Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Approximately two-dimensional harmonic (p_1, h_1)-(p_2, h_2)-convex functions and related integral inequalities

Saad Ihsan Butt[1], Artion Kashuri[2], Muhammad Nadeem[1], Adnan Aslam[3] and Wei Gao[4*]

*Correspondence:
gaowei@ynnu.edu.cn
[4]School of Information Science and Technology, Yunnan Normal University, 650500, Kunming, China
Full list of author information is available at the end of the article

Abstract

The aim of this study is to introduce the notion of two-dimensional approximately harmonic (p_1, h_1)-(p_2, h_2)-convex functions. We show that the new class covers many new and known extensions of harmonic convex functions. We formulate several new refinements of Hermite–Hadamard like inequalities involving two-dimensional approximately harmonic (p_1, h_1)-(p_2, h_2)-convex functions. We discuss in detail the special cases that can be deduced from the main results of the paper.

Keywords: Hermite–Hadamard inequality; Hölder inequality; Convexity

1 Introduction and preliminaries

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if

$$f\big((1-t)x + ty\big) \le (1-t)f(x) + tf(y), \quad \forall x, y \in I, t \in [0,1].$$

In recent years the classical concepts of convex functions have been extended and generalized in different directions using innovative and novel ideas. Işcan [1] introduced the class of harmonic convex functions.

A function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically convex if

$$f\left(\frac{xy}{(1-t)x + ty}\right) \le tf(x) + (1-t)f(y), \quad \forall x, y \in I, t \in [0,1].$$

Noor et al. [2] generalized the notion of harmonic convex functions and gave the definition of harmonically h-convex functions. This class contains several other classes of harmonic convex functions as well. In [3], the authors introduced the definition of p-harmonic convex function.

A function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be p-harmonically convex if

$$f\left(\frac{x^p y^p}{(1-t)x^p + ty^p}\right)^{\frac{1}{p}} \le tf(x) + (1-t)f(y), \quad \forall x, y \in I, t \in [0,1]$$

holds, where I is a p-harmonic convex set.

Noor et al. [4] also extended the class of harmonic convex functions on coordinates and introduced the class of coordinated harmonic convex functions.

Consider the rectangle $\Omega = [a,b] \times [c,d] \subset (0,\infty) \times (0,\infty)$. A function $f : \Omega \to \mathbb{R}$ is said to be two-dimensional harmonically convex function on Ω if

$$\begin{aligned} &f\left(\frac{xy}{tx+(1-t)y}, \frac{uw}{ru+(1-r)w}\right) \\ &\quad \le \operatorname{tr} f(y,w) + t(1-r)f(y,u) + (1-t)rf(x,w) + (1-t)(1-r)f(x,u), \end{aligned}$$

whenever $x, y \in [a,b], u, w \in [c,d]$, and $t, r \in [0,1]$. Recently, Awan et al. [5] gave the definition of approximately harmonic h-convex functions depending on a metric function $d : X \times X \to \mathbb{R}$ where $(X, \|\cdot\|)$ is a real normed space. Let $h : (0,1) \to \mathbb{R}$ and Θ be a harmonic convex subset of X. A function $f : \Theta \to \mathbb{R}$ is an approximately harmonic h-convex function if

$$f\left(\frac{xy}{(1-t)x+ty}\right) \le h(t)f(x) + h(1-t)f(y) + d(x,y), \quad \forall x, y \in \Theta, t \in [0,1].$$

For more and recent details on convexity and its generalizations, see [6–18].

Theory of convexity also played a significant role in the development of theory of inequalities. Many famously known results in inequalities theory can be obtained using the convexity property of the functions. Hermite–Hadamard double inequality is one of the most intensively studied results involving convex functions. This result provides us a necessary and sufficient condition for a function to be convex. For interesting details on Hermite–Hadamard inequality and its generalizations, see [15, 19].

The main motive of this article is to extend the notion of approximately harmonic h-convex functions on two dimensions and derive some new corresponding Hermite–Hadamard like inequalities.

2 New notions

In this section, we define the class of two-dimensional approximately harmonic (p_1,h_1)-(p_2,h_2)-convex functions. We also discuss that for suitable choices we get several other new classes of harmonic convexity.

Definition 1 Consider the rectangle $\Omega = [a,b] \times [c,d] \subset (0,\infty) \times (0,\infty)$. A function $f : \Omega \to \mathbb{R}$ is said to be a two-dimensional approximately harmonic (p_1,h_1)-(p_2,h_2)-convex function if

$$\begin{aligned} &f\left(\left[\frac{x^{p_1}y^{p_1}}{tx^{p_1}+(1-t)y^{p_1}}\right]^{\frac{1}{p_1}}, \left[\frac{u^{p_2}w^{p_2}}{ru^{p_2}+(1-r)w^{p_2}}\right]^{\frac{1}{p_2}}\right) \\ &\quad \le h_1(t)h_2(r)f(y,w) + h_1(t)h_2(1-r)f(y,u) \\ &\qquad + h_1(1-t)h_2(r)f(x,w) + h_1(1-t)h_2(1-r)f(x,u) + \Delta(x,y) + \Delta(u,w), \end{aligned}$$

whenever $x, y \in [a,b], u, w \in [c,d]$, and $t, r \in [0,1]$.

We now discuss some special cases of Definition 1.

I. If we take $\Delta(x,y) = \epsilon(\|x^{p_1} - y^{p_1}\|)^\gamma$ and $\Delta(u,w) = \epsilon(\|u^{p_2} - w^{p_2}\|)^\gamma$ for some $\epsilon \in \mathbb{R}$ and $\gamma > 1$ in Definition 1, we have a new definition of γ-paraharmonic (p_1,h_1)-(p_2,h_2)-convex function of higher order.

Definition 2 Consider the rectangle $\Omega = [a,b] \times [c,d] \subset (0,\infty) \times (0,\infty)$. A function $f : \Omega \to \mathbb{R}$ is said to be a two-dimensional γ-paraharmonic (p_1,h_1)-(p_2,h_2)-convex function of higher order if

$$\begin{aligned} &f\left(\left[\frac{x^{p_1}y^{p_1}}{tx^{p_1} + (1-t)y^{p_1}}\right]^{\frac{1}{p_1}}, \left[\frac{u^{p_2}w^{p_2}}{ru^{p_2} + (1-r)w^{p_2}}\right]^{\frac{1}{p_2}}\right) \\ &\quad \le h_1(t)h_2(r)f(y,w) + h_1(t)h_2(1-r)f(y,u) \\ &\qquad + h_1(1-t)h_2(r)f(x,w) + h_1(1-t)h_2(1-r)f(x,u) + \epsilon\left(\left\|x^{p_1} - y^{p_1}\right\|\right)^\gamma \\ &\qquad + \epsilon\left(\left\|u^{p_2} - w^{p_2}\right\|\right)^\gamma, \end{aligned}$$

whenever $x,y \in [a,b], u,w \in [c,d]$, and $t,r \in [0,1]$.

II. If we take $\Delta(x,y) = \epsilon(\|x^{p_1} - y^{p_1}\|)$ and $\Delta(u,w) = \epsilon(\|u^{p_2} - w^{p_2}\|)$ for some $\epsilon \in \mathbb{R}$ and in Definition 1, we have a new definition of ϵ-paraharmonic (p_1,h_1)-(p_2,h_2)-convex function.

Definition 3 Consider the rectangle $\Omega = [a,b] \times [c,d] \subset (0,\infty) \times (0,\infty)$. A function $f : \Omega \to \mathbb{R}$ is said to be a two-dimensional ϵ-paraharmonic (p_1,h_1)-(p_2,h_2)-convex function if

$$\begin{aligned} &f\left(\left[\frac{x^{p_1}y^{p_1}}{tx^{p_1} + (1-t)y^{p_1}}\right]^{\frac{1}{p_1}}, \left[\frac{u^{p_2}w^{p_2}}{ru^{p_2} + (1-r)w^{p_2}}\right]^{\frac{1}{p_2}}\right) \\ &\quad \le h_1(t)h_2(r)f(y,w) + h_1(t)h_2(1-r)f(y,u) \\ &\qquad + h_1(1-t)h_2(r)f(x,w) + h_1(1-t)h_2(1-r)f(x,u) \\ &\qquad + \epsilon\left(\left\|x^{p_1} - y^{p_1}\right\|\right) + \epsilon\left(\left\|u^{p_2} - w^{p_2}\right\|\right), \end{aligned}$$

whenever $x,y \in [a,b], u,w \in [c,d]$, and $t,r \in [0,1]$.

III. If we take $\Delta(x,y) = -\mu(t^\sigma(1-t) + t(1-t)^\sigma)(\|\frac{1}{y^{p_1}} - \frac{1}{x^{p_1}}\|)^\sigma$ and $\Delta(u,w) = -\mu(r^\sigma(1-r) + r(1-r)^\sigma)(\|\frac{1}{w^{p_2}} - \frac{1}{u^{p_2}}\|)^\sigma$ for some $\mu > 0$ and $\sigma > 0$ in Definition 1, we have a new definition of two-dimensional harmonically strong (p_1,h_1)-(p_2,h_2)-convex function of higher order.

Definition 4 Consider the rectangle $\Omega = [a,b] \times [c,d] \subset (0,\infty) \times (0,\infty)$. A function $f : \Omega \to \mathbb{R}$ is said to be a two-dimensional harmonically strong (p_1,h_1)-(p_2,h_2)-convex function of higher order if

$$\begin{aligned} &f\left(\left[\frac{x^{p_1}y^{p_1}}{tx^{p_1} + (1-t)y^{p_1}}\right]^{\frac{1}{p_1}}, \left[\frac{u^{p_2}w^{p_2}}{ru^{p_2} + (1-r)w^{p_2}}\right]^{\frac{1}{p_2}}\right) \\ &\quad \le h_1(t)h_2(r)f(y,w) + h_1(t)h_2(1-r)f(y,u) \\ &\qquad + h_1(1-t)h_2(r)f(x,w) + h_1(1-t)h_2(1-r)f(x,u) \end{aligned}$$

$$
\begin{aligned}
&- \mu\bigl(t^{\sigma}(1-t)+t(1-t)^{\sigma}\bigr)\left(\left\|\frac{1}{y^{p_1}}-\frac{1}{x^{p_1}}\right\|\right)^{\sigma} \\
&- \mu\bigl(r^{\sigma}(1-r)+r(1-r)^{\sigma}\bigr)\left(\left\|\frac{1}{w^{p_2}}-\frac{1}{u^{p_2}}\right\|\right)^{\sigma},
\end{aligned}
$$

whenever $x, y \in [a, b], u, w \in [c, d], \sigma > 0$, and $t, r \in [0, 1]$.

IV. If we take $\sigma = 2$ in Definition 4, then $\Delta(x, y) = -\mu t(1-t)(\|\frac{1}{y^{p_1}} - \frac{1}{x^{p_1}}\|)^2$ and $\Delta(u, w) = -\mu r(1-r)(\|\frac{1}{w^{p_2}} - \frac{1}{u^{p_2}}\|)^2$ for some $\mu > 0$ in Definition 1, then

Definition 5 Consider the rectangle $\Omega = [a, b] \times [c, d] \subset (0, \infty) \times (0, \infty)$. A function $f : \Omega \to \mathbb{R}$ is said to be a two-dimensional harmonically strong (p_1, h_1)-(p_2, h_2)-convex function if

$$
\begin{aligned}
&f\left(\left[\frac{x^{p_1}y^{p_1}}{tx^{p_1}+(1-t)y^{p_1}}\right]^{\frac{1}{p_1}}, \left[\frac{u^{p_2}w^{p_2}}{ru^{p_2}+(1-r)w^{p_2}}\right]^{\frac{1}{p_2}}\right) \\
&\quad\le h_1(t)h_2(r)f(y,w) + h_1(t)h_2(1-r)f(y,u) \\
&\qquad + h_1(1-t)h_2(r)f(x,w) + h_1(1-t)h_2(1-r)f(x,u) \\
&\qquad - \mu t(1-t)\left(\left\|\frac{1}{y^{p_1}}-\frac{1}{x^{p_1}}\right\|\right)^2 - \mu r(1-r)\left(\left\|\frac{1}{w^{p_2}}-\frac{1}{u^{p_2}}\right\|\right)^2,
\end{aligned}
$$

whenever $x, y \in [a, b], u, w \in [c, d]$, and $t, r \in [0, 1]$.

