii) \\ & & & & & & \Uparrow & & & & \\ & & & & & & \text{(iv)} & & & & \end{array}$$

Proof (i)⇒(ii) By [17, Theorem 1.60] assumption of (i) implies that f^{-1} is locally single-valued around $\bar{y}$ and strict differentiable at this point. Obviously, (ii) holds.

(ii)⇒(iii), (iii)⇒(v), (iv)⇒(v), and (vii)⇒(vi) immediately follow from the definition of compactness and construction of M.

(vi)⇒(v) We can find the result in [5, Proposition 2.5.9]. This completes the proof. □

Now we consider the coderivatives of the set-valued map $\mathcal{F}$.

Theorem 3.1 *Let $\bar{x} \in C(\bar{p})$ and $\bar{y} \in \mathcal{F}(\bar{p})$. Suppose that f is locally Lipschitz at $(\bar{p},\bar{x})$, $M(\bar{p},\bar{y}) = \{\bar{x}\}$, and C is locally compact around $\bar{p}$. If condition (9) holds, then we have the following results.*

(i) *For any $y^* \in \mathcal{R}^s$,*

$$D^*\mathcal{F}(\bar{p},\bar{y})(y^*) = D^*F(\bar{p},\bar{y})(y^*) \subset \bigcup_{(p^*,x^*)\in D^*f(\bar{p},\bar{x})(y^*)} p^* + D^*C(\bar{p},\bar{x})(x^*), \tag{11}$$

$$\bigcup_{(p^*,x^*)\in \hat{D}^*f(\bar{p},\bar{x})(y^*)} p^* + \hat{D}^*C(\bar{p},\bar{x})(x^*) \subset \hat{D}^*F(\bar{p},\bar{y})(y^*) = \hat{D}^*\mathcal{F}(\bar{p},\bar{y})(y^*).$$

Moreover, if f and C are regular at $(\bar{p},\bar{x})$ and $\bar{p}$, respectively, then $\mathcal{F}$ is regular at $(\bar{p},\bar{y})$, and for any $y^* \in \mathcal{R}^s$, (11) holds as equality.

(ii) *If f is Fréchet differentiable at $(\bar{p},\bar{x})$ with the derivative $\nabla f(\bar{p},\bar{x}) = (\nabla_p f(\bar{p},\bar{x}), \nabla_x f(\bar{p},\bar{x}))$, then for any $y^* \in \mathcal{R}^s$,*

$$\hat{D}^*\mathcal{F}(\bar{p},\bar{y})(y^*) = \hat{D}^*F(\bar{p},\bar{y})(y^*) = \nabla_p f(\bar{p},\bar{x})^*y^* + \hat{D}^*C(\bar{p},\bar{x})(\nabla_x f(\bar{p},\bar{x})^*y^*).$$

(iii) *If f is strictly differentiable at $(\bar{p},\bar{x})$, then for any $y^* \in \mathcal{R}^s$,*

$$D^*\mathcal{F}(\bar{p},\bar{y})(y^*) = D^*F(\bar{p},\bar{y})(y^*) \subset \nabla_p f(\bar{p},\bar{x})^*y^* + D^*C(\bar{p},\bar{x})(\nabla_x f(\bar{p},\bar{x})^*y^*). \tag{12}$$

Moreover, if C is regular at $\bar{p}$, then $\mathcal{F}$ is regular at $(\bar{p},\bar{y})$, and for any $y^* \in \mathcal{R}^s$, (12) holds as equality.

Proof First, we prove that $D^*\mathcal{F}(\bar{p},\bar{y})(y^*) = D^*F(\bar{p},\bar{y})(y^*)$. Let $p^* \in D^*\mathcal{F}(\bar{p},\bar{y})(y^*)$. By the definitions of normal coderivative and normal cone, there are sequences $(p_k, y_k) \xrightarrow{\mathcal{F}} (\bar{p},\bar{y})$ and $(p_k^*, y_k^*) \to (p^*, y^*)$ such that

$$\limsup_{(p_{k_i},y_{k_i})\xrightarrow{\mathcal{F}}(p_k,y_k)} \frac{\langle p_k^*, p_{k_i} - p_k\rangle - \langle y_k^*, y_{k_i} - y_k\rangle}{\|p_{k_i} - p_k\| + \|y_{k_i} - y_k\|} \leq 0.$$

