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On post quantum estimates of upper bounds involving twice (p,q)-differentiable preinvex function

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Abstract

The main objective of this paper is to derive a new post quantum integral identity using twice (p,q)-differentiable functions. Using this identity as an auxiliary result, we obtain some new post quantum estimates of upper bounds involving twice (p,q)-differentiable preinvex functions.

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1 Introduction and preliminaries

The quantum calculus is often regarded as calculus without limits, we obtain q-analogues of mathematical objects which can be recaptured by taking $q \to 1^-$. Historically the subject of quantum calculus can be traced back to Euler and Jacobi, but in recent decades it has experienced a rapid development. This can be attributed to the fact that it serves as a bridge between mathematics and physics. It is also pertinent to mention here that quantum calculus is a subfield of time scale calculus. In quantum calculus, we are concerned with a specific time scale, called the q-time scale. In the twentieth century Jackson [8] introduced the notion of q-definite integrals in quantum calculus. This motivated many quantum calculus analysts, and consequently a number of articles have been written in this area. It is worth to mention here for interested readers that it is possible that sometimes more than one q-analogue exists. In [9] interested readers may find some basic and interesting details on some recent developments of basic theory of quantum calculus. While studying quantum calculus, Tariboon et al. [23] introduced the notions of q-derivatives and q-integrals on finite intervals and developed several new q-analogues of classical inequalities. This particular article inspired many researchers working in the field of inequalities, particulary inequalities involving convexity and its generalizations. Resultantly, several new quantum analogues of classical results have been obtained. For example, Noor et al. [21] obtained the quantum analogues of Hermite–Hadamard's inequality using the class of preinvex functions. Sudsutad et al. [22] and Noor et al. [20]

obtained new quantum analogues of trapezium like inequalities involving q-differentiable convex functions. Noor et al. [19] obtained quantum analogues of Ostrowski's inequality. Zhang et al. [26] obtained a new generalized q-integral identity, and utilizing this as an auxiliary result, they have obtained several new q-analogues of classical inequalities. Liu and Zhuang [15] obtained certain new q-analogues of Hermite–Hadamard's inequality using two times q-differentiable convex functions. Alp et al. [3] obtained some new refined q-analogues of Hermite–Hadamard's inequality. For more details, see [4, 10, 11, 13]

A recent development in the study of quantum calculus is the introduction of post quantum calculus. In quantum calculus we deal with q-number with one base q; however, post quantum calculus includes p and q-numbers with two independent variables p and q. This was first considered by Chakarabarti and Jagannathan [7]. For some interesting applications, see [1, 2, 6, 12, 17, 18]. Motivated by the research work going on, Tunc and Gov [24] introduced the concepts of (p,q)-derivatives and (p,q)-integrals on finite intervals.

Since the appearance of this article, a number of new post quantum analogues of classical inequalities have been obtained. For example, Kunt et al. [14] obtained new post quantum analogues of Hermite–Hadamard's inequality. Luo et al. [16] obtained some new variants of parameterized (p,q)-integral inequalities using a generalized integral identity involving (p,q)-differentiable functions.

The main idea behind the study of this paper is to obtain a new general post quantum integral inequality using twice (p,q)-differentiable functions. We then establish some new estimates of post quantum bounds essentially using the class of preinvex functions. We hope that the ideas and techniques of this paper will inspire interested readers working in this field.

Before we move to our next section of the paper, let us recall the definitions of invex set and preinvex function.

Definition 1.1 ([24]) Let $\mathcal{K} \subseteq \mathbb{R}$ be a nonempty set such that $a \in \mathcal{K}$, $0 < q < p \leq 1$, and let $f : \mathcal{K} \to \mathbb{R}$ be a continuous function. Then the (p,q)-derivative ${}_a\mathcal{D}_{p,q}f(x)$ of f at $x \in \mathcal{K}$ is defined by

$$ {}_a\mathcal{D}_{p,q}f(x) = \frac{f(px + (1-p)a) - f(qx + (1-q)a)}{(p-q)(x-a)} \quad (x \neq a). $$

Definition 1.2 ([24]) Let $\mathcal{K} \subseteq \mathbb{R}$ be a nonempty set such that $a \in \mathcal{K}$, $0 < q < p \leq 1$, and let $f : \mathcal{K} \to \mathbb{R}$ be a continuous function. Then (p,q)-integral on $\mathcal{K}$ is defined by

$$ \int_a^x f(t)\, {}_a\mathrm{d}_{p,q}t = (p-q)(x-a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}x + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right) $$

for $x \in \mathcal{K}$.

Definition 1.3 ([5]) A nonempty set $\mathcal{K} \subseteq \mathbb{R}$ is said to be invex with respect to the bivariate function $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ if

$$ a + \mu\zeta(b,a) \in \mathcal{K} $$

for all $a, b \in \mathcal{K}$ and $\mu \in [0,1]$.