V. If we take $\Delta(x, y) = \mu t(1-t)(\frac{1}{y^{p_1}} - \frac{1}{x^{p_1}})^2$ and $\Delta(u, w) = \mu r(1-r)(\frac{1}{w^{p_2}} - \frac{1}{u^{p_2}})^2$ for some $\mu > 0$ in Definition 1, then

Definition 6 Consider the rectangle $\Omega = [a, b] \times [c, d] \subset (0, \infty) \times (0, \infty)$. A function $f : \Omega \to \mathbb{R}$ is said to be a two-dimensional harmonically relaxed (p_1, h_1)-(p_2, h_2)-convex function if

$$
\begin{aligned}
&f\left(\left[\frac{x^{p_1}y^{p_1}}{tx^{p_1}+(1-t)y^{p_1}}\right]^{\frac{1}{p_1}}, \left[\frac{u^{p_2}w^{p_2}}{ru^{p_2}+(1-r)w^{p_2}}\right]^{\frac{1}{p_2}}\right) \\
&\quad\le h_1(t)h_2(r)f(y,w) + h_1(t)h_2(1-r)f(y,u) + h_1(1-t)h_2(r)f(x,w) \\
&\qquad + h_1(1-t)h_2(1-r)f(x,u) \\
&\qquad + \mu t(1-t)\left(\frac{1}{y^{p_1}}-\frac{1}{x^{p_1}}\right)^2 + \mu r(1-r)\left(\frac{1}{w^{p_2}}-\frac{1}{u^{p_2}}\right)^2,
\end{aligned}
$$

whenever $x, y \in [a, b], u, w \in [c, d]$, and $t, r \in [0, 1]$.

VI. If we take $\Delta(x, y) = -t(1-t)(\frac{x^{p_1}y^{p_1}}{x^{p_1}-y^{p_1}})^2$ and $\Delta(u, w) = -r(1-r)(\frac{u^{p_2}w^{p_2}}{u^{p_2}-w^{p_2}})^2$ in Definition 1, we have a new definition of two-dimensional strongly F harmonic (p_1, h_1)-(p_2, h_2)-convex function.

Definition 7 Consider the rectangle $\Omega = [a, b] \times [c, d] \subset (0, \infty) \times (0, \infty)$. A function $f : \Omega \to \mathbb{R}$ is said to be a two-dimensional strongly F harmonic (p_1, h_1)-(p_2, h_2)-convex function if

$$\begin{aligned} &f\left(\left[\frac{x^{p_1}y^{p_1}}{tx^{p_1}+(1-t)y^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{u^{p_2}w^{p_2}}{ru^{p_2}+(1-r)w^{p_2}}\right]^{\frac{1}{p_2}}\right)\\ &\quad\le h_1(t)h_2(r)f(y,w)+h_1(t)h_2(1-r)f(y,u)\\ &\qquad+h_1(1-t)h_2(r)f(x,w)+h_1(1-t)h_2(1-r)f(x,u)\\ &\qquad-t(1-t)\left(\frac{x^{p_1}y^{p_1}}{x^{p_1}-y^{p_1}}\right)^2-r(1-r)\left(\frac{u^{p_2}w^{p_2}}{u^{p_2}-w^{p_2}}\right)^2, \end{aligned}$$

whenever $x,y\in[a,b], u,w\in[c,d]$, and $t,r\in[0,1]$.

3 Main results

In this section, we discuss our main results.

Theorem 1 *Let $f:\Omega\to\mathbb{R}$ be an integrable function. If f is an approximately two-dimensional harmonically (p_1,h_1)-(p_2,h_2)-convex function, then*

$$\begin{aligned} &\frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})}\left[f\left(\left[\frac{2a^{p_1}b^{p_1}}{a^{p_1}+b^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{2c^{p_2}d^{p_2}}{c^{p_2}+d^{p_2}}\right]^{\frac{1}{p_2}}\right)\right.\\ &\quad-\frac{p_1a^{p_1}b^{p_1}}{b^{p_1}-a^{p_1}}\int_a^b\frac{\Delta(x,((a^{p_1})^{-1}+(b^{p_1})^{-1}-(x^{p_1})^{-1})^{-1})}{x^{1+p_1}}\,\mathrm{d}x\\ &\quad\left.-\frac{p_2c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\int_c^d\frac{\Delta(u,((c^{p_2})^{-1}+(d^{p_2})^{-1}-(u^{p_2})^{-1})^{-1})}{u^{1+p_2}}\,\mathrm{d}u\right]\\ &\quad\le p_1p_2\left(\frac{a^{p_1}b^{p_1}}{b^{p_1}-a^{p_1}}\right)\left(\frac{c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\right)\int_a^b\int_c^d\frac{f(x,u)}{x^{1+p_1}u^{1+p_2}}\,\mathrm{d}u\,\mathrm{d}x\\ &\quad\le\big[f(a,c)+f(a,d)+f(b,c)+f(b,d)\big]\int_0^1\int_0^1h_1(t)h_2(r)\,\mathrm{d}t\,\mathrm{d}r+\Delta(a,b)+\Delta(c,d). \end{aligned}$$

Proof Since f is an approximately two-dimensional harmonic (p_1,h_1)-(p_2,h_2)-convex function, we have

$$\begin{aligned} &f\left(\left[\frac{2a^{p_1}b^{p_1}}{a^{p_1}+b^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{2c^{p_2}d^{p_2}}{c^{p_2}+d^{p_2}}\right]^{\frac{1}{p_2}}\right)\\ &\quad\le h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\\ &\qquad\times\left[f\left(\left[\frac{a^{p_1}b^{p_1}}{ta^{p_1}+(1-t)b^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}d^{p_2}}{rc^{p_2}+(1-r)d^{p_2}}\right]^{\frac{1}{p_2}}\right)\right.\\ &\qquad+f\left(\left[\frac{a^{p_1}b^{p_1}}{ta^{p_1}+(1-t)b^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}d^{p_2}}{rd^{p_2}+(1-r)c^{p_2}}\right]^{\frac{1}{p_2}}\right)\\ &\qquad+f\left(\left[\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}d^{p_2}}{rc^{p_2}+(1-r)d^{p_2}}\right]^{\frac{1}{p_2}}\right)\\ &\qquad\left.+f\left(\left[\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}d^{p_2}}{rd^{p_2}+(1-r)c^{p_2}}\right]^{\frac{1}{p_2}}\right)\right] \end{aligned}$$

$$+\Delta\left(\left[\frac{a^{p_1}b^{p_1}}{ta^{p_1}+(1-t)b^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}\right]^{\frac{1}{p_1}}\right)$$
$$+\Delta\left(\left[\frac{c^{p_2}d^{p_2}}{rc^{p_2}+(1-r)d^{p_2}}\right]^{\frac{1}{p_2}},\left[\frac{c^{p_2}d^{p_2}}{rd^{p_2}+(1-r)c^{p_2}}\right]^{\frac{1}{p_2}}\right)\bigg].$$

Integrating the above inequality with respect to (t,r) on $[0,1]\times[0,1]$, we have

$$\begin{aligned}&\frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})}\bigg[f\left(\left[\frac{2a^{p_1}b^{p_1}}{a^{p_1}+b^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{2c^{p_2}d^{p_2}}{c^{p_2}+d^{p_2}}\right]^{\frac{1}{p_2}}\right)\\&\quad-\frac{p_1a^{p_1}b^{p_1}}{b^{p_1}-a^{p_1}}\int_a^b\frac{\Delta(x,((a^{p_1})^{-1}+(b^{p_1})^{-1}-(x^{p_1})^{-1})^{-1})}{x^{1+p_1}}\,\mathrm{d}x\\&\quad-\frac{p_2c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\int_c^d\frac{\Delta(u,((c^{p_2})^{-1}+(d^{p_2})^{-1}-(u^{p_2})^{-1})^{-1})}{u^{1+p_2}}\,\mathrm{d}u\bigg]\\&\quad\le p_1p_2\left(\frac{a^{p_1}b^{p_1}}{b^{p_1}-a^{p_1}}\right)\left(\frac{c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\right)\int_a^b\int_c^d\frac{f(x,u)}{x^{1+p_1}u^{1+p_2}}\,\mathrm{d}u\,\mathrm{d}x.\end{aligned}$$

Also

$$\begin{aligned}&f\left(\left[\frac{a^{p_1}b^{p_1}}{ta^{p_1}+(1-t)b^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}d^{p_2}}{rc^{p_2}+(1-r)d^{p_2}}\right]^{\frac{1}{p_2}}\right)\\&\quad\le h_1(t)h_2(r)f(b,d)+h_1(t)h_2(1-r)f(b,c)+h_1(1-t)h_2(r)f(a,d)\\&\qquad+h_1(1-t)h_2(1-r)f(a,c)+\Delta(a,b)+\Delta(c,d).\end{aligned}$$

Integrating both sides of the above inequality with respect to (t,r) on $[0,1]\times[0,1]$, we have

$$\begin{aligned}&p_1p_2\left(\frac{a^{p_1}b^{p_1}}{b^{p_1}-a^{p_1}}\right)\left(\frac{c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\right)\int_a^b\int_c^d\frac{f(x,u)}{x^{1+p_1}u^{1+p_2}}\,\mathrm{d}u\,\mathrm{d}x\\&\quad\le\big(f(a,c)+f(a,d)+f(b,c)+f(b,d)\big)\int_0^1\int_0^1 h_1(t)h_2(r)\,\mathrm{d}t\,\mathrm{d}r+\Delta(a,b)+\Delta(c,d).\end{aligned}$$

This completes the proof. □

We now discuss some special cases of Theorem 1.

I. If $h_1(t)=t$ and $h_2(r)=r$ in Theorem 1, then

Corollary 1 *Under the assumptions of Theorem 1, if f is an approximately two-dimensional harmonic (p_1,p_2)-convex function, then*

$$\begin{aligned}&f\left(\left[\frac{2a^{p_1}b^{p_1}}{a^{p_1}+b^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{2c^{p_2}d^{p_2}}{c^{p_2}+d^{p_2}}\right]^{\frac{1}{p_2}}\right)\\&\quad-\frac{p_1a^{p_1}b^{p_1}}{b^{p_1}-a^{p_1}}\int_a^b\frac{\Delta(x,((a^{p_1})^{-1}+(b^{p_1})^{-1}-(x^{p_1})^{-1})^{-1})}{x^{1+p_1}}\,\mathrm{d}x\\&\quad-\frac{p_2c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\int_c^d\frac{\Delta(u,((c^{p_2})^{-1}+(d^{p_2})^{-1}-(u^{p_2})^{-1})^{-1})}{u^{1+p_2}}\,\mathrm{d}u\end{aligned}$$

$$\le p_1p_2\left(\frac{a^{p_1}b^{p_1}}{b^{p_1}-a^{p_1}}\right)\left(\frac{c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\right)\int_a^b\int_c^d\frac{f(x,u)}{x^{1+p_1}u^{1+p_2}}\,\mathrm{d}u\,\mathrm{d}x$$
$$\le\frac{[f(a,c)+f(a,d)+f(b,c)+f(b,d)]}{4}+\Delta(a,b)+\Delta(c,d).$$