We only need to show that in a neighborhood U of $(\bar{p},\bar{y})$, for any $(p_k, y_k) \in U$, $(p_{k_i}, y_{k_i}) \xrightarrow{\mathcal{F}} (p_k, y_k)$ if and only if $(p_{k_i}, y_{k_i}) \xrightarrow{F} (p_k, y_k)$. Suppose the contrary. Then there is a sequence $(p'_{k_i}, y'_{k_i}) \in \operatorname{gph} F \setminus \operatorname{gph}\mathcal{F}$ such that $(p'_{k_i}, y'_{k_i}) \longrightarrow (p_k, y_k)$. Since C is locally compact around $\bar{p}$, there exists a neighborhood U of $\bar{p}$ such that for any $p \in U$, C is compact at p. For any $p_k \to p$ and $y_k \in F(p_k)$, by the definition of F there exists $x_k \in C(p_k)$ with $y_k = f(p_k, x_k)$. Because C is compact at p, there is a subsequence $(p_{k_i}, x_{k_i}) \to (p, x)$ with $x \in C(p)$. Thus by the Lipschitzian continuity assumption of f we have $(p_{k_i}, y_{k_i}) \to (p, y)$ along with $y = f(p, x) \in F(p)$. So F is also locally compact around $\bar{p}$. Case (iv) of Proposition 4.3 in [13] implies that $\mathcal{F}$ is locally order semicontinuous around $(\bar{p},\bar{y})$. So, for the above (p'_{k_i}, y'_{k_i}), there is a sequence $(p'_{k_i}, y''_{k_i}) \in \operatorname{gph}\mathcal{F}$ such that $y''_{k_i} \leq_K y'_{k_i}$. However, under the assumption, we have $D_*F(\bar{p},\bar{y})(0) = \{0\}$, which implies that there exists at most one element $y(p)$ in a neighborhood of $\bar{p}$ (see [9, Theorem 1.3]). This is a contradiction, and thus the equality relation holds.

From [24, Propositions 3.4 and 3.5, Corollaries 3.1 and 3.2] it follows that we only need to show that the set-valued map M admits a locally upper Lipschitzian selection at $(\bar{p},\bar{y},\bar{x})$. Following [9, Theorem 1.3], the nonsingularity condition of C implies that there exist neighborhoods U of $\bar{p}$ and V of $\bar{x}$ and a constant $L > 0$ such that there is at most one element $x(p)$ in the local image set $C(p) \cap V$ for $p \in U$, and it satisfies

$$\left|x(p) - x(p')\right| \leq L\left|p - p'\right| \quad \text{for } p, p' \in U.$$

We claim here that for any $(p_n, y_n) \to (\bar{p},\bar{y})$ and $x_n \in M(p_n, y_n)$, $x_n \in V$ as $n \to \infty$. If not, there would be a subsequence $x_{n_k} \in M(p_{n_k}, y_{n_k}) \subset C(p_{n_k})$ such that $x_{n_k} \notin V$. By the compactness of C we may assume without loss of generality that $x_{n_k} \to \hat{x} \in C(\bar{p})$, $\hat{x} \neq \bar{x}$, by taking a subsequence if necessary. By the definition of a set-valued map M we have $\hat{x} \in M(\bar{p},\bar{y})$, which contradicts to the fact that $M(\bar{p},\bar{y}) = \{\bar{x}\}$. Thus M has at most one element in a neighborhood of $(\bar{p},\bar{y})$ and Lipschitzian around $(\bar{p},\bar{y})$, which implies the locally upper Lipschitzian selection and inner semicontinuous property of M at $(\bar{p},\bar{y},\bar{x})$. This completes the proof. □

If f is strictly differentiable with surjective derivative, then we have the following result.