Definition 1.4 ([25]) Let $\mathcal{K} \subseteq \mathbb{R}$ be an invex set with respect to the bivariate function $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Then the real-valued function $f : \mathcal{K} \to \mathbb{R}$ is said to be preinvex with respect to ζ if

$$f\big(a + \mu\zeta(b,a)\big) \leq (1-t)f(a) + tf(b)$$

for all $a, b \in \mathcal{K}$ and $t \in [0,1]$.

2 Results and discussions

In this section, we derive our main results. First of all we derive our new post quantum integral identity involving twice (p,q)-differentiable function.

Lemma 2.1 *Let $0 < q < p \leq 1$, $f : \mathcal{K} \to \mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$ (where $\mathcal{K}^\circ$ is the interior of $\mathcal{K}$), and ${}_a\mathcal{D}^2_{p,q}f$ be continuous and (p,q)-integrable on $\mathcal{K}$. Then*

$$\begin{aligned} &\frac{qf(a) + pf(a + p\zeta(b,a))}{p+q} - \frac{1}{p^2\zeta(b,a)} \int_a^{a+p^2\zeta(b,a)} f(x)\, {}_0\mathrm{d}_{p,q}x \\ &\quad = \frac{pq^2\zeta^2(b,a)}{p+q} \int_0^1 t(1-qt)\, {}_a\mathcal{D}^2_{p,q} f\big(a + t\zeta(b,a)\big)\, {}_a\mathrm{d}_{p,q}t. \end{aligned}$$

Proof It suffices to prove that

$$\begin{aligned} &{}_a\mathcal{D}^2_{p,q} f\big(a + t\zeta(b,a)\big) \\ &\quad = {}_a\mathcal{D}_{p,q}\big({}_a\mathcal{D}_{p,q} f\big(a + t\zeta(b,a)\big)\big) \\ &\quad = \frac{{}_a\mathcal{D}_{p,q} f(a + pt\zeta(b,a)) - {}_a\mathcal{D}_{p,q} f(a + qt\zeta(b,a))}{t(p-q)\zeta(b,a)} \\ &\quad = \frac{1}{t(p-q)\zeta(b,a)} \bigg[\frac{f(a + p^2t\zeta(b,a)) - f(a + pqt\zeta(b,a))}{tp(p-q)\zeta(b,a)} \\ &\qquad - \frac{f(a + pqt\zeta(b,a)) - f(a + q^2t\zeta(b,a))}{tq(p-q)\zeta(b,a)} \bigg] \\ &\quad = \frac{qf(a + p^2t\zeta(b,a)) - (p+q)f(a + pqt\zeta(b,a)) + pf(a + q^2t\zeta(b,a))}{pqt^2(p-q)^2\zeta^2(b,a)}. \end{aligned}$$

Elaborated computation leads to

$$\begin{aligned} &\int_0^1 t(1-qt)\, {}_a\mathcal{D}^2_{p,q} f\big(a + t\zeta(b,a)\big)\, {}_0\mathrm{d}_{p,q}t \\ &\quad = \int_0^1 t(1-qt) \\ &\qquad \times \frac{qf(a + p^2t\zeta(b,a)) - (p+q)f(a + pqt\zeta(b,a)) + pf(a + q^2t\zeta(b,a))}{t^2pq(p-q)^2\zeta^2(b,a)}\, {}_0\mathrm{d}_{p,q}t \\ &\quad = \frac{1}{pq(p-q)\zeta^2(b,a)} \Bigg[q\sum_{n=0}^{\infty} f\bigg(a + p^2\frac{q^n}{p^{n+1}}\zeta(b,a)\bigg) \\ &\qquad - (p+q)\sum_{n=0}^{\infty} f\bigg(a + p\frac{q^{n+1}}{p^{n+1}}\zeta(b,a)\bigg) + p\sum_{n=0}^{\infty} f\bigg(a + \frac{q^{n+2}}{p^{n+1}}\zeta(b,a)\bigg)\Bigg] \end{aligned}$$