II. If $h_1(t)=t^{s_1}$ and $h(r)=r^{s_2}$ in Theorem 1, then

Corollary 2 *Under the assumptions of Theorem 1, if f is a Breckner type approximately two-dimensional harmonic (p_1,s_1)-(p_2,s_2)-convex function, then*

$$\frac{2^{s_1+s_2}}{4}\Bigg[f\left(\left[\frac{2a^{p_1}b^{p_1}}{a^{p_1}+b^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{2c^{p_2}d^{p_2}}{c^{p_2}+d^{p_2}}\right]^{\frac{1}{p_2}}\right)$$
$$-\frac{p_1a^{p_1}b^{p_1}}{b^{p_1}-a^{p_1}}\int_a^b\frac{\Delta(x,((a^{p_1})^{-1}+(b^{p_1})^{-1}-(x^{p_1})^{-1})^{-1})}{x^{1+p_1}}\,\mathrm{d}x$$
$$-\frac{p_2c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\int_c^d\frac{\Delta(u,((c^{p_2})^{-1}+(d^{p_2})^{-1}-(u^{p_2})^{-1})^{-1})}{u^{1+p_2}}\,\mathrm{d}u\Bigg]$$
$$\le p_1p_2\left(\frac{a^{p_1}b^{p_1}}{b^{p_1}-a^{p_1}}\right)\left(\frac{c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\right)\int_a^b\int_c^d\frac{f(x,u)}{x^{1+p_1}u^{1+p_2}}\,\mathrm{d}u\,\mathrm{d}x$$
$$\le\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{(s_1+1)(s_2+1)}+\Delta(a,b)+\Delta(c,d).$$

III. If $h_1(t)=t^{-s_1}$ and $h_2(r)=r^{-s_2}$ in Theorem 1, then

Corollary 3 *Under the assumptions of Theorem 1, if f is a Godunova–Levin type approximately two-dimensional harmonic (p_1,s_1)-(p_2,s_2)-convex function, then*

$$\frac{1}{4.2^{s_1+s_2}}\Bigg[f\left(\left[\frac{2a^{p_1}b^{p_1}}{a^{p_1}+b^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{2c^{p_2}d^{p_2}}{c^{p_2}+d^{p_2}}\right]^{\frac{1}{p_2}}\right)$$
$$-\frac{p_1a^{p_1}b^{p_1}}{b^{p_1}-a^{p_1}}\int_a^b\frac{\Delta(x,((a^{p_1})^{-1}+(b^{p_1})^{-1}-(x^{p_1})^{-1})^{-1})}{x^{1+p_1}}\,\mathrm{d}x$$
$$-\frac{p_2c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\int_c^d\frac{\Delta(u,((c^{p_2})^{-1}+(d^{p_2})^{-1}-(u^{p_2})^{-1})^{-1})}{u^{1+p_2}}\,\mathrm{d}u\Bigg]$$
$$\le p_1p_2\left(\frac{a^{p_1}b^{p_1}}{b^{p_1}-a^{p_1}}\right)\left(\frac{c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\right)\int_a^b\int_c^d\frac{f(x,u)}{x^{1+p_1}u^{1+p_2}}\,\mathrm{d}u\,\mathrm{d}x$$
$$\le\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{(1-s_1)(1-s_2)}+\Delta(a,b)+\Delta(c,d).$$

IV. If $h_1(t)=1$ and $h_2(r)=1$ in Theorem 1, then

Corollary 4 *Under the assumptions of Theorem 1, if f is an approximately two-dimensional harmonic (p_1,P)-(p_2,P)-convex function, then*

$$\frac{1}{4}\Bigg[f\left(\left[\frac{2a^{p_1}b^{p_1}}{a^{p_1}+b^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{2c^{p_2}d^{p_2}}{c^{p_2}+d^{p_2}}\right]^{\frac{1}{p_2}}\right)$$
$$-\frac{p_1a^{p_1}b^{p_1}}{b^{p_1}-a^{p_1}}\int_a^b\frac{\Delta(x,((a^{p_1})^{-1}+(b^{p_1})^{-1}-(x^{p_1})^{-1})^{-1})}{x^{1+p_1}}\,\mathrm{d}x$$

$$
\begin{aligned}
&-\frac{p_2c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\int_c^d \frac{\Delta(u,((c^{p_2})^{-1}+(d^{p_2})^{-1}-(u^{p_2})^{-1})^{-1})}{u^{1+p_2}}\,\mathrm{d}u\Bigg]\\
&\le p_1p_2\left(\frac{a^{p_1}b^{p_1}}{b^{p_1}-a^{p_1}}\right)\left(\frac{c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\right)\int_a^b\int_c^d \frac{f(x,u)}{x^{1+p_1}u^{1+p_2}}\,\mathrm{d}u\,\mathrm{d}x\\
&\le \big[f(a,c)+f(a,d)+f(b,c)+f(b,d)\big]+\Delta(a,b)+\Delta(c,d).
\end{aligned}
$$

Now we introduce a generalized identity that will play a significant role in the development of our next results.

Lemma 1 *Let $f:\Omega\to\mathbb{R}$ be a partial differential function on $\Omega=[a,b]\times[c,d]\subset(0,\infty)\times(0,\infty)$ with $a^{p_1}<b^{p_1}$ and $c^{p_2}<d^{p_2}$. If $\frac{\partial^2 f}{\partial t\,\partial r}\in L_1(\Omega)$, then*

$$
\begin{aligned}
&\Xi\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\big)\\
&=\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}\int_0^1\int_0^1\left((1-2t)\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right)\\
&\quad\times\left((1-2r)\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right)\\
&\quad\times\frac{\partial^2 f}{\partial r\,\partial t}\left(\left[\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}d^{p_2}}{rd^{p_2}+(1-r)c^{p_2}}\right]^{\frac{1}{p_2}}\right)\mathrm{d}r\,\mathrm{d}t,
\end{aligned}
$$

where

$$
\begin{aligned}
&\Xi\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\big)\\
&=\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}-\frac{1}{2}\left[\frac{p_1a^{p_1}b^{p_1}}{b^{p_1}-a^{p_1}}\left[\int_a^b\frac{f(x,c)}{x^{1+p_1}}\,\mathrm{d}x+\int_a^b\frac{f(x,d)}{x^{1+p_1}}\,\mathrm{d}x\right]\right.\\
&\quad\left.+\frac{p_2c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\left[\int_c^d\frac{f(a,u)}{u^{1+p_2}}\,\mathrm{d}u+\int_c^d\frac{f(b,u)}{u^{1+p_2}}\,\mathrm{d}u\right]\right]\\
&\quad+\frac{p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d\frac{f(x,u)}{x^{1+p_1}u^{1+p_2}}\,\mathrm{d}u\,\mathrm{d}x.
\end{aligned}
$$

Proof It suffices to note that we can write

$$
\begin{aligned}
&\Xi\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\big)\\
&=\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}\int_0^1\left((1-2t)\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right)\\
&\quad\times\left[\int_0^1\left((1-2r)\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right)\right.\\
&\quad\left.\times\frac{\partial^2 f}{\partial r\,\partial t}\left(\left[\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}d^{p_2}}{rd^{p_2}+(1-r)c^{p_2}}\right]^{\frac{1}{p_2}}\right)\mathrm{d}r\right]\mathrm{d}t.
\end{aligned}
$$

Now, integrating by parts, we have

$$
I_1=\int_0^1\left((1-2r)\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right)
$$

$$
\begin{aligned}
&\times \frac{\partial^2 f}{\partial r\,\partial t}\left(\left[\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}d^{p_2}}{rd^{p_2}+(1-r)c^{p_2}}\right]^{\frac{1}{p_2}}\right)\mathrm{d}r \\
&= \frac{p_2c^{p_2}d^{p_2}}{(d^{p_2}-c^{p_2})}\frac{\partial f}{\partial t}\left(\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}]^{\frac{1}{p_1}},c\right)+\frac{p_2c^{p_2}d^{p_2}}{(d^{p_2}-c^{p_2})}\frac{\partial f}{\partial t}\left(\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}]^{\frac{1}{p_1}},d\right) \\
&\quad - \frac{2p_2c^{p_2}d^{p_2}}{(d^{p_2}-c^{p_2})}\int_0^1 \frac{\partial f}{\partial t}\left(\left[\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}d^{p_2}}{rd^{p_2}+(1-r)c^{p_2}}\right]^{\frac{1}{p_2}}\right)\mathrm{d}r, \\
I_2 &= \int_0^1\left((1-2t)\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right)\frac{\partial f}{\partial t}\left(\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}]^{\frac{1}{p_1}},c\right)\mathrm{d}t \\
&= \frac{p_1a_1^pb_1^p}{(b^{p_1}-a^{p_1})}\{f(a,c)+f(b,c)\}-\frac{2p_1^2(a_1^pb_1^p)^2}{(b^{p_1}-a^{p_1})^2}\int_a^b\frac{f(x,c)}{x^{1+p_1}}\,\mathrm{d}x, \\
I_3 &= \int_0^1\left((1-2t)\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right)\frac{\partial f}{\partial t}\left(\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}]^{\frac{1}{p_1}},d\right)\mathrm{d}t \\
&= \frac{p_1a_1^pb_1^p}{(b^{p_1}-a^{p_1})}\{f(a,d)+f(b,d)\}-\frac{2p_1^2(a_1^pb_1^p)^2}{(b^{p_1}-a^{p_1})^2}\int_a^b\frac{f(x,d)}{x^{1+p_1}}\,\mathrm{d}x, \\
I_4 &= \int_0^1\left\{\int_0^1\left((1-2t)\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right)\right. \\
&\quad\left.\times\frac{\partial f}{\partial t}\left(\left[\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}d^{p_2}}{rd^{p_2}+(1-r)c^{p_2}}\right]^{\frac{1}{p_2}}\right)\mathrm{d}t\right\}\mathrm{d}r \\
&= \frac{\{f(a,c)+f(b,c)+f(a,d)+f(b,d)\}}{4}-\frac{1}{2}\left[\frac{p_1a_1^pb_1^p}{b^{p_1}-a^{p_1}}\left\{\int_a^b\frac{f(x,c)}{x^{1+p_1}}\,\mathrm{d}x+\int_a^b\frac{f(x,d)}{x^{1+p_1}}\,\mathrm{d}x\right\}\right. \\
&\quad\left.+\frac{p_2c^{p_2}d^{p_2}}{d^{p_2}-c^{p_2}}\left\{\int_c^d\frac{f(a,u)}{u^{1+p_2}}\,\mathrm{d}u+\int_c^d\frac{f(b,u)}{u^{1+p_2}}\,\mathrm{d}u\right\}\right] \\
&\quad+\frac{p_1p_2(a_1^pb_1^pa_1^pb_1^p)}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d\frac{f(x,u)}{x^{1+p_1}u^{1+p_2}}\,\mathrm{d}x\,\mathrm{d}u.
\end{aligned}
$$

Summing up integrals I_1 to I_4 and using the change of variable technique will give the required result. □

Remark 1 Letting $p_1 = 1 = p_2$ in the above lemma, we get the lemma proved in [4].

In order to obtain next results, we need gamma, beta, and hypergeometric functions. Gamma $\Gamma(\cdot)$ and beta function $B(\cdot,\cdot)$ are defined respectively as follows:

$$
\begin{aligned}
\Gamma(x) &= \int_0^\infty e^{-x}t^{x-1}\,\mathrm{d}t, \\
\mathrm{B}(x,y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.
\end{aligned}
$$

The integral form of the hypergeometric function is

$$
{}_2F_1(x,y;c;z) = \frac{1}{\mathrm{B}(y,c-y)}\int_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}\,\mathrm{d}t
$$

for $|z| < 1, c > y > 0$. Now, using Lemma 1, we prove the next results of the article.