Corollary 3.1 *Let $\bar{x} \in C(\bar{p})$ and $\bar{y} \in \mathcal{F}(\bar{p})$. Suppose that f is strictly differentiable at $(\bar{p},\bar{x})$, $\nabla f(\bar{p},\bar{x})$ is surjective, and C is locally compact around $\bar{p}$. If $M(\bar{p},\bar{y}) = \{\bar{x}\}$ and condition (9) holds, then for any $y^* \in \mathcal{R}^s$, (12) holds as equality.*

Proof The inclusion relation (12) directly follows from Theorem 3.1. We only need to prove the converse inclusion. By [17, Corollary 1.59] the assumption on f implies that f^{-1}

has the Aubin property at $(\bar{y},\bar{p},\bar{x})$. Noting that $\{p\}\times C(p) = (f^{-1}\circ F)(p)$ and in this case

$$M'(p,x) = \{y \mid (p,x)\in f^{-1}(y), y\in F(p)\} = f(p,x)\cap F(p) = f(p,x)$$

is inner semicontinuous at $(\bar{p},\bar{x},\bar{y})$ naturally, using [17, Corollary 3.15], we have that for any $(p^*,x^*)\in\mathcal{R}^m\times\mathcal{R}^n$,

$$\begin{aligned} D^*(1\times C)(\bar{p},\bar{p},\bar{x})(p^*,x^*) &= D^*(f^{-1}\circ F)(\bar{p},\bar{p},\bar{x})(p^*,x^*) \\ &\subset D^*F(\bar{p},\bar{y})\circ D^*f^{-1}(\bar{y},\bar{p},\bar{x})(p^*,x^*). \end{aligned}$$

By [12, Lemma 49] we have that for any $(p^*,x^*)\in\mathcal{R}^m\times\mathcal{R}^n$,

$$D^*(1\times C)(\bar{p},\bar{p},\bar{x})(p^*,x^*) = p^* + D^*C(\bar{p},\bar{x})(x^*).$$

Note that

$$\begin{aligned} &y^*\in D^*f^{-1}(\bar{y},\bar{p},\bar{x})(p^*,x^*) \\ &\quad\Leftrightarrow\quad (-p^*,-x^*)\in D^*f(\bar{p},\bar{x},\bar{y})(-y^*) = (-\nabla_p f(\bar{p},\bar{x})^*y^*, -\nabla_x f(\bar{p},\bar{x})^*y^*) \\ &\quad\Leftrightarrow\quad p^* = \nabla_p f(\bar{p},\bar{x})^*y^*,\qquad x^* = \nabla_x f(\bar{p},\bar{x})^*y^*. \end{aligned}$$

So we have that for any $(p^*,x^*)\in\mathcal{R}^m\times\mathcal{R}^n$,

$$p^* + D^*C(\bar{p},\bar{x})(x^*)\subset \bigcup_{p^*=\nabla_p f(\bar{p},\bar{x})^*y_1^*, x^*=\nabla_x f(\bar{p},\bar{x})^*y_1^*} D^*F(\bar{p},\bar{y})(y_1^*).$$

Since $\nabla f(\bar{p},\bar{x})$ is surjective, by [17, Lemma 1.18] $\nabla f(\bar{p},\bar{x})^*$ is injective, which ensures that

$$\nabla_p f(\bar{p},\bar{x})^*y^* = \nabla_p f(\bar{p},\bar{x})^*y_1^*,\qquad \nabla_x f(\bar{p},\bar{x})^*y^* = \nabla_x f(\bar{p},\bar{x})^*y_1^* \quad\Rightarrow\quad y^* = y_1^*.$$

Thus, for any $y^*\in\mathcal{R}^s$,

$$\nabla_p f(\bar{p},\bar{x})^*y^* + D^*C(\bar{p},\bar{x})(\nabla_x f(\bar{p},\bar{x})^*y^*)\subset D^*F(\bar{p},\bar{y})(y^*) = D^*\mathcal{F}(\bar{p},\bar{y})(y^*).$$

This completes the proof. □

Remark 3.2 We know that the paratingent cone is large. So condition (9) is rather strong. However, the results we obtained are more interesting, because the independent variable can rake values in the whole space. This ensures us to obtain the Aubin property of $\mathcal{F}$ and the coderivatives of efficient solution map S. In this view, our results are better than the analogous ones in [4, Theorem 3.6], [3, Theorem 3.6], and [24, Theorem 3.6], where the independent variable can only take values in $K_{up}^* := \{y^*\in\mathcal{R}^s \mid \exists\beta>0, \langle y^*,k\rangle\geq\beta\|k\|, \forall k\in K\}$.