$$
\begin{aligned}
&-q\left\{\frac{q(p-q)\zeta(b,a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f(a+p^2\frac{q^n}{p^{n+1}}\zeta(b,a))}{pq(p-q)^2\zeta^3(b,a)}\right.\\
&-\frac{(p+q)(p-q)\zeta(b,a)\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+1}}f(a+p\frac{q^{n+1}}{p^{n+1}}\zeta(b,a))}{pq^2(p-q)^2\zeta^3(b,a)}\\
&\left.+\frac{p(p-q)\zeta(b,a)\sum_{n=0}^{\infty}\frac{q^{n+2}}{p^{n+1}}f(a+\frac{q^{n+2}}{p^{n+1}}\zeta(b,a))}{pq^3(p-q)^2\zeta^3(b,a)}\right\}\\
={}&\frac{q[\sum_{n=0}^{\infty}f(a+p^2\frac{q^n}{p^{n+1}}\zeta(b,a))-\sum_{n=0}^{\infty}f(a+p\frac{q^{n+1}}{p^{n+1}}\zeta(b,a))]}{pq(p-q)\zeta^2(b,a)}\\
&-\frac{p[\sum_{n=0}^{\infty}f(a+p\frac{q^{n+1}}{p^{n+1}}\zeta(b,a))-\sum_{n=0}^{\infty}f(a+\frac{q^{n+2}}{p^{n+1}}\zeta(b,a))]}{pq(p-q)\zeta^2(b,a)}\\
&-q\left\{\frac{q(p-q)\zeta(b,a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f(a+p^2\frac{q^n}{p^{n+1}}\zeta(b,a))}{pq(p-q)^2\zeta^3(b,a)}\right.\\
&-\frac{p(p+q)(p-q)\zeta(b,a)\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+2}}f(a+p^2\frac{q^{n+1}}{p^{n+2}}\zeta(b,a))}{pq^2(p-q)^2\zeta^3(b,a)}\\
&\left.+\frac{p^3(p-q)\zeta(b,a)\sum_{n=0}^{\infty}\frac{q^{n+2}}{p^{n+3}}f(a+p^2\frac{q^{n+2}}{p^{n+3}}\zeta(b,a))}{pq^3(p-q)^2\zeta^3(b,a)}\right\}\\
={}&\frac{q[f(a+p\zeta(b,a))-f(a)]-p[f(a+q\zeta(b,a))-f(a)]}{pq(p-q)\zeta^2(b,a)}\\
&-\frac{p+q}{p^3q^2\zeta^3(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_0\mathrm{d}_{p,q}t\\
&-\frac{q^2+pq-p^2}{pq^2(p-q)\zeta^2(b,a)}f\big(a+p\zeta(b,a)\big)+\frac{f(a+q\zeta(b,a))}{q(p-q)\zeta^2(b,a)}\\
={}&\frac{f(a)}{pq\zeta^2(b,a)}+\frac{f(a+p\zeta(b,a))}{q^2\zeta^2(b,a)}\\
&-\frac{p+q}{p^3q^2\zeta^3(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x.
\end{aligned}
$$

Multiplying both sides of the above equality by $\frac{pq^2\zeta^2(b,a)}{p+q}$, we get the required result. □

Theorem 2.2 *Let $0<q<p\le 1$, $\mathcal{K}\subseteq\mathbb{R}$ be an invex set with respect to the bivariate function $\zeta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and $f:\mathcal{K}\to\mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$ such that ${}_a\mathcal{D}_{p,q}^2f$ is continuous and (p,q)-integrable on $\mathcal{K}$. Then the inequality*

$$
\begin{aligned}
&\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right|\\
&\quad\le\frac{pq^2\zeta^2(b,a)((p^4-p^3+p^2q^2)|{}_a\mathcal{D}_{p,q}^2f(a)|+p^3|{}_a\mathcal{D}_{p,q}^2f(b)|)}{(p+q)^2(p^2+q^2)(q^2+pq+p^2)}
\end{aligned}
$$

holds for all $a,b\in\mathcal{K}$ if $|{}_a\mathcal{D}_{p,q}^2f|$ is a preinvex function with respect to ζ.

Proof It follows from Lemma 2.1 and the property of the modulus together with the preinvexity of $|{}_a\mathcal{D}^2_{p,q}f|$ that

$$\begin{aligned}
&\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\quad\le\frac{pq^2\zeta^2(b,a)}{p+q}\int_0^1 t(1-qt)\left|{}_a\mathcal{D}^2_{p,q}f\big(a+t\zeta(b,a)\big)\right|{}_0\mathrm{d}_{p,q}t \\
&\quad\le\frac{pq^2\zeta^2(b,a)}{p+q}\left(\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|\int_0^1 t(1-t)(1-qt)\,{}_0\mathrm{d}_{p,q}t\right. \\
&\qquad\left.+\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|\int_0^1 t^2(1-qt)\,{}_0\mathrm{d}_{p,q}t\right) \\
&\quad=\frac{pq^2\zeta^2(b,a)((p^4-p^3+p^2q^2)|{}_a\mathcal{D}^2_{p,q}f(a)|+p^3|{}_a\mathcal{D}^2_{p,q}f(b)|)}{(p+q)^2(p^2+q^2)(q^2+pq+p^2)}.
\end{aligned}$$

□

Theorem 2.3 *Let $0<q<p\le 1$, $r>1$, $\mathcal{K}\subseteq\mathbb{R}$ be an invex set with respect to the bivariate function $\zeta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and $f:\mathcal{K}\to\mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$ such that ${}_a\mathcal{D}^2_{p,q}f$ is continuous and (p,q)-integrable on $\mathcal{K}$. Then the inequality*

$$\begin{aligned}
&\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\quad\le\frac{pq^2\zeta^2(b,a)}{(p+q)^{2-\frac{1}{r}}}\left(d_1\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r+d_2\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\right)^{\frac{1}{r}}
\end{aligned}$$

is valid for all $a,b\in\mathcal{K}$ if $|{}_a\mathcal{D}^2_{p,q}f|^r$ is preinvex with respect to ζ, where

$$d_1=(p-q)\sum_{n=0}^{\infty}\left(\frac{q^{2n}}{p^{2n+2}}-\frac{q^{3n}}{p^{3n+3}}\right)\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^r$$

and

$$d_2=(p-q)\sum_{n=0}^{\infty}\frac{q^{3n}}{p^{3n+3}}\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^r.$$