Theorem 2 *Let $f:\Omega\rightarrow\mathbb{R}$ be a partial differentiable function on $\Omega=[a,b]\times[c,d]\subset(0,\infty)\times(0,\infty)$ with $a^{p_1}<b^{p_1}$ and $c^{p_2}<d^{p_2}$ and $\frac{\partial^2 f}{\partial t\,\partial r}\in L_1(\Omega)$. If $|\frac{\partial^2 f}{\partial r\,\partial t}|$ is an approximately two-dimensional harmonic (p_1,h_1)-(p_2,h_2)-convex function, then*

$$\begin{aligned}
&\bigl|\Xi\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\bigr)\bigr| \\
&\quad\leq\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}\biggl[\vartheta_1\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr)\biggl|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\biggr| \\
&\qquad+\vartheta_2\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr)\biggl|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\biggr|+\vartheta_3\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr)\biggl|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\biggr| \\
&\qquad+\vartheta_4\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr)\biggl|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\biggr|+\vartheta_5\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr) \\
&\qquad+\vartheta_6\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr)\biggr],
\end{aligned}$$

where

$$\begin{aligned}
&\vartheta_1\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr) \\
&\quad=\int_0^1\int_0^1\biggl[|1-2t|h_1(t)\biggl[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\biggr]^{1+\frac{1}{p_1}}\biggr] \\
&\qquad\times\biggl[|1-2r|h_2(r)\biggl[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\biggr]^{1+\frac{1}{p_2}}\biggr]\,\mathrm{d}t\,\mathrm{d}r, \\
&\vartheta_2\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr) \\
&\quad=\int_0^1\int_0^1\biggl[|1-2t|h_1(1-t)\biggl[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\biggr]^{1+\frac{1}{p_1}}\biggr] \\
&\qquad\times\biggl[|1-2r|h_2(r)\biggl[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\biggr]^{1+\frac{1}{p_2}}\biggr]\,\mathrm{d}t\,\mathrm{d}r, \\
&\vartheta_3\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr) \\
&\quad=\int_0^1\int_0^1\biggl[|1-2t|h_1(t)\biggl[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\biggr]^{1+\frac{1}{p_1}}\biggr] \\
&\qquad\times\biggl[|1-2r|h_2(1-r)\biggl[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\biggr]^{1+\frac{1}{p_2}}\biggr]\,\mathrm{d}t\,\mathrm{d}r, \\
&\vartheta_4\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr) \\
&\quad=\int_0^1\int_0^1\biggl[|1-2t|h_1(1-t)\biggl[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\biggr]^{1+\frac{1}{p_1}}\biggr] \\
&\qquad\times\biggl[|1-2r|h_2(1-r)\biggl[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\biggr]^{1+\frac{1}{p_2}}\biggr]\,\mathrm{d}t\,\mathrm{d}r, \\
&\vartheta_5\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr) \\
&\quad=\Delta(a,b)\int_0^1\int_0^1\biggl[|1-2t|\biggl[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\biggr]^{1+\frac{1}{p_1}}\biggr]
\end{aligned}$$

$$\times\left[|1-2r|\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right]\mathrm{d}t\,\mathrm{d}r$$
$$=\Delta(a,b)\left[\left(b^{p_1}\right)^{1+\frac{1}{p_1}}{}_2F_1\left(1+\frac{1}{p_1},2,3,1-\frac{b^{p_1}}{a^{p_1}}\right)\right.$$
$$-\left(b^{p_1}\right)^{1+\frac{1}{p_1}}{}_2F_1\left(1+\frac{1}{p_1},1,2,1-\frac{b^{p_1}}{a^{p_1}}\right)$$
$$\left.+\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2}{}_2F_1\left(1+\frac{1}{p_1},1,3,\frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right)\right]$$
$$\times\left[\left(d^{p_2}\right)^{1+\frac{1}{p_2}}{}_2F_1\left(1+\frac{1}{p_2},2,3,1-\frac{d^{p_2}}{c^{p_2}}\right)\right.$$
$$-\left(d^{p_2}\right)^{1+\frac{1}{p_2}}{}_2F_1\left(1+\frac{1}{p_2},1,2,1-\frac{d^{p_2}}{c^{p_2}}\right)$$
$$\left.+\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2}{}_2F_1\left(1+\frac{1}{p_2},1,3,\frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right)\right],$$

and

$$\vartheta_6\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)$$
$$=\Delta(c,d)\int_0^1\int_0^1\left[|1-2t|\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right]$$
$$\times\left[|1-2r|\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right]\mathrm{d}t\,\mathrm{d}r$$
$$=\Delta(c,d)\left[\left(b^{p_1}\right)^{1+\frac{1}{p_1}}{}_2F_1\left(1+\frac{1}{p_1},2,3,1-\frac{b^{p_1}}{a^{p_1}}\right)-\left(b^{p_1}\right)^{1+\frac{1}{p_1}}{}_2F_1\left(1+\frac{1}{p_1},1,2,1-\frac{b^{p_1}}{a^{p_1}}\right)\right.$$
$$\left.+\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2}{}_2F_1\left(1+\frac{1}{p_1},1,3,\frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right)\right]\left[\left(d^{p_2}\right)^{1+\frac{1}{p_2}}{}_2F_1\left(1+\frac{1}{p_2},2,3,1-\frac{d^{p_2}}{c^{p_2}}\right)\right.$$
$$-\left(d^{p_2}\right)^{1+\frac{1}{p_2}}{}_2F_1\left(1+\frac{1}{p_2},1,2,1-\frac{d^{p_2}}{c^{p_2}}\right)$$
$$\left.+\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2}{}_2F_1\left(1+\frac{1}{p_2},1,3,\frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right)\right].$$

Proof Using Lemma 1 and the fact that $|\frac{\partial^2 f}{\partial r\,\partial t}|$ is an approximately two-dimensional harmonic (p_1,h_1)-(p_2,h_2)-convex function, then

$$\left|\Xi\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\right)\right|$$
$$=\left|\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}\int_0^1\int_0^1\left((1-2t)\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right)\right.$$
$$\times\left((1-2r)\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right)$$
$$\left.\times\frac{\partial^2 f}{\partial r\,\partial t}\left(\left[\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}d^{p_2}}{rd^{p_2}+(1-r)c^{p_2}}\right]^{\frac{1}{p_2}}\right)\mathrm{d}r\,\mathrm{d}t\right|$$

$$\leq \frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}\int_0^1\int_0^1\left[|1-2t|\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right]$$
$$\times\left[|1-2r|\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right]$$
$$\times\left|\frac{\partial^2 f}{\partial r\,\partial t}\left(\left[\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}d^{p_2}}{rd^{p_2}+(1-r)c^{p_2}}\right]^{\frac{1}{p_2}}\right)\right|\mathrm{d}r\,\mathrm{d}t$$
$$\leq \frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}\int_0^1\int_0^1\left[|1-2t|\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right]$$
$$\times\left[|1-2r|\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right]$$
$$\times\left[h_1(t)h_2(r)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\right|+h_1(1-t)h_2(r)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\right|\right.$$
$$+h_1(t)h_2(1-r)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\right|$$
$$\left.+h_1(1-t)h_2(1-r)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\right|+\Delta(a,b)+\Delta(c,d)\right]\mathrm{d}r\,\mathrm{d}t.$$

This completes the proof. □

Corollary 5 *Under the assumptions of Theorem 2, if* $|\frac{\partial^2 f}{\partial r\,\partial t}|\leq M$ *is a bounded function for* $M>0$ *on* Ω*, then*

$$\left|\Xi\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\right)\right|$$
$$\leq \frac{M(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}\left[\vartheta_1\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\right.$$
$$+\vartheta_2\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)+\vartheta_3\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)$$
$$+\vartheta_4\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)+\frac{1}{M}\vartheta_5\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)$$
$$\left.+\frac{1}{M}\vartheta_6\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\right],$$

where $\vartheta_1(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)$ *to* $\vartheta_6(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)$ *are given in Theorem 2.*

We now discuss some special cases of Theorem 2.

I. If we take $h_1(t)=t$ and $h_2(r)=r$ in Theorem 2, then

Corollary 6 *Under the assumptions of Theorem 2, if* $|\frac{\partial^2 f}{\partial r\,\partial t}|$ *is an approximately two-dimensional harmonic* (p_1,p_2)*-convex function, then*

$$\left|\Xi\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\right)\right|$$
$$\leq \frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}\left[\vartheta_1^*\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\right|\right.$$
$$+\vartheta_2^*\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\right|+\vartheta_3^*\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\right|$$

$$+ \vartheta_4^*\big(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\big)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\right| + \vartheta_5\big(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\big) + \vartheta_6\big(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\big)\bigg],$$

where

$$\begin{aligned}
&\vartheta_1^*\big(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\big)\\
&= \int_0^1\int_0^1\left[|1-2t|t\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right]\\
&\quad\times\left[|1-2r|r\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right]\mathrm{d}t\,\mathrm{d}r\\
&= \bigg[\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{12}{}_2F_1\bigg(1+\frac{1}{p_1},2,4,\frac{1}{2}\bigg(1-\frac{b^{p_1}}{a^{p_1}}\bigg)\bigg) + \frac{2(b^{p_1})^{1+\frac{1}{p_1}}}{3}{}_2F_1\bigg(1+\frac{1}{p_1},3,4,1-\frac{b^{p_1}}{a^{p_1}}\bigg)\\
&\quad - \frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2}{}_2F_1\bigg(1+\frac{1}{p_1},2,3,1-\frac{b^{p_1}}{a^{p_1}}\bigg)\bigg]\bigg[\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{12}{}_2F_1\bigg(1+\frac{1}{p_2},2,4,\frac{1}{2}\bigg(1-\frac{d^{p_2}}{c^{p_2}}\bigg)\bigg)\\
&\quad + \frac{2(d^{p_2})^{1+\frac{1}{p_2}}}{3}{}_2F_1\bigg(1+\frac{1}{p_2},3,4,1-\frac{d^{p_2}}{c^{p_2}}\bigg) - \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2}{}_2F_1\bigg(1+\frac{1}{p_2},2,3,1-\frac{d^{p_2}}{c^{p_2}}\bigg)\bigg],\\
&\vartheta_2^*\big(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\big)\\
&= \int_0^1\int_0^1\left[|1-2t|(1-t)\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right]\\
&\quad\times\left[|1-2r|r\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right]\mathrm{d}t\,\mathrm{d}r\\
&= \bigg[\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2}{}_2F_1\bigg(1+\frac{1}{p_1},1,3,\frac{1}{2}\bigg(1-\frac{b^{p_1}}{a^{p_1}}\bigg)\bigg)\\
&\quad - \frac{(b^{p_1})^{1+\frac{1}{p_1}}}{12}{}_2F_1\bigg(1+\frac{1}{p_1},2,4,\frac{1}{2}\bigg(1-\frac{b^{p_1}}{a^{p_1}}\bigg)\bigg)\\
&\quad + \frac{(b^{p_1})^{1+\frac{1}{p_1}}}{3}{}_2F_1\bigg(1+\frac{1}{p_1},2,4,1-\frac{b^{p_1}}{a^{p_1}}\bigg) - \frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2}{}_2F_1\bigg(1+\frac{1}{p_1},1,3,1-\frac{b^{p_1}}{a^{p_1}}\bigg)\bigg]\\
&\quad\times\bigg[\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{12}{}_2F_1\bigg(1+\frac{1}{p_2},2,4,\frac{1}{2}\bigg(1-\frac{d^{p_2}}{c^{p_2}}\bigg)\bigg)\\
&\quad + \frac{2(d^{p_2})^{1+\frac{1}{p_2}}}{3}{}_2F_1\bigg(1+\frac{1}{p_2},3,4,1-\frac{d^{p_2}}{c^{p_2}}\bigg) - \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2}{}_2F_1\bigg(1+\frac{1}{p_2},2,3,1-\frac{d^{p_2}}{c^{p_2}}\bigg)\bigg],\\
&\vartheta_3^*\big(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\big)\\
&= \int_0^1\int_0^1\left[|1-2t|t\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right]\\
&\quad\times\left[|1-2r|(1-r)\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right]\mathrm{d}t\,\mathrm{d}r
\end{aligned}$$

$$= \left[\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{12} {}_2F_1\left(1+\frac{1}{p_1}, 2, 4, \frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right) + \frac{2(b^{p_1})^{1+\frac{1}{p_1}}}{3} {}_2F_1\left(1+\frac{1}{p_1}, 3, 4, 1-\frac{b^{p_1}}{a^{p_1}}\right) \right.$$

$$\left. - \frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2} {}_2F_1\left(1+\frac{1}{p_1}, 2, 3, 1-\frac{b^{p_1}}{a^{p_1}}\right) \right]$$

$$\times \left[\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2} {}_2F_1\left(1+\frac{1}{p_2}, 1, 3, \frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right) \right.$$

$$- \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{12} {}_2F_1\left(1+\frac{1}{p_2}, 2, 4, \frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right)$$