Example 3.2 Let $K=\mathcal{R}_+$, and let $C:\mathcal{R}\rightrightarrows\mathcal{R}$ and $f:\mathcal{R}\times\mathcal{R}\to\mathcal{R}$ be defined by

$$C(p) = \begin{cases} \{-p,1\}, & p\geq 0, \\ \{p,1\}, & \text{else} \end{cases}$$

and

$$f(p,x) = p + x,$$

respectively. Then we have

$$F(p) = \begin{cases} \{0, p+1\}, & p \geq 0, \\ \{2p, p+1\}, & \text{else} \end{cases}$$

and

$$\mathcal{F}(p) = \begin{cases} \{0\}, & p \geq 0, \\ \{2p\}, & \text{else.} \end{cases}$$

Consider $(\bar{p},\bar{x}) = (0,0)$ and $\bar{y} = f(\bar{p},\bar{x}) = 0$. By direct computation we obtain

$$D_*C(\bar{p},\bar{x})(p) = \begin{cases} [-p, p], & p \geq 0, \\ [p, -p], & \text{else} \end{cases}$$

and

$$M(\bar{p},\bar{y}) = \{\bar{x}\}.$$

Obviously, C is compact around $\bar{p}$, so all the conditions of Corollary 3.1 hold. In fact, we can see that

$$D^*C(0,0)(x^*) = \begin{cases} \{x^*, -x^*\}, & x^* \geq 0, \\ [x^*, -x^*], & \text{else} \end{cases}$$

and

$$D^*\mathcal{F}(0,0)(y^*) = D^*F(0,0)(y^*) = \begin{cases} \{0, 2y^*\}, & y^* \geq 0, \\ [2y^*, 0], & \text{else.} \end{cases}$$

Thus, for any y^*, we have

$$D^*\mathcal{F}(0,0)(y^*) = D^*F(0,0)(y^*) = \nabla_p f(\bar{p},\bar{x})^* y^* + D^*C(\bar{p},\bar{x})(\nabla_x f(\bar{p},\bar{x})^* y^*).$$

We are now ready to obtain the Aubin property of $\mathcal{F}$. Recall that a set-valued map $F : \mathcal{R}^n \rightrightarrows \mathcal{R}^s$ is said to be locally continuous around $\bar{x}$ for $\bar{y}$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that, for any $x \in U$ and $\epsilon > 0$, we can find $\delta > 0$ with

$$F(x') \cap V \subset F(x) + \epsilon\mathcal{B} \quad \text{when } x' \in U \cap \mathcal{B}(x,\delta).$$

Corollary 3.2 *Let $\bar{x} \in C(\bar{p})$ and $\bar{y} \in \mathcal{F}(\bar{p})$. Suppose that f is locally Lipschitz at $(\bar{p},\bar{x})$, $M(\bar{p},\bar{y}) = \{\bar{x}\}$, and C is locally compact around $\bar{p}$. Assume that condition (9) holds and that C is locally continuous around $\bar{p}$ for $\bar{x}$. Then $\mathcal{F}$ has the Aubin property at $(\bar{p},\bar{x})$.*

Proof It follows from the first part of Theorem 3.1 that

$$D^*\mathcal{F}(\bar{p},\bar{y})(0) = D^*F(\bar{p},\bar{y})(0) \subset \bigcup_{(p^*,x^*)\in D^*f(\bar{p},\bar{x})(0)} p^* + D^*C(\bar{p},\bar{x})(x^*).$$

Since f is locally Lipschitz at $(\bar{p},\bar{x})$, by using [17, Theorem 1.44] we have

$$D^*f(\bar{p},\bar{x})(0) = \{(0,0)\},$$

and thus

$$D^*\mathcal{F}(\bar{p},\bar{y})(0) = D^*F(\bar{p},\bar{y})(0) \subset D^*C(\bar{p},\bar{x})(0).$$

By case (a) of [21, Theorem 9.54] the continuity assumption of C and the nonsingularity condition (9) imply that

$$D^*C(\bar{p},\bar{x})(0) = \{0\}.$$

So we have $D^*\mathcal{F}(\bar{p},\bar{y})(0) = \{0\}$, which means that $\mathcal{F}$ has the Aubin property at $(\bar{p},\bar{x})$. This completes the proof. □

4 Coderivative of solution map

In this section, we establish verifiable formulas for upper estimating and precise computing the coderivatives of the efficient solution map S in the parametric multiobjective problem (4).