Proof From Lemma 2.1, Hölder's inequality, and the preinvexity of $|{}_a\mathcal{D}^2_{p,q}f|^r$, we get

$$\begin{aligned}
&\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\quad\le\frac{pq^2\zeta^2(b,a)}{p+q}\int_0^1 t(1-qt)\left|{}_a\mathcal{D}^2_{p,q}f\big(a+t\zeta(b,a)\big)\right|{}_0\mathrm{d}_{p,q}t \\
&\quad\le\frac{pq^2\zeta^2(b,a)}{p+q}\left(\int_0^1 t\,{}_0\mathrm{d}_{p,q}t\right)^{1-\frac{1}{r}}\left(\int_0^1 t(1-qt)^r\left|{}_a\mathcal{D}^2_{p,q}f\big(a+t\zeta(b,a)\big)\right|^r{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}} \\
&\quad\le\frac{pq^2\zeta^2(b,a)}{p+q}\left(\frac{1}{p+q}\right)^{1-\frac{1}{r}}
\end{aligned}$$

$$\times\left(\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r\int_0^1 t(1-t)(1-qt)^r\,{}_0\mathrm{d}_{p,q}t+\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\int_0^1 t^2(1-qt)^r\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}}$$
$$=\frac{pq^2\zeta^2(b,a)}{(p+q)^{2-\frac{1}{r}}}\left(d_1\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r+d_2\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\right)^{\frac{1}{r}}.\qquad\square$$

Theorem 2.4 *Let $0<q<p\leq 1$, $r,s>1$ with $1/r+1/s=1$, $\mathcal{K}\subseteq\mathbb{R}$ be an invex set with respect to the bivariate function $\zeta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and $f:\mathcal{K}\to\mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$ such that ${}_a\mathcal{D}^2_{p,q}f$ is continuous and (p,q)-integrable on $\mathcal{K}$. Then the inequality*

$$\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right|$$
$$\leq\frac{pq^2\zeta^2(b,a)}{(p+q)}h^{\frac{1}{s}}\left(\frac{(q^2+p^2+pq-p-q)|{}_a\mathcal{D}^2_{p,q}f(a)|^r+(p+q)|{}_a\mathcal{D}^2_{p,q}f(b)|^r}{(p+q)(q^2+pq+p^2)}\right)^{\frac{1}{r}}$$

takes place for all $a,b\in\mathcal{K}$ if $|{}_a\mathcal{D}^2_{p,q}f|^r$ is preinvex with respect to ζ, where

$$h=(p-q)\sum_{n=0}^{\infty}\frac{q^{2n}}{p^{2n+2}}\left(1-\frac{q^n}{p^{n+1}}\right)^s.$$

Proof Using Lemma 2.1, Hölder's inequality, and the preinvexity of $|{}_a\mathcal{D}^2_{p,q}f|^r$, we have

$$\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right|$$
$$\leq\frac{pq^2\zeta^2(b,a)}{p+q}\int_0^1 t(1-qt)\left|{}_a\mathcal{D}^2_{p,q}f\left(a+t\zeta(b,a)\right)\right|\,{}_0\mathrm{d}_{p,q}t$$
$$\leq\frac{pq^2\zeta^2(b,a)}{p+q}\left(\int_0^1 t(1-qt)^s\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{s}}\left(\int_0^1 t\left|{}_a\mathcal{D}^2_{p,q}f\left(a+t\zeta(b,a)\right)\right|^r\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}}$$
$$\leq\frac{pq^2\zeta^2(b,a)}{p+q}$$
$$\times\left(\int_0^1 t(1-qt)^s\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{s}}$$
$$\times\left(\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r\int_0^1 t(1-t)\,{}_0\mathrm{d}_{p,q}t+\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\int_0^1 t^2\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}}$$
$$=\frac{pq^2\zeta^2(b,a)}{(p+q)}h^{\frac{1}{s}}\left(\frac{(q^2+p^2+pq-p-q)|{}_a\mathcal{D}^2_{p,q}f(a)|^r+(p+q)|{}_a\mathcal{D}^2_{p,q}f(b)|^r}{(p+q)(q^2+pq+p^2)}\right)^{\frac{1}{r}}.\qquad\square$$

Theorem 2.5 *Let $0<q<p\leq 1$, $r>1$, $\mathcal{K}\subseteq\mathbb{R}$ be an invex set with respect to the bivariate function $\zeta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and $f:\mathcal{K}\to\mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$ such that ${}_a\mathcal{D}^2_{p,q}f$ is continuous and (p,q)-integrable on $\mathcal{K}$. Then one has*

$$\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right|$$
$$\leq\frac{pq^2\zeta^2(b,a)}{(p+q)}\left(k_1\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r+k_2\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\right)^{\frac{1}{r}}$$

for all $a,b\in\mathcal{K}$ *if* $|{}_a\mathcal{D}^2_{p,q}f|^r$ *is a preinvex function with respect to* ζ, *where*

$$k_1=(p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{r+1}\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^r$$

and

$$k_2=(p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{r+3}\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^r.$$