$$\left. + \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{3} {}_2F_1\left(1+\frac{1}{p_2}, 2, 4, 1-\frac{d^{p_2}}{c^{p_2}}\right) - \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2} {}_2F_1\left(1+\frac{1}{p_2}, 1, 3, 1-\frac{d^{p_2}}{c^{p_2}}\right) \right],$$

$$\vartheta_4^*\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right)$$

$$= \int_0^1 \int_0^1 \left[|1-2t|(1-t) \left[\frac{a^{p_1} b^{p_1}}{(tb^{p_1} + (1-t)a^{p_1})} \right]^{1+\frac{1}{p_1}} \right]$$

$$\times \left[|1-2r|(1-r) \left[\frac{c^{p_2} d^{p_2}}{(rd^{p_2} + (1-r)c^{p_2})} \right]^{1+\frac{1}{p_2}} \right] \mathrm{d}t\, \mathrm{d}r$$

$$= \left[\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2} {}_2F_1\left(1+\frac{1}{p_1}, 1, 3, \frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right) \right.$$

$$- \frac{(b^{p_1})^{1+\frac{1}{p_1}}}{12} {}_2F_1\left(1+\frac{1}{p_1}, 2, 4, \frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right)$$

$$\left. + \frac{(b^{p_1})^{1+\frac{1}{p_1}}}{3} {}_2F_1\left(1+\frac{1}{p_1}, 2, 4, 1-\frac{b^{p_1}}{a^{p_1}}\right) - \frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2} {}_2F_1\left(1+\frac{1}{p_1}, 1, 3, 1-\frac{b^{p_1}}{a^{p_1}}\right) \right]$$

$$\times \left[\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2} {}_2F_1\left(1+\frac{1}{p_2}, 1, 3, \frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right) \right.$$

$$- \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{12} {}_2F_1\left(1+\frac{1}{p_2}, 2, 4, \frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right)$$

$$\left. + \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{3} {}_2F_1\left(1+\frac{1}{p_2}, 2, 4, 1-\frac{d^{p_2}}{c^{p_2}}\right) - \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2} {}_2F_1\left(1+\frac{1}{p_2}, 1, 3, 1-\frac{d^{p_2}}{c^{p_2}}\right) \right],$$

$\vartheta_5(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega)$ and $\vartheta_6(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega)$ are given in Theorem 2.

II. If we take $h_1(t) = t^{s_1}$ and $h_2(r) = r^{s_2}$ in Theorem 2, then

Corollary 7 *Under the assumptions of Theorem 2, if $|\frac{\partial^2 f}{\partial r\,\partial t}|$ is a Breckner type approximately two-dimensional harmonic (p_1, s_1)-(p_2, s_2)-convex function, then*

$$\left|\Xi\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2}, x, y : \Omega\right)\right|$$

$$\leq \frac{(b^{p_1} - a^{p_1})(d^{p_2} - c^{p_2})}{4 p_1 p_2 a^{p_1} b^{p_1} c^{p_2} d^{p_2}} \left[\vartheta_1^{**}\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right) \left| \frac{\partial^2 f}{\partial r\,\partial t}(a,c) \right| \right.$$

$$+ \vartheta_2^{**}\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right) \left| \frac{\partial^2 f}{\partial r\,\partial t}(b,c) \right| + \vartheta_3^{**}\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right) \left| \frac{\partial^2 f}{\partial r\,\partial t}(a,d) \right|$$

$$+ \vartheta_4^{**}\big(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\big)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\right| + \vartheta_5\big(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\big) + \vartheta_6\big(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\big)\Bigg],$$

where

$$\begin{aligned}
&\vartheta_1^{**}\big(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\big)\\
&= \int_0^1\int_0^1 \left[|1-2t|t^{s_1}\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right]\\
&\quad\times\left[|1-2r|r^{s_2}\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right]\mathrm{d}t\,\mathrm{d}r\\
&= \left[\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2^{s_1}(s_1+1)(s_1+2)}\,{}_2F_1\left(1+\frac{1}{p_1}, s_1+1, s_1+3, \frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right)\right.\\
&\quad+\frac{2(b^{p_1})^{1+\frac{1}{p_1}}}{(s_1+2)}\,{}_2F_1\left(1+\frac{1}{p_1}, s_1+2, s_1+3, 1-\frac{b^{p_1}}{a^{p_1}}\right)\\
&\quad\left.-\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{(s_1+1)}\,{}_2F_1\left(1+\frac{1}{p_1}, s_1+1, s_1+2, 1-\frac{b^{p_1}}{a^{p_1}}\right)\right]\\
&\quad\times\left[\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2^{s_2}(s_2+1)(s_2+2)}\,{}_2F_1\left(1+\frac{1}{p_2}, s_2+1, s_2+3, \frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_1}}\right)\right)\right.\\
&\quad+\frac{2(d^{p_2})^{1+\frac{1}{p_2}}}{(s_2+2)}\,{}_2F_1\left(1+\frac{1}{p_2}, s_2+2, s_2+3, 1-\frac{d^{p_2}}{c^{p_2}}\right)\\
&\quad\left.-\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{(1-s_1)}\,{}_2F_1\left(1+\frac{1}{p_2}, s_2+1, s_2+2, 1-\frac{d^{p_2}}{c^{p_2}}\right)\right],\\
&\vartheta_2^{**}\big(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\big)\\
&= \int_0^1\int_0^1 \left[|1-2t|(1-t)^{s_1}\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right]\\
&\quad\times\left[|1-2r|r^{s_2}\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right]\mathrm{d}t\,\mathrm{d}r\\
&= \left[\frac{4(a^{p_1})^{1+\frac{1}{p_1}}}{(s_1+2)}\,{}_2F_1\left(1+\frac{1}{p_1}, s_1+2, s_1+3, 1-\frac{a^{p_1}}{b^{p_1}}\right)\right.\\
&\quad-\frac{2(a^{p_1})^{1+\frac{1}{p_1}}}{(s_1+1)}\,{}_2F_1\left(1+\frac{1}{p_1}, s_1+1, s_1+2, 1-\frac{a^{p_1}}{b^{p_1}}\right)\\
&\quad+\frac{(a^{p_1})^{1+\frac{1}{p_1}}}{2^{s_1}(s_1+1)(s_1+2)}\,{}_2F_1\left(1+\frac{1}{p_1}, s_1+1, s_1+3, \frac{1}{2}\left(1-\frac{a^{p_1}}{b^{p_1}}\right)\right)\\
&\quad+\frac{2(b^{p_1})^{1+\frac{1}{p_1}}}{(s_1+1)(s_1+2)}\,{}_2F_1\left(1+\frac{1}{p_1}, 2, s_1+3, 1-\frac{b^{p_1}}{a^{p_1}}\right)\\
&\quad\left.-\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{(s_1+1)}\,{}_2F_1\left(1+\frac{1}{p_1}, 1, s_1+2, 1-\frac{b^{p_1}}{a^{p_1}}\right)\right]
\end{aligned}$$

$$\times \left[\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2^{s_2}(s_2+1)(s_2+2)} {}_2F_1\left(1+\frac{1}{p_2}, s_2+1, s_2+3, \frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right) \right.$$
$$+ \frac{2(d^{p_2})^{1+\frac{1}{p_2}}}{(s_2+2)} {}_2F_1\left(1+\frac{1}{p_2}, s_2+2, s_2+3, 1-\frac{d^{p_2}}{c^{p_2}}\right)$$
$$\left. - \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{(s_2+1)} {}_2F_1\left(1+\frac{1}{p_2}, s_2+1, s_2+2, 1-\frac{d^{p_2}}{c^{p_2}}\right) \right],$$

$$\vartheta_3^{**}\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right)$$
$$= \int_0^1 \int_0^1 \left[|1-2t| t^{s_1} \left[\frac{a^{p_1} b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})} \right]^{1+\frac{1}{p_1}} \right]$$
$$\times \left[|1-2r|(1-r)^{s_2} \left[\frac{c^{p_2} d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})} \right]^{1+\frac{1}{p_2}} \right] \mathrm{d}t\,\mathrm{d}r$$
$$= \left[\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2^{s_1}(s_1+1)(s_1+2)} {}_2F_1\left(1+\frac{1}{p_1}, s_1+1, s_1+3, \frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right) \right.$$
$$+ \frac{2(b^{p_1})^{1+\frac{1}{p_1}}}{(s_1+2)} {}_2F_1\left(1+\frac{1}{p_1}, s_1+2, s_1+3, 1-\frac{b^{p_1}}{a^{p_1}}\right)$$
$$\left. - \frac{(b^{p_1})^{1+\frac{1}{p_1}}}{(1-s_1)} {}_2F_1\left(1+\frac{1}{p_1}, s_1+1, s_1+2, 1-\frac{b^{p_1}}{a^{p_1}}\right) \right]$$
$$\times \left[\frac{4(c^{p_2})^{1+\frac{1}{p_2}}}{(s_2+2)} {}_2F_1\left(1+\frac{1}{p_2}, s_2+2, s_2+3, 1-\frac{c^{p_2}}{d^{p_2}}\right) \right.$$
$$- \frac{2(c^{p_2})^{1+\frac{1}{p_2}}}{(s_2+1)} {}_2F_1\left(1+\frac{1}{p_2}, s_2+1, s_2+2, 1-\frac{c^{p_2}}{d^{p_2}}\right)$$
$$+ \frac{(c^{p_2})^{1+\frac{1}{p_2}}}{2^{s_2}(s_2+1)(s_2+2)} {}_2F_1\left(1+\frac{1}{p_2}, s_2+1, s_2+3, \frac{1}{2}\left(1-\frac{c^{p_2}}{d^{p_2}}\right)\right)$$
$$+ \frac{2(d^{p_2})^{1+\frac{1}{p_2}}}{(s_2+1)(s_2+2)} {}_2F_1\left(1+\frac{1}{p_2}, 2, s_2+3, 1-\frac{d^{p_2}}{c^{p_2}}\right)$$
$$\left. - \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{(s_2+1)} {}_2F_1\left(1+\frac{1}{p_2}, 1, s_2+2, 1-\frac{d^{p_2}}{c^{p_2}}\right) \right],$$

$$\vartheta_4^{**}\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right)$$
$$= \int_0^1 \int_0^1 \left[|1-2t|(1-t)^{s_1} \left[\frac{a^{p_1} b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})} \right]^{1+\frac{1}{p_1}} \right]$$
$$\times \left[|1-2r|(1-r)^{s_2} \left[\frac{c^{p_2} d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})} \right]^{1+\frac{1}{p_2}} \right] \mathrm{d}t\,\mathrm{d}r$$
$$= \left[\frac{4(a^{p_1})^{1+\frac{1}{p_1}}}{(s_1+2)} {}_2F_1\left(1+\frac{1}{p_1}, s_1+2, s_1+3, 1-\frac{a^{p_1}}{b^{p_1}}\right) \right.$$
$$- \frac{2(a^{p_1})^{1+\frac{1}{p_1}}}{(s_1+1)} {}_2F_1\left(1+\frac{1}{p_1}, s_1+1, s_1+2, 1-\frac{a^{p_1}}{b^{p_1}}\right)$$

$$+\frac{(a^{p_1})^{1+\frac{1}{p_1}}}{2^{s_1}(s_1+1)(s_1+2)}{}_2F_1\left(1+\frac{1}{p_1},s_1+1,s_1+3,\frac{1}{2}\left(1-\frac{a^{p_1}}{b^{p_1}}\right)\right)$$
$$+\frac{2(b^{p_1})^{1+\frac{1}{p_1}}}{(s_1+1)(s_1+2)}{}_2F_1\left(1+\frac{1}{p_1},2,s_1+3,1-\frac{b^{p_1}}{a^{p_1}}\right)$$
$$-\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{(s_1+1)}{}_2F_1\left(1+\frac{1}{p_1},1,s_1+2,1-\frac{b^{p_1}}{a^{p_1}}\right)\Bigg]$$
$$\times\Bigg[\frac{4(c^{p_2})^{1+\frac{1}{p_2}}}{(s_2+2)}{}_2F_1\left(1+\frac{1}{p_2},s_2+2,s_2+3,1-\frac{c^{p_2}}{d^{p_2}}\right)$$
$$-\frac{2(c^{p_2})^{1+\frac{1}{p_2}}}{(s_2+1)}{}_2F_1\left(1+\frac{1}{p_2},s_2+1,s_2+2,1-\frac{c^{p_2}}{d^{p_2}}\right)$$
$$+\frac{(c^{p_2})^{1+\frac{1}{p_2}}}{2^{s_2}(s_2+1)(s_2+2)}{}_2F_1\left(1+\frac{1}{p_2},s_2+1,s_2+3,\frac{1}{2}\left(1-\frac{c^{p_2}}{d^{p_2}}\right)\right)$$
$$+\frac{2(d^{p_2})^{1+\frac{1}{p_2}}}{(s_2+1)(s_2+2)}{}_2F_1\left(1+\frac{1}{p_2},2,s_2+3,1-\frac{d^{p_2}}{c^{p_2}}\right)$$
$$-\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{(s_2+1)}{}_2F_1\left(1+\frac{1}{p_2},1,s_2+2,1-\frac{d^{p_2}}{c^{p_2}}\right)\Bigg],$$

$\vartheta_5(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)$ and $\vartheta_6(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)$ are given in Theorem 2.