Theorem 4.1 *Let S in (4) be the efficient solution map for the multiobjective optimization problem (1). Let $\bar{x} \in S(\bar{p})$ with $\bar{y} = f(\bar{p},\bar{x})$. Let the map $N : \mathcal{R}^m \times \mathcal{R}^n \rightrightarrows \mathcal{R}^s$ be defined by $N(p,x) = f(p,x) \cap \mathcal{F}(p)$. Suppose that $\mathcal{F}$ has locally closed-graph around $(\bar{p},\bar{y})$.*

(i) *Assume that N is inner semicontinuous at $(\bar{p},\bar{x},\bar{y})$ and that one of the following constraint qualification conditions* (a) *and* (b) *holds*:

(a)

$$D^*N(\bar{p},\bar{x},\bar{y})(0) \subset \bigcup_{y^*\in\mathcal{R}^s} D^*f(\bar{p},\bar{x})(-y^*) + (D^*\mathcal{F}(\bar{p},\bar{y})(y^*),0); \tag{13}$$

(b)

$$\begin{aligned} &0 \in p^* + D^*\mathcal{F}(\bar{p},\bar{y})(y^*),\\ &(p^*,0) \in D^*f(\bar{p},\bar{x})(-y^*) \quad \Rightarrow \quad p^* = y^* = 0 \end{aligned} \tag{14}$$

and

$$\begin{aligned} &p^* \in -D^*C(\bar{p},\bar{x})(x^*) \cap \bigcup_{(p'^*,x^*)\in D^*f(\bar{p},\bar{x})(-y^*)} [p'^* + D^*\mathcal{F}(\bar{p},\bar{y})(y^*)]\\ &\quad\Rightarrow \quad p^* = x^* = 0. \end{aligned} \tag{15}$$

Then for any $x^* \in \mathcal{R}^n$,

$$D^*S(\bar{p},\bar{x})(x^*) \subset \bigcup_{x_1^* \in \mathcal{R}^n} \bigcup_{(p^*,x_1^*-x^*)\in D^*f(\bar{p},\bar{x})(-y^*)} D^*C(\bar{p},\bar{x})(x_1^*) + p^* + D^*\mathcal{F}(\bar{p},\bar{y})(y^*). \tag{16}$$

Moreover, suppose that N admits a locally upper Lipschitzian selection around $(\bar{p},\bar{x},\bar{y})$ *and that* f, C, *and* $\mathcal{F}$ *are regular at* $(\bar{p},\bar{x})$, $(\bar{p},\bar{x})$, *and* $(\bar{p},\bar{y})$, *respectively. Then* S *is regular at* $(\bar{p},\bar{x})$, *and* (16) *holds as equality.*

(ii) *Suppose that* f *is locally Lipschitz at* $(\bar{p},\bar{x})$, $M(\bar{p},\bar{y}) = \{\bar{x}\}$, *and* C *is locally compact around* $\bar{p}$. *Assume that condition* (9) *holds and that the following constraint qualification conditions hold*:

$$\begin{aligned} &-p^* \in \bigcup_{(p_1^*,x^*)\in D^*f(\bar{p},\bar{x})(y^*)} p_1^* + D^*C(\bar{p},\bar{x})(x^*), \\ &(p^*,0) \in D^*f(\bar{p},\bar{x})(-y^*) \quad \Rightarrow \quad p^* = y^* = 0; \end{aligned} \tag{17}$$

$$p^* \in -D^*C(\bar{p},\bar{x})(x^*) \cap \bigcup_{\substack{(p'^*,x^*)\in D^*f(\bar{p},\bar{x})(-y^*)\\ (p_1^*,x_1^*)\in D^*f(\bar{p},\bar{x})(y^*)}} \left[p'^* + p_1^* + D^*C(\bar{p},\bar{x})(x_1^*)\right]$$

$$\Rightarrow \quad p^* = x^* = 0. \tag{18}$$

Then for any $x^* \in \mathcal{R}^n$,

$$\begin{aligned} &D^*S(\bar{p},\bar{x})(x^*) \\ &\subset \bigcup_{x_1^* \in \mathcal{R}^n} \bigcup_{\substack{(p^*,x_1^*-x^*)\in D^*f(\bar{p},\bar{x})(-y^*)\\ (p_1^*,x_2^*)\in D^*f(\bar{p},\bar{x})(y^*)}} D^*C(\bar{p},\bar{x})(x_1^*) + p^* + p_1^* + D^*C(\bar{p},\bar{x})(x_2^*). \end{aligned} \tag{19}$$