Proof It follows from Lemma 2.1 and Hölder's inequality together with the preinvexity of $|{}_a\mathcal{D}^2_{p,q}f|^r$ that

$$\begin{aligned}&\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right|\\&\quad\leq\frac{pq^2\zeta^2(b,a)}{p+q}\int_0^1 t(1-qt)\left|{}_a\mathcal{D}^2_{p,q}f\left(a+t\zeta(b,a)\right)\right|\,{}_0\mathrm{d}_{p,q}t\\&\quad\leq\frac{pq^2\zeta^2(b,a)}{p+q}\left(\int_0^1 1\,{}_0\mathrm{d}_{p,q}t\right)^{1-\frac{1}{r}}\left(\int_0^1 t^r(1-qt)^r\left|{}_a\mathcal{D}^2_{p,q}f\left(a+t\zeta(b,a)\right)\right|^r\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}}\\&\quad\leq\frac{pq^2\zeta^2(b,a)}{p+q}\\&\qquad\times\left(\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r\int_0^1 t^r(1-t)(1-qt)^r\,{}_0\mathrm{d}_{p,q}t+\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\right.\\&\qquad\left.\times\int_0^1 t^{r+2}(1-qt)^r\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}}\\&\quad=\frac{pq^2\zeta^2(b,a)}{(p+q)}\left(k_1\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r+k_2\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\right)^{\frac{1}{r}}.\end{aligned}$$

□

Theorem 2.6 *Let* $0<q<p\leq 1$, $r,s>1$ *with* $1/r+1/s=1$, $\mathcal{K}\subseteq\mathbb{R}$ *be an invex set with respect to the bivariate function* $\zeta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, *and* $f:\mathcal{K}\to\mathbb{R}$ *be a twice* (p,q)*-differentiable function on* $\mathcal{K}^\circ$ *such that* ${}_a\mathcal{D}^2_{p,q}f$ *is continuous and* (p,q)*-integrable on* $\mathcal{K}$. *Then the inequality*

$$\begin{aligned}&\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right|\\&\quad\leq\frac{pq^2\zeta^2(b,a)}{(p+q)}m^{\frac{1}{s}}\left(\frac{q|{}_a\mathcal{D}^2_{p,q}f(a)|^r+|{}_a\mathcal{D}^2_{p,q}f(b)|^r}{(1+q)}\right)^{\frac{1}{r}},\end{aligned}$$

holds for all $a,b\in\mathcal{K}$ *if* $|{}_a\mathcal{D}^2_{p,q}f|^r$ *is preinvex with respect to* ζ, *where*

$$m=(p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{s+1}\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^s.$$

Proof Making use of Lemma 2.1, Hölder's inequality, and the preinvexity of $|_a\mathcal{D}^2_{p,q}f|^r$, we have

$$\begin{aligned}
&\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\quad\leq\frac{pq^2\zeta^2(b,a)}{p+q}\int_0^1 t(1-qt)\left|{}_a\mathcal{D}^2_{p,q}f\left(a+t\zeta(b,a)\right)\right|{}_0\mathrm{d}_{p,q}t \\
&\quad\leq\frac{pq^2\zeta^2(b,a)}{p+q}\left(\int_0^1 t^s(1-qt)^s\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{s}}\left(\int_0^1\left|{}_a\mathcal{D}^2_{p,q}f\left(a+t\zeta(b,a)\right)\right|^r{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}} \\
&\quad\leq\frac{pq^2\zeta^2(b,a)}{p+q}\left(\int_0^1 t^s(1-qt)^s\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{s}} \\
&\qquad\times\left(\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r\int_0^1(1-t)\,{}_0\mathrm{d}_{p,q}t+\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\int_0^1 t\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}} \\
&\quad=\frac{pq^2\zeta^2(b,a)}{(p+q)}m^{\frac{1}{s}}\left(\frac{(p+q-1)|{}_a\mathcal{D}^2_{p,q}f(a)|^r+|{}_a\mathcal{D}^2_{p,q}f(b)|^r}{(p+q)}\right)^{\frac{1}{r}}.
\end{aligned}$$

□

Theorem 2.7 *Let $0<q<p\leq 1$, $r,s>1$ with $1/r+1/s=1$, $\mathcal{K}\subseteq\mathbb{R}$ be an invex set with respect to the bivariate function $\zeta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and $f:\mathcal{K}\to\mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$ such that ${}_a\mathcal{D}^2_{p,q}f$ is continuous and (p,q)-integrable on $\mathcal{K}$. Then*

$$\begin{aligned}
&\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\quad\leq\frac{pq^2\zeta^2(b,a)}{(p+q)}\left(\frac{p-q}{p^{s+1}-q^{s+1}}\right)^{\frac{1}{s}}\left(u_1\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r+u_2\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\right)^{\frac{1}{r}}
\end{aligned}$$

for all $a,b\in\mathcal{K}$ if $|{}_a\mathcal{D}^2_{p,q}f|^r$ is a preinvex function with respect to ζ, where

$$u_1=(p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}-\frac{q^{2n}}{p^{2n+2}}\right)\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^r$$

and

$$u_2=(p-q)\sum_{n=0}^{\infty}\frac{q^{2n}}{p^{2n+2}}\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^r.$$