III. If we take $h_1(t)=t^{-s_1}$ and $h_2(r)=r^{-s_2}$ in Theorem 2, then

Corollary 8 *Under the assumptions of Theorem 2, if $|\frac{\partial^2 f}{\partial r\,\partial t}|$ is a Godunova–Levin type approximately two-dimensional harmonic (p_1,s_1)-(p_2,s_2)-convex function, then*

$$\begin{aligned}&\left|\Xi\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\right)\right|\\&\quad\le\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}\left[\vartheta_1'\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\right|\right.\\&\qquad+\vartheta_2'\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\right|+\vartheta_3'\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\right|\\&\qquad+\vartheta_4'\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\right|+\vartheta_5\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\\&\qquad\left.+\vartheta_6\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\right],\end{aligned}$$

where

$$\begin{aligned}&\vartheta_1'\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\\&\quad=\int_0^1\int_0^1\left[|1-2t|t^{-s_1}\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right]\\&\qquad\times\left[|1-2r|r^{-s_2}\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right]\mathrm{d}t\,\mathrm{d}r\end{aligned}$$

$$= \left[\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2^{-s_1}(-s_1+1)(-s_1+2)} {}_2F_1\left(1+\frac{1}{p_1}, -s_1+1, -s_1+3, \frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right) + \frac{2(b^{p_1})^{1+\frac{1}{p_1}}}{(-s_1+2)} {}_2F_1\left(1+\frac{1}{p_1}, -s_1+2, -s_1+3, 1-\frac{b^{p_1}}{a^{p_1}}\right) - \frac{(b^{p_1})^{1+\frac{1}{p_1}}}{(-s_1+1)} {}_2F_1\left(1+\frac{1}{p_1}, -s_1+1, -s_1+2, 1-\frac{b^{p_1}}{a^{p_1}}\right) \right]$$
$$\times \left[\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2^{-s_2}(-s_2+1)(-s_2+2)} {}_2F_1\left(1+\frac{1}{p_2}, -s_2+1, -s_2+3, \frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_1}}\right)\right) + \frac{2(d^{p_2})^{1+\frac{1}{p_2}}}{(-s_2+2)} {}_2F_1\left(1+\frac{1}{p_2}, -s_2+2, -s_2+3, 1-\frac{d^{p_2}}{c^{p_2}}\right) - \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{(-s_1+1)} {}_2F_1\left(1+\frac{1}{p_2}, -s_2+1, -s_2+2, 1-\frac{d^{p_2}}{c^{p_2}}\right) \right],$$

$$\vartheta_2'\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right)$$
$$= \int_0^1 \int_0^1 \left[|1-2t|(1-t)^{-s_1} \left[\frac{a^{p_1} b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})} \right]^{1+\frac{1}{p_1}} \right] \times \left[|1-2r| r^{-s_2} \left[\frac{c^{p_2} d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})} \right]^{1+\frac{1}{p_2}} \right] \mathrm{d}t\,\mathrm{d}r$$
$$= \left[\frac{4(a^{p_1})^{1+\frac{1}{p_1}}}{(-s_1+2)} {}_2F_1\left(1+\frac{1}{p_1}, -s_1+2, -s_1+3, 1-\frac{a^{p_1}}{b^{p_1}}\right) - \frac{2(a^{p_1})^{1+\frac{1}{p_1}}}{(-s_1+1)} {}_2F_1\left(1+\frac{1}{p_1}, -s_1+1, -s_1+2, 1-\frac{a^{p_1}}{b^{p_1}}\right) + \frac{(a^{p_1})^{1+\frac{1}{p_1}}}{2^{-s_1}(-s_1+1)(-s_1+2)} {}_2F_1\left(1+\frac{1}{p_1}, -s_1+1, -s_1+3, \frac{1}{2}\left(1-\frac{a^{p_1}}{b^{p_1}}\right)\right) + \frac{2(b^{p_1})^{1+\frac{1}{p_1}}}{(-s_1+1)(-s_1+2)} {}_2F_1\left(1+\frac{1}{p_1}, 2, -s_1+3, 1-\frac{b^{p_1}}{a^{p_1}}\right) - \frac{(b^{p_1})^{1+\frac{1}{p_1}}}{(-s_1+1)} {}_2F_1\left(1+\frac{1}{p_1}, 1, -s_1+2, 1-\frac{b^{p_1}}{a^{p_1}}\right) \right]$$
$$\times \left[\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2^{-s_2}(-s_2+1)(-s_2+2)} {}_2F_1\left(1+\frac{1}{p_2}, -s_2+1, -s_2+3, \frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right) + \frac{2(d^{p_2})^{1+\frac{1}{p_2}}}{(-s_2+2)} {}_2F_1\left(1+\frac{1}{p_2}, -s_2+2, -s_2+3, 1-\frac{d^{p_2}}{c^{p_2}}\right) - \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{(-s_2+1)} {}_2F_1\left(1+\frac{1}{p_2}, -s_2+1, -s_2+2, 1-\frac{d^{p_2}}{c^{p_2}}\right) \right],$$

$$\vartheta_3'\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right)$$
$$= \int_0^1 \int_0^1 \left[|1-2t| t^{-s_1} \left[\frac{a^{p_1} b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})} \right]^{1+\frac{1}{p_1}} \right]$$

$$\times \left[|1-2r|(1-r)^{-s_2} \left[\frac{c^{p_2} d^{p_2}}{(rd^{p_2} + (1-r)c^{p_2})} \right]^{1+\frac{1}{p_2}} \right] \mathrm{d}t\, \mathrm{d}r$$

$$= \left[\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2^{-s_1}(-s_1+1)(-s_1+2)} {}_2F_1\left(1+\frac{1}{p_1}, -s_1+1, -s_1+3, \frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right) \right.$$

$$+ \frac{2(b^{p_1})^{1+\frac{1}{p_1}}}{(-s_1+2)} {}_2F_1\left(1+\frac{1}{p_1}, -s_1+2, -s_1+3, 1-\frac{b^{p_1}}{a^{p_1}}\right)$$

$$\left. - \frac{(b^{p_1})^{1+\frac{1}{p_1}}}{(-s_1+1)} {}_2F_1\left(1+\frac{1}{p_1}, -s_1+1, -s_1+2, 1-\frac{b^{p_1}}{a^{p_1}}\right) \right]$$

$$\times \left[\frac{4(c^{p_2})^{1+\frac{1}{p_2}}}{(-s_2+2)} {}_2F_1\left(1+\frac{1}{p_2}, -s_2+2, -s_2+3, 1-\frac{c^{p_2}}{d^{p_2}}\right) \right.$$

$$- \frac{2(c^{p_2})^{1+\frac{1}{p_2}}}{(-s_2+1)} {}_2F_1\left(1+\frac{1}{p_2}, -s_2+1, -s_2+2, 1-\frac{c^{p_2}}{d^{p_2}}\right)$$

$$+ \frac{(c^{p_2})^{1+\frac{1}{p_2}}}{2^{-s_2}(-s_2+1)(-s_2+2)} {}_2F_1\left(1+\frac{1}{p_2}, -s_2+1, -s_2+3, \frac{1}{2}\left(1-\frac{c^{p_2}}{d^{p_2}}\right)\right)$$

$$+ \frac{2(d^{p_2})^{1+\frac{1}{p_2}}}{(-s_2+1)(-s_2+2)} {}_2F_1\left(1+\frac{1}{p_2}, 2, -s_2+3, 1-\frac{d^{p_2}}{c^{p_2}}\right)$$

$$\left. - \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{(-s_2+1)} {}_2F_1\left(1+\frac{1}{p_2}, 1, -s_2+2, 1-\frac{d^{p_2}}{c^{p_2}}\right) \right],$$

$$\vartheta_4'\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right)$$

$$= \int_0^1 \int_0^1 \left[|1-2t|(1-t)^{-s_1} \left[\frac{a^{p_1} b^{p_1}}{(tb^{p_1} + (1-t)a^{p_1})} \right]^{1+\frac{1}{p_1}} \right]$$

$$\times \left[|1-2r|(1-r)^{-s_2} \left[\frac{c^{p_2} d^{p_2}}{(rd^{p_2} + (1-r)c^{p_2})} \right]^{1+\frac{1}{p_2}} \right] \mathrm{d}t\, \mathrm{d}r$$

$$= \left[\frac{4(a^{p_1})^{1+\frac{1}{p_1}}}{(-s_1+2)} {}_2F_1\left(1+\frac{1}{p_1}, -s_1+2, -s_1+3, 1-\frac{a^{p_1}}{b^{p_1}}\right) \right.$$

$$- \frac{2(a^{p_1})^{1+\frac{1}{p_1}}}{(-s_1+1)} {}_2F_1\left(1+\frac{1}{p_1}, -s_1+1, -s_1+2, 1-\frac{a^{p_1}}{b^{p_1}}\right)$$

$$+ \frac{(a^{p_1})^{1+\frac{1}{p_1}}}{2^{-s_1}(-s_1+1)(-s_1+2)} {}_2F_1\left(1+\frac{1}{p_1}, -s_1+1, -s_1+3, \frac{1}{2}\left(1-\frac{a^{p_1}}{b^{p_1}}\right)\right)$$

$$+ \frac{2(b^{p_1})^{1+\frac{1}{p_1}}}{(-s_1+1)(-s_1+2)} {}_2F_1\left(1+\frac{1}{p_1}, 2, -s_1+3, 1-\frac{b^{p_1}}{a^{p_1}}\right)$$

$$\left. - \frac{(b^{p_1})^{1+\frac{1}{p_1}}}{(-s_1+1)} {}_2F_1\left(1+\frac{1}{p_1}, 1, -s_1+2, 1-\frac{b^{p_1}}{a^{p_1}}\right) \right]$$

$$\times \left[\frac{4(c^{p_2})^{1+\frac{1}{p_2}}}{(-s_2+2)} {}_2F_1\left(1+\frac{1}{p_2}, -s_2+2, -s_2+3, 1-\frac{c^{p_2}}{d^{p_2}}\right) \right.$$

$$- \frac{2(c^{p_2})^{1+\frac{1}{p_2}}}{(-s_2+1)} {}_2F_1\left(1+\frac{1}{p_2}, -s_2+1, -s_2+2, 1-\frac{c^{p_2}}{d^{p_2}}\right)$$

$$+\frac{(c^{p_2})^{1+\frac{1}{p_2}}}{2^{-s_2}(-s_2+1)(-s_2+2)}{}_2F_1\left(1+\frac{1}{p_2},-s_2+1,-s_2+3,\frac{1}{2}\left(1-\frac{c^{p_2}}{d^{p_2}}\right)\right)$$
$$+\frac{2(d^{p_2})^{1+\frac{1}{p_2}}}{(-s_2+1)(-s_2+2)}{}_2F_1\left(1+\frac{1}{p_2},2,-s_2+3,1-\frac{d^{p_2}}{c^{p_2}}\right)$$
$$-\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{(-s_2+1)}{}_2F_1\left(1+\frac{1}{p_2},1,-s_2+2,1-\frac{d^{p_2}}{c^{p_2}}\right)\Bigg],$$

$\vartheta_5(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)$ and $\vartheta_6(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)$ are given in Theorem 2.