If, moreover, f *and* C *are regular at* $(\bar{p},\bar{x})$, *then the equality relation holds,* S *is regular at* $(\bar{p},\bar{x})$, *and the qualification condition* (17) *can be replaced by*

$$\begin{aligned} &D^*N(\bar{p},\bar{x},\bar{y})(0) \\ &\subset \bigcup_{y^*\in\mathcal{R}^s} D^*f(\bar{p},\bar{x})(-y^*) + \Bigl(\bigcup_{(p^*,x^*)\in D^*f(\bar{p},\bar{x})(y^*)} p^* + D^*C(\bar{p},\bar{x})(x^*), 0\Bigr); \end{aligned} \tag{20}$$

(iii) *Suppose that* f *is strictly differentiable at* $(\bar{p},\bar{x})$, $M(\bar{p},\bar{y}) = \{\bar{x}\}$, *and* C *is locally compact around* $\bar{p}$. *Assume that condition* (9) *holds and that one of the following constraint qualification conditions* (a) *and* (b) *holds*:

(a)

$$\left[p^* = -\nabla_p f(\bar{p},\bar{x})^* y^*, 0 = \nabla_x f(\bar{p},\bar{x})^* y^*\right] \quad \Rightarrow \quad p^* = y^* = 0; \tag{21}$$

(b) *The partial derivative operator* $\nabla_x f(\bar{p},\bar{x})$ *is surjective, and*

$$\Big[p^* \in -D^*C(\bar{p},\bar{x})(x^*) \cap \bigcup_{x^*=-\nabla_x f(\bar{p},\bar{x})^* y^*} D^*C(\bar{p},\bar{x})(-x^*)\Big] \Rightarrow \quad p^* = x^* = 0. \tag{22}$$

Then for any $x^* \in \mathcal{R}^n$,

$$D^*S(\bar{p},\bar{x})(x^*) \subset \bigcup_{x_1^* \in \mathcal{R}^n} \bigcup_{x^*-x_1^*=\nabla_x f(\bar{p},\bar{x})^* y^*} D^*C(\bar{p},\bar{x})(x_1^*) + D^*C(\bar{p},\bar{x})(x^* - x_1^*). \tag{23}$$

Moreover, if C *is regular at* $(\bar{p},\bar{x})$, *then the equality relation holds,* S *is regular at* $(\bar{p},\bar{x})$, *and the qualification condition* (21) *can be replaced by*

$$D^*N(\bar{p},\bar{x},\bar{y})(0) \subset \bigcup_{y^* \in \mathcal{R}^s} \big(D^*C(\bar{p},\bar{x})\big(\nabla_x f(\bar{p},\bar{x})^* y^*\big), -\nabla_x f(\bar{p},\bar{x})^* y^*\big). \tag{24}$$

Proof Let $G(p) = \{p\} \times \mathcal{F}(p)$ and $H(p,y) = \{x \mid y = f(p,x)\}$. Then we have $S(p) = C(p) \cap H \circ G(p)$.

(i) First, we compute the coderivative of $H \circ G$. On one hand, by the definition of coderivative, $(p^*, y^*) \in D^*H(\bar{p},\bar{y},\bar{x})(x^*)$ if and only if $(p^*, -x^*) \in D^*f(\bar{p},\bar{x})(-y^*)$. On the other hand, [12, Lemma 49] ensures that for any $(p^*, y^*) \in \mathcal{R}^m \times \mathcal{R}^s$,

$$D^*G(\bar{p},\bar{p},\bar{y})(p^*, y^*) = p^* + D^*\mathcal{F}(\bar{p},\bar{y})(y^*).$$

Thus by [12, Theorem 39], for any $x^* \in \mathcal{R}^n$, we have

$$\begin{aligned} D^*H \circ G(\bar{p},\bar{x})(x^*) &\subset D^*G(\bar{p},\bar{p},\bar{y}) \circ D^*H(\bar{p},\bar{y},\bar{x})(x^*) \\ &= \bigcup_{(p^*,-x^*) \in D^*f(\bar{p},\bar{x})(-y^*)} p^* + D^*\mathcal{F}(\bar{p},\bar{y})(y^*), \end{aligned} \tag{25}$$

provided that N is inner semicontinuous at $(\bar{p},\bar{x},\bar{y})$ and (13) holds. By the expressions of G, H, and N we can easily check that the qualification condition (14) is equivalent to

$$D^*G(\bar{p},\bar{p},\bar{y})^{-1}(0) \cap D^*H(\bar{p},\bar{y},\bar{x})(0) = \{0\}, \tag{26}$$

which implies (13).