Proof From Lemma 2.1, Hölder's inequality, and the preinvexity of $|_a\mathcal{D}^2_{p,q}f|^r$, we have

$$\begin{aligned}
&\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\quad\leq\frac{pq^2\zeta^2(b,a)}{p+q}\int_0^1 t(1-qt)\left|{}_a\mathcal{D}^2_{p,q}f\left(a+t\zeta(b,a)\right)\right|{}_0\mathrm{d}_{p,q}t \\
&\quad\leq\frac{pq^2\zeta^2(b,a)}{p+q}\left(\int_0^1 t^s\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{s}}\left(\int_0^1(1-qt)^r\left|{}_a\mathcal{D}^2_{p,q}f\left(a+t\zeta(b,a)\right)\right|^r{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}} \\
&\quad\leq\frac{pq^2\zeta^2(b,a)}{p+q}\left(\frac{p-q}{p^{s+1}-q^{s+1}}\right)^{\frac{1}{s}}
\end{aligned}$$

$$\times\left(\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r\int_0^1(1-t)(1-qt)^r\,{}_0\mathrm{d}_{p,q}t+\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\int_0^1 t(1-qt)^r\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}}$$
$$=\frac{pq^2\zeta^2(b,a)}{(p+q)}\left(\frac{p-q}{p^{s+1}-q^{s+1}}\right)^{\frac{1}{s}}\left(u_1\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r+u_2\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\right)^{\frac{1}{r}}. \qquad \square$$

Theorem 2.8 *Let $0<q<p\leq 1$, $r,s>1$ with $1/r+1/s=1$, $\mathcal{K}\subseteq\mathbb{R}$ be an invex set with respect to the bivariate function $\zeta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and $f:\mathcal{K}\to\mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$ such that ${}_a\mathcal{D}^2_{p,q}f$ is continuous and (p,q)-integrable on $\mathcal{K}$. Then the inequality*

$$\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right|$$
$$\leq\frac{pq^2\zeta^2(b,a)}{(p+q)}w^{\frac{1}{s}}\left(\left(\frac{p-q}{p^{r+1}-q^{r+1}}-\frac{p-q}{p^{r+2}-q^{r+2}}\right)\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r\right.$$
$$\left.+\frac{p-q}{p^{r+2}-q^{r+2}}\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\right)^{\frac{1}{r}}$$

holds for all $a,b\in\mathcal{K}$ if $|{}_a\mathcal{D}^2_{p,q}f|^r$ is a preinvex function with respect to ζ, where

$$w=(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^s.$$

Proof According to Lemma 2.1, Hölder's inequality, and the preinvexity of $|{}_a\mathcal{D}^2_{p,q}f|^r$, one has

$$\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right|$$
$$\leq\frac{pq^2\zeta^2(b,a)}{p+q}\int_0^1(1-qt)\left|{}_a\mathcal{D}^2_{p,q}f\bigl(a+t\zeta(b,a)\bigr)\right|\,{}_0\mathrm{d}_{p,q}t$$
$$\leq\frac{pq^2\zeta^2(b,a)}{p+q}\left(\int_0^1(1-qt)^s\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{s}}\left(\int_0^1 t^r\left|{}_a\mathcal{D}^2_{p,q}f\bigl(a+t\zeta(b,a)\bigr)\right|^r\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}}$$
$$\leq\frac{pq^2\zeta^2(b,a)}{p+q}\left(\int_0^1(1-qt)^s\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{s}}$$
$$\times\left(\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r\int_0^1 t^r(1-t)\,{}_0\mathrm{d}_{p,q}t+\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\int_0^1 t^{r+1}\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}}$$
$$=\frac{pq^2\zeta^2(b,a)}{(p+q)}w^{\frac{1}{s}}\left(\left(\frac{p-q}{p^{r+1}-q^{r+1}}-\frac{p-q}{p^{r+2}-q^{r+2}}\right)\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r\right.$$
$$\left.+\frac{p-q}{p^{r+2}-q^{r+2}}\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\right)^{\frac{1}{r}}. \qquad \square$$

Theorem 2.9 *Let $0<q<p\leq 1$, $r>1$, $\mathcal{K}\subseteq\mathbb{R}$ be an invex set with respect to the bivariate function $\zeta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and $f:\mathcal{K}\to\mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$ such* *that ${}_a\mathcal{D}^2_{p,q}f$ is continuous and (p,q)-integrable on $\mathcal{K}$. Then*

$$\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right| \leq \frac{p^{2-\frac{1}{r}}q^2\zeta^2(b,a)}{(p+q)^{2-\frac{1}{r}}}\left(\frac{p^2}{q^2+pq+p^2}\right)^{1-\frac{1}{r}} \times\left(\frac{(p^4-p^3+p^2q^2)|{}_a\mathcal{D}^2_{p,q}f(a)|^r+p^3|{}_a\mathcal{D}^2_{p,q}f(b)|^r}{(p+q)(p^2+q^2)(q^2+pq+p^2)}\right)^{\frac{1}{r}}$$

for all $a,b\in\mathcal{K}$ if $|{}_a\mathcal{D}^2_{p,q}f|^r$ is preinvex with respect to ζ.