IV. If we take $h(t)=1$ and $h(r)=1$ in Theorem 2, then

Corollary 9 *Under the assumptions of Theorem 2, if $|\frac{\partial^2 f}{\partial r\,\partial t}|$ is an approximately two-dimensional harmonic (p_1,P)-(p_2,P)-convex function, then*

$$\begin{aligned}&\left|\Xi\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\right)\right|\\&\quad\le\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}\vartheta''\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left[\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\right|\right.\\&\quad\quad\left.+\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\right|+\Delta(a,b)+\Delta(c,d)\right],\end{aligned}$$

where

$$\begin{aligned}&\vartheta''\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\\&\quad=\int_0^1\int_0^1\left[|1-2t|\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right]\\&\quad\quad\times\left[|1-2r|\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right]\mathrm{d}t\,\mathrm{d}r\\&\quad=\left[(b^{p_1})^{1+\frac{1}{p_1}}{}_2F_1\left(1+\frac{1}{p_1},2,3,1-\frac{b^{p_1}}{a^{p_1}}\right)-(b^{p_1})^{1+\frac{1}{p_1}}{}_2F_1\left(1+\frac{1}{p_1},1,2,1-\frac{b^{p_1}}{a^{p_1}}\right)\right.\\&\quad\quad\left.+\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2}{}_2F_1\left(1+\frac{1}{p_1},1,3,\frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right)\right]\left[(d^{p_2})^{1+\frac{1}{p_2}}{}_2F_1\left(1+\frac{1}{p_2},2,3,1-\frac{d^{p_2}}{c^{p_2}}\right)\right.\\&\quad\quad-(d^{p_2})^{1+\frac{1}{p_2}}{}_2F_1\left(1+\frac{1}{p_2},1,2,1-\frac{d^{p_2}}{c^{p_2}}\right)\\&\quad\quad\left.+\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2}{}_2F_1\left(1+\frac{1}{p_2},1,3,\frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right)\right]\end{aligned}$$

V. If we take $\Delta(a,b)=-\mu(t^{\sigma}(1-t)+t(1-t)^{\sigma})(\|\frac{1}{b^{p_1}}-\frac{1}{a^{p_1}}\|)^{\sigma}$ and $\Delta(c,d)=-\mu(r^{\sigma}(1-r)+r(1-r)^{\sigma})(\|\frac{1}{d^{p_2}}-\frac{1}{c^{p_2}}\|)^{\sigma}$ for some $\mu>0$ in Theorem 2, then

Corollary 10 *Under the assumptions of Theorem 2, if $|\frac{\partial^2 f}{\partial r\,\partial t}|$ is a two-dimensional harmonically strong (p_1,h_1)-(p_2,h_2)-convex function with $\mu>0$ of higher order, then*

$$\left|\Xi\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\right)\right|$$

$$\leq \frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}\bigg[\vartheta_1\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\bigg|$$
$$+\vartheta_2\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\bigg|+\vartheta_3\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\bigg|$$
$$+\vartheta_4\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\bigg|+\vartheta_5^*\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)$$
$$+\vartheta_6^*\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg],$$

where $\vartheta_1(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega),\vartheta_2(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega),\vartheta_3(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega),\vartheta_4(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)$ *are given in Theorem 2 and*

$$\vartheta_5^*\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\big)$$
$$=-\mu\left(\left\|\frac{1}{b^{p_1}}-\frac{1}{a^{p_1}}\right\|\right)^{\sigma}$$
$$\times\int_0^1\int_0^1\left(|1-2t|\big(t^{\sigma}(1-t)+t(1-t)^{\sigma}\big)\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right)$$
$$\times\left(|1-2r|\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right)\mathrm{d}t\,\mathrm{d}r$$
$$=-\mu\left(\left\|\frac{1}{b^{p_1}}-\frac{1}{a^{p_1}}\right\|\right)^{\sigma}\left[\frac{2(b^{p_1})^{1+\frac{1}{p_1}}}{(\sigma+2)(\sigma+3)}{}_2F_1\left(1+\frac{1}{p_1},\sigma+2,\sigma+4,1-\frac{b^{p_1}}{a^{p_1}}\right)\right.$$
$$+\frac{2(b^{p_1})^{1+\frac{1}{p_1}}}{(\sigma+3)(\sigma+2)}{}_2F_1\left(1+\frac{1}{p_1},3,\sigma+4,1-\frac{b^{p_1}}{a^{p_1}}\right)$$
$$-\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{(\sigma+1)(\sigma+2)}{}_2F_1\left(1+\frac{1}{p_1},\sigma+1,\sigma+3,1-\frac{b^{p_1}}{a^{p_1}}\right)$$
$$-\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{(\sigma+1)(\sigma+2)}{}_2F_1\left(1+\frac{1}{p_1},2,\sigma+3,1-\frac{b^{p_1}}{a^{p_1}}\right)$$
$$+\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2^{\sigma}(\sigma+1)(\sigma+2)}{}_2F_1\left(1+\frac{1}{p_1},\sigma+1,\sigma+3,\frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right)$$
$$-\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2^{\sigma+1}(\sigma+2)(\sigma+3)}{}_2F_1\left(1+\frac{1}{p_1},\sigma+2,\sigma+4,\frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right)$$
$$+\frac{4(a^{p_1})^{1+\frac{1}{p_1}}}{(\sigma+2)(\sigma+3)}{}_2F_1\left(1+\frac{1}{p_1},\sigma+2,\sigma+4,1-\frac{a^{p_1}}{b^{p_1}}\right)$$
$$-\frac{2(a^{p_1})^{1+\frac{1}{p_1}}}{(\sigma+1)(\sigma+2)}{}_2F_1\left(1+\frac{1}{p_1},\sigma+1,\sigma+3,1-\frac{a^{p_1}}{b^{p_1}}\right)$$
$$+\frac{(a^{p_1})^{1+\frac{1}{p_1}}}{2^{\sigma}(\sigma+1)(\sigma+2)}{}_2F_1\left(1+\frac{1}{p_1},\sigma+1,\sigma+3,\frac{1}{2}\left(1-\frac{a^{p_1}}{b^{p_1}}\right)\right)$$
$$\left.-\frac{(a^{p_1})^{1+\frac{1}{p_1}}}{2^{\sigma+1}(\sigma+2)(\sigma+3)}{}_2F_1\left(1+\frac{1}{p_1},\sigma+2,\sigma+4,\frac{1}{2}\left(1-\frac{a^{p_1}}{b^{p_1}}\right)\right)\right]$$

$$\times\left[\left(d^{p_2}\right)^{1+\frac{1}{p_2}}{}_2F_1\left(1+\frac{1}{p_2},2,3,1-\frac{d^{p_2}}{c^{p_2}}\right)-\left(d^{p_2}\right)^{1+\frac{1}{p_2}}{}_2F_1\left(1+\frac{1}{p_2},1,2,1-\frac{d^{p_2}}{c^{p_2}}\right)\right.$$
$$\left.+\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2}{}_2F_1\left(1+\frac{1}{p_2},1,3,\frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right)\right],$$

$$\vartheta_6^*\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\right)$$
$$=-\mu\left(\left\|\frac{1}{d^{p_2}}-\frac{1}{c^{p_2}}\right\|\right)^{\sigma}\times\int_0^1\int_0^1\left(|1-2t|\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right)$$
$$\times\left(|1-2r|\left(r^{\sigma}(1-r)+r(1-r)^{\sigma}\right)\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right)\mathrm{d}t\,\mathrm{d}r$$
$$=-\mu\left(\left\|\frac{1}{d^{p_2}}-\frac{1}{c^{p_2}}\right\|\right)^{\sigma}\left[\left(b^{p_1}\right)^{1+\frac{1}{p_1}}{}_2F_1\left(1+\frac{1}{p_1},2,3,1-\frac{b^{p_1}}{a^{p_1}}\right)\right.$$
$$\left.-\left(b^{p_1}\right)^{1+\frac{1}{p_1}}{}_2F_1\left(1+\frac{1}{p_1},1,2,1-\frac{b^{p_1}}{a^{p_1}}\right)+\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2}{}_2F_1\left(1+\frac{1}{p_1},1,3,\frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right)\right]$$
$$\times\left[\frac{2(d^{p_2})^{1+\frac{1}{p_2}}}{(\sigma+2)(\sigma+3)}{}_2F_1\left(1+\frac{1}{p_2},\sigma+2,\sigma+4,1-\frac{d^{p_2}}{c^{p_2}}\right)\right.$$
$$+\frac{2(d^{p_2})^{1+\frac{1}{p_2}}}{(\sigma+3)(\sigma+2)}{}_2F_1\left(1+\frac{1}{p_2},3,\sigma+4,1-\frac{d^{p_2}}{c^{p_2}}\right)$$
$$-\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{(\sigma+1)(\sigma+2)}{}_2F_1\left(1+\frac{1}{p_2},\sigma+1,\sigma+3,1-\frac{d^{p_2}}{c^{p_2}}\right)$$
$$-\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{(\sigma+1)(\sigma+2)}{}_2F_1\left(1+\frac{1}{p_2},2,\sigma+3,1-\frac{d^{p_2}}{c^{p_2}}\right)$$
$$+\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2^{\sigma}(\sigma+1)(\sigma+2)}{}_2F_1\left(1+\frac{1}{p_2},\sigma+1,\sigma+3,\frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right)$$
$$-\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2^{\sigma+1}(\sigma+2)(\sigma+3)}{}_2F_1\left(1+\frac{1}{p_2},\sigma+2,\sigma+4,\frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right)$$
$$+\frac{4(c^{p_2})^{1+\frac{1}{p_2}}}{(\sigma+2)(\sigma+3)}{}_2F_1\left(1+\frac{1}{p_2},\sigma+2,\sigma+4,1-\frac{c^{p_2}}{d^{p_2}}\right)$$
$$-\frac{2(c^{p_2})^{1+\frac{1}{p_2}}}{(\sigma+1)(\sigma+2)}{}_2F_1\left(1+\frac{1}{p_2},\sigma+1,\sigma+3,1-\frac{c^{p_2}}{d^{p_2}}\right)$$
$$+\frac{(c^{p_2})^{1+\frac{1}{p_2}}}{2^{\sigma}(\sigma+1)(\sigma+2)}{}_2F_1\left(1+\frac{1}{p_2},\sigma+1,\sigma+3,\frac{1}{2}\left(1-\frac{c^{p_2}}{d^{p_2}}\right)\right)$$
$$\left.-\frac{(c^{p_2})^{1+\frac{1}{p_2}}}{2^{\sigma+1}(\sigma+2)(\sigma+3)}{}_2F_1\left(1+\frac{1}{p_2},\sigma+2,\sigma+4,\frac{1}{2}\left(1-\frac{c^{p_2}}{d^{p_2}}\right)\right)\right].$$