Then by Proposition 27 of [12] we get

$$D^*S(\bar{p},\bar{x})(x^*) \subset \bigcup_{x_1^* \in \mathcal{R}^n} D^*C(\bar{p},\bar{x})(x_1^*) + D^*H \circ G(\bar{p},\bar{x})(x^* - x_1^*), \tag{27}$$

provided that

$$p^* \in \big(-D^*C(\bar{p},\bar{x})(x^*)\big) \cap D^*H \circ G(\bar{p},\bar{x})(-x^*) \quad \Longrightarrow \quad p^* = x^* = 0. \tag{28}$$

By the above coderivative expressions of $H \circ G$ and C, it is a simple matter to check that (15) and (27) imply (28) and (16), respectively.

Moreover, if N admits a locally upper Lipschitzian selection around $(\bar{p},\bar{x},\bar{y})$ and f and $\mathcal{F}$ are regular at $(\bar{p},\bar{x})$ and $(\bar{p},\bar{y})$, respectively, then by [12, Propositions 36 and 37] the inclusion relation (25) becomes equality, and $H \circ G$ is regular at $(\bar{p},\bar{x})$. Furthermore, if C is regular at $(\bar{p},\bar{x})$, then by [12, Proposition 27], (27) becomes equality, and S is regular at $(\bar{p},\bar{x})$.

(ii) Since f is locally Lipschitz at $(\bar{p},\bar{y})$, by the definition of N we can easily check that N admits a locally upper Lipschitzian selection around $(\bar{p},\bar{x},\bar{y})$. Then by Theorem 3.1, $\mathcal{F}$ has the upper estimate (11), and thus (17), (18), and (16) imply (14), (15), and (19), respectively.

Moreover, if f and C are regular at $(\bar{p},\bar{x})$, then by Theorem 3.1 and the above coderivative expressions of $H \circ G$, $\mathcal{F}$, G, and $H \circ G$ are regular at $(\bar{p},\bar{y})$, $(\bar{p},\bar{p},\bar{y})$, and $(\bar{p},\bar{x})$, respectively. Thus (11) and (25) become equalities, and therefore (27) becomes equality, and S is regular at $(\bar{p},\bar{x})$. Meanwhile, qualification (20) is equivalent to (13), and therefore the qualification condition (17) can be replaced by (20).

(iii) Since f is strictly differentiable at $(\bar{p},\bar{x})$, we have that H is N-regular at $(\bar{p},\bar{y},\bar{x})$ and $p^* = -\nabla_p f(\bar{p},\bar{x})^* y^*$, $x^* = \nabla_x f(\bar{p},\bar{x})^* y^*$. If the partial derivative operator $\nabla_x f(\bar{p},\bar{x})$ is surjective, then [17, Lemma 1.17] implies that $\nabla_x f(\bar{p},\bar{x})^*$ is injective. Thus $0 = \nabla_x f(\bar{p},\bar{x})^* y^*$, implying $y^* = 0$, and therefore qualification conditions (21) hold naturally. Note that (21) and (22) are equivalent to (17) and (18), respectively. Similarly to the proof of case (ii), we can obtain the results in case (iii) by using Theorem 3.1 and Corollary 3.1. This completes the proof. □

For a strictly differentiable map with surjective derivative, we have the following results.