Proof Lemma 2.1, Hölder's inequality, and the preinvexity of $|{}_a\mathcal{D}^2_{p,q}f|^r$ lead to

$$\begin{aligned}
&\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\leq \frac{pq^2\zeta^2(b,a)}{p+q}\int_0^1 t(1-qt)\left|{}_a\mathcal{D}^2_{p,q}f\big(a+t\zeta(b,a)\big)\right|{}_0\mathrm{d}_{p,q}t \\
&\leq \frac{pq^2\zeta^2(b,a)}{p+q}\left(\int_0^1 t(1-qt)\,{}_0\mathrm{d}_{p,q}t\right)^{1-\frac{1}{r}} \\
&\quad\times\left(\int_0^1 t(1-qt)\left|{}_a\mathcal{D}^2_{p,q}f\big(a+t\zeta(b,a)\big)\right|^r{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}} \\
&\leq \frac{pq^2\zeta^2(b,a)}{p+q}\left(\int_0^1 t(1-qt)\,{}_0\mathrm{d}_{p,q}t\right)^{1-\frac{1}{r}} \\
&\quad\times\left(\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r\int_0^1 t(1-t)(1-qt)\,{}_0\mathrm{d}_{p,q}t+\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\int_0^1 t^2(1-qt)\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}} \\
&= \frac{p^{2-\frac{1}{r}}q^2\zeta^2(b,a)}{(p+q)^{2-\frac{1}{r}}}\left(\frac{p^2}{q^2+pq+p^2}\right)^{1-\frac{1}{r}} \\
&\quad\times\left(\frac{(p^4-p^3+p^2q^2)|{}_a\mathcal{D}^2_{p,q}f(a)|^r+p^3|{}_a\mathcal{D}^2_{p,q}f(b)|^r}{(p+q)(p^2+q^2)(q^2+pq+p^2)}\right)^{\frac{1}{r}}.
\end{aligned}$$

□

Theorem 2.10 *Let $0<q<p\leq 1$, $r>1$, $\mathcal{K}\subseteq\mathbb{R}$ be an invex set with respect to the bivariate function $\zeta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and $f:\mathcal{K}\to\mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$ such that ${}_a\mathcal{D}^2_{p,q}f$ is continuous and (p,q)-integrable on $\mathcal{K}$. Then one has*

$$\begin{aligned}
&\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\leq \frac{p^{2-\frac{1}{r}}q^2\zeta^2(b,a)}{(p+q)^{2-\frac{1}{r}}}\left(\left[\frac{p-q}{p^{r+1}-q^{r+1}}-\frac{(p-q)(1+q)}{p^{r+2}-q^{r+2}}+\frac{q(p-q)}{p^{r+3}-q^{r+3}}\right]\left|{}_a\mathcal{D}^2_{p,q}f(a)\right|^r\right. \\
&\quad\left.+\left[\frac{p-q}{p^{r+2}-q^{r+2}}-\frac{q(p-q)}{p^{r+3}-q^{r+3}}\right]\left|{}_a\mathcal{D}^2_{p,q}f(b)\right|^r\right)^{\frac{1}{r}}
\end{aligned}$$

for all $a,b\in\mathcal{K}$ if $|{}_a\mathcal{D}^2_{p,q}f|^r$ is a preinvex function with respect to ζ.

Proof Making use of Lemma 2.1, Hölder's inequality, and the preinvexity of $|{}_a\mathcal{D}^2_{p,q}f|^r$, we have