Corollary 4.1 *Let $\bar{x} \in C(\bar{p})$ and $\bar{y} \in \mathcal{F}(\bar{p})$. Suppose that f is strictly differentiable at $(\bar{p},\bar{x})$, $\nabla f(\bar{p},\bar{x})$ is surjective, and C is locally compact around $\bar{p}$. If $M(\bar{p},\bar{y}) = \{\bar{x}\}$ and condition (9) and (22) hold, then for any $x^* \in \mathcal{R}^n$, (23) holds.*

Proof It directly follows from Corollary 3.1 and Theorem 3.1. □

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References

1. Aubin, J.P., Frankowska, H.: Set-Valued Analysis. Birkhäuser, Boston (1990)
2. Bednarczuk, E.M., Song, W.: Contingent epiderivative and its applications to set-valued maps. Control Cybern. **27**, 375–386 (1998)
3. Chuong, T.D.: Clarke coderivatives of efficient point multifunctions in parametric vector optimization. Nonlinear Anal. **74**, 273–285 (2011)
4. Chuong, T.D., Yao, J.C.: Coderivatives of efficient point multifunctions in parametric vector optimization. Taiwan. J. Math. **13**(6), 1671–1693 (2009)
5. Göpfert, A., Riahi, H., Tammer, C., Zălinescu, C.: Variational Methods in Partially Ordered Spaces. Springer, New York (2003)
6. Huy, N.Q., Mordukhovich, B.S., Yao, J.C.: Coderivatives of frontier and solution maps in parametric multiobjective optimization. Taiwan. J. Math. **12**(8), 2083–2111 (2008)
7. Kuk, H., Tanino, T., Tanaka, M.: Sensitivity analysis in vector optimization. J. Optim. Theory Appl. **89**, 713–730 (1996)
8. Lee, G.M., Huy, N.Q.: On sensitivity analysis in vector optimization. Taiwan. J. Math. **11**, 945–958 (2007)
9. Levy, A.B.: Nonsingularity conditions for multifunctions. Set-Valued Anal. **7**, 89–99 (1999)
10. Levy, A.B., Mordukhovich, B.S.: Coderivatives in parametric optimization. Math. Program., Ser. A **99**, 311–327 (2004)
11. Li, M.H., Li, S.J.: Second-order differential and sensitivity properties of weak vector variational inequalities. J. Optim. Theory Appl. **144**, 76–87 (2010)
12. Li, S.J., Penot, J.-P., Xue, X.W.: Codifferential calculus. Set-Valued Var. Anal. **19**, 505–536 (2011)
13. Li, S.J., Xue, X.W.: Sensitivity analysis of gap functions for vector variational inequality via coderivatives. Optimization **63**(7), 1075–1098 (2014)
14. Li, S.J., Yan, H., Chen, G.Y.: Differential and sensitivity properties of gap functions for vector variational inequalities. Math. Methods Oper. Res. **57**, 377–391 (2003)
15. Lucet, Y., Ye, J.J.: Sensitivity analysis of the value function for optimization problems with variational inequality constraints. SIAM J. Control Optim. **40**, 699–723 (2001). Erratum: SIAM J. Control Optim. **41**, 1315–1319 (2002)
16. Meng, K.W., Li, S.J.: Differential and sensitivity properties of gap functions for Minty vector variational inequalities. J. Math. Anal. Appl. **337**, 386–398 (2008)
17. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, Vol. I: Basic Theory, Vol. II: Applications. Springer, Berlin (2006)
18. Mordukhovich, B.S.: Variational Analysis and Applications. Springer, Berlin (2018)
19. Mordukhovich, B.S., Nam, N.M., Yen, N.D.: Subgradients of marginal functions in parametric mathematical programming. Math. Program. **116**, 369–396 (2009)
20. Outrata, J.V.: Optimality conditions for a class of mathematical programs with equilibrium constraints. Math. Oper. Res. **24**, 627–644 (1999)
21. Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis. Springer, Berlin (1998)
22. Shi, D.S.: Contingent derivative of the perturbation map in multiobjective optimization. J. Optim. Theory Appl. **70**, 385–396 (1991)
23. Tanino, T.: Stability and sensitivity analysis in convex vector optimization. SIAM J. Control Optim. **26**, 521–536 (1988)
24. Xue, X.W., Li, S.J., Liao, C.M., Yao, J.C.: Sensitivity analysis of parametric vector set-valued optimization problems via coderivatives. Taiwan. J. Math. **15**(6), 2533–2554 (2011)
25. Zhang, R.: Weakly upper Lipschitzian multifunctions and applications to parametric optimization. Math. Program., Ser. A **102**, 153–166 (2005)