$$\begin{aligned}
&\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\quad\leq \frac{pq^2\zeta^2(b,a)}{p+q}\int_0^1 t(1-qt)\big|{}_a\mathcal{D}^2_{p,q}f\big(a+t\zeta(b,a)\big)\big|\,{}_0\mathrm{d}_{p,q}t \\
&\quad\leq \frac{pq^2\zeta^2(b,a)}{p+q}\left(\int_0^1 (1-qt)\,{}_0\mathrm{d}_{p,q}t\right)^{1-\frac{1}{r}}\left(\int_0^1 t^r(1-qt)\big|{}_a\mathcal{D}^2_{p,q}f\big(a+t\zeta(b,a)\big)\big|^r\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}} \\
&\quad\leq \frac{pq^2\zeta^2(b,a)}{p+q}\left(\int_0^1 (1-qt)\,{}_0\mathrm{d}_{p,q}t\right)^{1-\frac{1}{r}} \\
&\qquad\times\left(\big|{}_a\mathcal{D}^2_{p,q}f(a)\big|^r\int_0^1 t^r(1-t)(1-qt)\,{}_0\mathrm{d}_{p,q}t+\big|{}_a\mathcal{D}^2_{p,q}f(b)\big|^r\int_0^1 t^{r+1}(1-qt)\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}} \\
&\quad=\frac{p^{2-\frac{1}{r}}q^2\zeta^2(b,a)}{(p+q)^{2-\frac{1}{r}}}\left(\left[\frac{p-q}{p^{r+1}-q^{r+1}}-\frac{(p-q)(1+q)}{p^{r+2}-q^{r+2}}+\frac{q(p-q)}{p^{r+3}-q^{r+3}}\right]\big|{}_a\mathcal{D}^2_{p,q}f(a)\big|^r\right. \\
&\qquad\left.+\left[\frac{p-q}{p^{r+2}-q^{r+2}}-\frac{q(p-q)}{p^{r+3}-q^{r+3}}\right]\big|{}_a\mathcal{D}^2_{p,q}f(b)\big|^r\right)^{\frac{1}{r}}. \qquad \square
\end{aligned}$$

Theorem 2.11 *Let $0<q<p\leq 1$, $r,s>1$ with $1/r+1/s=1$, $\mathcal{K}\subseteq\mathbb{R}$ be an invex set with respect to the bivariate function $\zeta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and $f:\mathcal{K}\to\mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$ such that ${}_a\mathcal{D}^2_{p,q}f$ is continuous and (p,q)-integrable on $\mathcal{K}$. Then the inequality*

$$\begin{aligned}
&\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\quad\leq \frac{pq^2\zeta^2(b,a)}{(p+q)}\lambda^{\frac{1}{s}}\left(\frac{(p^3-p^2+pq^2+p^2q)|{}_a\mathcal{D}^2_{p,q}f(a)|^r+p^2|{}_a\mathcal{D}^2_{p,q}f(b)|^r}{(p+q)(q^2+pq+p^2)}\right)^{\frac{1}{r}}
\end{aligned}$$

holds for all $a,b\in\mathcal{K}$ if $|{}_a\mathcal{D}^2_{p,q}f|^r$ is preinvex function with respect to ζ, where

$$\lambda=\frac{p-q}{p^{s+1}-q^{s+1}}-\frac{q(p-q)}{p^{s+2}-q^{s+2}}.$$

Proof It follows from Lemma 2.1, Hölder's inequality, and the preinvexity of $|{}_a\mathcal{D}^2_{p,q}f|^r$ that

$$\begin{aligned}
&\left|\frac{qf(a)+pf(a+p\zeta(b,a))}{p+q}-\frac{1}{p^2\zeta(b,a)}\int_a^{a+p^2\zeta(b,a)}f(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\quad\leq \frac{pq^2\zeta^2(b,a)}{p+q}\int_0^1 t(1-qt)\big|{}_a\mathcal{D}^2_{p,q}f\big(a+t\zeta(b,a)\big)\big|\,{}_0\mathrm{d}_{p,q}t \\
&\quad\leq \frac{pq^2\zeta^2(b,a)}{p+q}\left(\int_0^1 t^s(1-qt)\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{s}}\left(\int_0^1 (1-qt)\big|{}_a\mathcal{D}^2_{p,q}f\big(a+t\zeta(b,a)\big)\big|^r\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}} \\
&\quad\leq \frac{pq^2\zeta^2(b,a)}{p+q}\left(\int_0^1 t^s(1-qt)\,{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{s}}
\end{aligned}$$

$$\times \left(\left|{}_a\mathcal{D}^2_{p,q} f(a)\right|^r \int_0^1 (1-t)(1-qt)\,{}_0\mathrm{d}_{p,q}t + \left|{}_a\mathcal{D}^2_{p,q} f(b)\right|^r \int_0^1 t(1-qt)\,{}_0\mathrm{d}_{p,q}t \right)^{\frac{1}{r}}$$

$$= \frac{pq^2\zeta^2(b,a)}{(p+q)}\lambda^{\frac{1}{s}} \left(\frac{(p^3-p^2+pq^2+p^2q)|{}_a\mathcal{D}^2_{p,q} f(a)|^r + p^2|{}_a\mathcal{D}^2_{p,q} f(b)|^r}{(p+q)(q^2+pq+p^2)} \right)^{\frac{1}{r}}. \qquad \square$$

3 Conclusion

We have derived a new generalized post quantum integral identity using twice (p,q)-differentiable functions. Utilizing this new identity as an auxiliary result, we have obtained several new post quantum estimates of upper bounds using the class of preinvex functions. We would like to mention here that if $\zeta(b,a) = b-a$, then all the main results of this paper reduce to the results for classical convex functions, and it is pertinent to mention here that these results are also new in the literature. We hope that the ideas and techniques of this paper will inspire interested readers working in this field.

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