VI. If we take $\sigma = 2$ in Corollary 10, then

Corollary 11 *Under the assumptions of Corollary 10, if* $|\frac{\partial^2 f}{\partial r\,\partial t}|$ *is a two-dimensional harmonically strong* (p_1,h_1)*-*(p_2,h_2)*-convex function with* $\mu > 0$*, then*

$$\left|\Xi\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\right)\right|$$

$$\leq \frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}\bigg[\vartheta_1\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\bigg|$$
$$+\vartheta_2\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\bigg|+\vartheta_3\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\bigg|$$
$$+\vartheta_4\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\bigg|+\vartheta_5^{**}\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)$$
$$+\vartheta_6^{**}\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg],$$

where $\vartheta_1(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega),\vartheta_2(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega),\vartheta_3(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega),\vartheta_4(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)$ *are given in Theorem* 2 *and*

$$\vartheta_5^{**}\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\big)$$
$$=-\mu\bigg(\bigg\|\frac{1}{b^{p_1}}-\frac{1}{a^{p_1}}\bigg\|\bigg)^2\times\int_0^1\int_0^1\bigg(|1-2t|\bigg(t(1-t)\bigg[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\bigg]^{1+\frac{1}{p_1}}\bigg)$$
$$\times\bigg(|1-2r|\bigg[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\bigg]^{1+\frac{1}{p_2}}\bigg)\,\mathrm{d}t\,\mathrm{d}r$$
$$=-\mu\bigg(\bigg\|\frac{1}{b^{p_1}}-\frac{1}{a^{p_1}}\bigg\|\bigg)^2\bigg[\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{4}{}_2F_1\bigg(1+\frac{1}{p_1},2,3,\frac{1}{2}\bigg(1-\frac{b^{p_1}}{a^{p_1}}\bigg)\bigg)$$
$$-\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{4}{}_2F_1\bigg(1+\frac{1}{p_1},3,4,\frac{1}{2}\bigg(1-\frac{b^{p_1}}{a^{p_1}}\bigg)\bigg)$$
$$+\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{16}{}_2F_1\bigg(1+\frac{1}{p_1},4,5,\frac{1}{2}\bigg(1-\frac{b^{p_1}}{a^{p_1}}\bigg)\bigg)$$
$$+(b^{p_1})^{1+\frac{1}{p_1}}{}_2F_1\bigg(1+\frac{1}{p_1},3,4,\bigg(1-\frac{b^{p_1}}{a^{p_1}}\bigg)\bigg)$$
$$-\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2}{}_2F_1\bigg(1+\frac{1}{p_1},4,5,\bigg(1-\frac{b^{p_1}}{a^{p_1}}\bigg)\bigg)$$
$$-\frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2}{}_2F_1\bigg(1+\frac{1}{p_1},2,3,\bigg(1-\frac{b^{p_1}}{a^{p_1}}\bigg)\bigg)\bigg]$$
$$\times\bigg[(d^{p_2})^{1+\frac{1}{p_2}}{}_2F_1\bigg(1+\frac{1}{p_2},2,3,1-\frac{d^{p_2}}{c^{p_2}}\bigg)-(d^{p_2})^{1+\frac{1}{p_2}}{}_2F_1\bigg(1+\frac{1}{p_2},1,2,1-\frac{d^{p_2}}{c^{p_2}}\bigg)$$
$$+\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2}{}_2F_1\bigg(1+\frac{1}{p_2},1,3,\frac{1}{2}\bigg(1-\frac{d^{p_2}}{c^{p_2}}\bigg)\bigg)\bigg],$$

$$\vartheta_6^{**}\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\big)$$
$$=-\mu\bigg(\bigg\|\frac{1}{d^{p_2}}-\frac{1}{c^{p_2}}\bigg\|\bigg)^2\times\int_0^1\int_0^1\bigg(|1-2t|\bigg[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\bigg]^{1+\frac{1}{p_1}}\bigg)$$
$$\times\bigg(|1-2r|r(1-r)\bigg[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\bigg]^{1+\frac{1}{p_2}}\bigg)\,\mathrm{d}t\,\mathrm{d}r$$
$$-\mu\bigg(\bigg\|\frac{1}{d^{p_2}}-\frac{1}{c^{p_2}}\bigg\|\bigg)^2\times\bigg[(b^{p_1})^{1+\frac{1}{p_1}}{}_2F_1\bigg(1+\frac{1}{p_1},2,3,1-\frac{b^{p_1}}{a^{p_1}}\bigg)$$

$$
\begin{aligned}
&- \left(d^{p_1}\right)^{1+\frac{1}{p_1}} {}_2F_1\left(1+\frac{1}{p_1}, 1, 2, 1-\frac{b^{p_1}}{a^{p_1}}\right) \\
&+ \frac{(b^{p_1})^{1+\frac{1}{p_1}}}{2} {}_2F_1\left(1+\frac{1}{p_1}, 1, 3, \frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right)\bigg] \\
&\times \left[\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{4} {}_2F_1\left(1+\frac{1}{p_2}, 2, 3, \frac{1}{2}\left(1-\frac{d^{p_2}}{a^{p_2}}\right)\right)\right. \\
&- \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{4} {}_2F_1\left(1+\frac{1}{p_2}, 3, 4, \frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right) \\
&+ \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{16} {}_2F_1\left(1+\frac{1}{p_2}, 4, 5, \frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right) \\
&+ \left(d^{p_2}\right)^{1+\frac{1}{p_2}} {}_2F_1\left(1+\frac{1}{p_2}, 3, 4, \left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right) \\
&- \frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2} {}_2F_1\left(1+\frac{1}{p_2}, 4, 5, \left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right) \\
&- \left.\frac{(d^{p_2})^{1+\frac{1}{p_2}}}{2} {}_2F_1\left(1+\frac{1}{p_2}, 2, 3, \left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right)\right].
\end{aligned}
$$

Theorem 3 *Let $f : \Omega \to \mathbb{R}$ be a partial differentiable function on $\Omega = [a,b] \times [c,d] \subset (0,\infty) \times (0,\infty)$ with $a < b$ and $c < d$ and $\frac{\partial^2 f}{\partial t \partial r} \in L_1(\Omega)$. If $|\frac{\partial^2 f}{\partial r \partial t}|^q$ is an approximately two-dimensional harmonic (p_1,h_1)-(p_2,h_2)-convex function, where $\frac{1}{p} + \frac{1}{q} = 1$ and $q > 1$, we have*

$$
\begin{aligned}
&\left|\Xi\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2}, x, y : \Omega\right)\right| \\
&\quad \le \frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}(p+1)^{\frac{2}{p}}}\left[\varphi_1\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\right|^q\right. \\
&\qquad + \varphi_2\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\right|^q + \varphi_3\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\right|^q \\
&\qquad + \varphi_4\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\right|^q + \varphi_5\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right) \\
&\qquad \left. + \varphi_6\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2} : \Omega\right)\right]^{\frac{1}{q}},
\end{aligned}
$$

where

$$
\begin{aligned}
&\varphi_1\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2}; \Omega\right) \\
&\quad = \int_0^1\int_0^1\left[h_1(t)\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\right] \\
&\qquad \times \left[h_2(r)\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right]\mathrm{d}t\,\mathrm{d}r, \\
&\varphi_2\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2}; \Omega\right) \\
&\quad = \int_0^1\int_0^1\left[h_1(1-t)\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\right]
\end{aligned}
$$

$$\times \left[h_2(r)\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right]\mathrm{d}t\,\mathrm{d}r,$$

$$\varphi_3\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\big)$$
$$=\int_0^1\int_0^1\left[h_1(t)\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\right]$$
$$\times \left[h_2(1-r)\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right]\mathrm{d}t\,\mathrm{d}r,$$

$$\varphi_4\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\big)$$
$$=\int_0^1\int_0^1\left[h_1(1-t)\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\right]$$
$$\times \left[h_2(1-r)\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right]\mathrm{d}t\,\mathrm{d}r,$$

$$\varphi_5\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\big)$$
$$=\Delta(a,b)\int_0^1\int_0^1\left[\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\right]$$
$$\times \left[\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right]\mathrm{d}t\,\mathrm{d}r$$
$$=\Delta(a,b)\left(\left[\big(b^{p_1}\big)^{q(1+\frac{1}{p_1})}{}_2F_1\left(q\left(1+\frac{1}{p_1}\right),1,2,1-\frac{b^{p_1}}{a^{p_1}}\right)\right]\right.$$
$$\left.\times\left[\big(d^{p_2}\big)^{q(1+\frac{1}{p_2})}{}_2F_1\left(q\left(1+\frac{1}{p_2}\right),1,2,1-\frac{d^{p_2}}{c^{p_2}}\right)\right]\right),$$

and

$$\varphi_6\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\big)$$
$$=\int_0^1\int_0^1\left[\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\right]\left[\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right]\mathrm{d}t\,\mathrm{d}r$$
$$=\Delta(c,d)\left(\left[\big(b^{p_1}\big)^{q(1+\frac{1}{p_1})}{}_2F_1\left(q\left(1+\frac{1}{p_1}\right),1,2,1-\frac{b^{p_1}}{a^{p_1}}\right)\right]\right.$$
$$\left.\times\left[\big(d^{p_2}\big)^{q(1+\frac{1}{p_2})}{}_2F_1\left(q\left(1+\frac{1}{p_2}\right),1,2,1-\frac{d^{p_2}}{c^{p_2}}\right)\right]\right).$$

Proof Using Lemma 1, well-known Hölder's inequality, and the fact that $|\frac{\partial^2 f}{\partial r\,\partial t}|^q$ is an approximately two-dimensional harmonic (p_1,h_1)-(p_2,h_2)-convex function, we have

$$\big|\Xi\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\big)\big|$$
$$=\left|\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}\int_0^1\int_0^1\left((1-2t)\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{1+\frac{1}{p_1}}\right)\right.$$
$$\times\left((1-2r)\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{1+\frac{1}{p_2}}\right)$$

$$\times \frac{\partial^2 f}{\partial r\,\partial t}\left(\left[\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}d^{p_2}}{rd^{p_2}+(1-r)c^{p_2}}\right]^{\frac{1}{p_2}}\right)\mathrm{d}r\,\mathrm{d}t\bigg|$$

$$\le \frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}}\int_0^1\int_0^1\left(\left|(1-2t)(1-2r)\right|^p\,\mathrm{d}r\,\mathrm{d}t\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right.$$

$$\left.\times\left|\frac{\partial^2 f}{\partial r\,\partial t}\left(\left[\frac{a^{p_1}b^{p_1}}{tb^{p_1}+(1-t)a^{p_1}}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}d^{p_2}}{rd^{p_2}+(1-r)c^{p_2}}\right]^{\frac{1}{p_2}}\right)\right|^q\mathrm{d}r\,\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\le \frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}(p+1)^{\frac{2}{p}}}$$

$$\times\left(\int_0^1\int_0^1\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right.$$

$$\times\left[h_1(t)h_2(r)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\right|^q+h_1(1-t)h_2(r)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\right|^q\right.$$

$$+h_1(t)h_2(1-r)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\right|^q$$

$$\left.\left.+h_1(1-t)h_2(1-r)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\right|^q+\Delta(a,b)+\Delta(c,d)\right]\mathrm{d}r\,\mathrm{d}t\right)^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 12 *Under the assumptions of Theorem 2, if* $|\frac{\partial^2 f}{\partial r\,\partial t}|\le M$ *is a bounded function for* $M>0$ *on* Ω*, then*

$$\left|\Xi\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\right)\right|$$

$$\le \frac{M(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}(p+1)^{\frac{2}{p}}}\left[\varphi_1\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\right.$$

$$+\varphi_2\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)+\varphi_3\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)$$

$$+\varphi_4\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)+\frac{1}{M^q}\varphi_5\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)$$

$$\left.+\frac{1}{M^q}\varphi_6\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\right]^{\frac{1}{q}},$$

where $\varphi_1(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)$ *to* $\varphi_6(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)$ *are given in Theorem 3.*

We now discuss some special cases of Theorem 3.

I. If we take $h_1(t)=t$ and $h_2(r)=r$ in Theorem 3, then

Corollary 13 *Under the assumptions of Theorem 3, if* $|\frac{\partial^2 f}{\partial r\,\partial t}|^q$ *is an approximately two-dimensional harmonic* (p_1,p_2)*-convex function, where* $\frac{1}{p}+\frac{1}{q}=1$ *and* $q>1$*, then*

$$\left|\Xi\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\right)\right|$$

$$\begin{aligned}
&\leq \frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}(p+1)^{\frac{2}{p}}}\bigg[\varphi_1^*\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\bigg|^q \\
&\quad+\varphi_2^*\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\bigg|^q+\varphi_3^*\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\bigg|^q \\
&\quad+\varphi_4^*\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\bigg|^q+\varphi_5\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big) \\
&\quad+\varphi_6\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\big)\bigg]^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
&\varphi_1^*\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\big) \\
&\quad=\int_0^1\int_0^1\bigg[t\bigg[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\bigg]^{q(1+\frac{1}{p_1})}\bigg]\bigg[r\bigg[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\bigg]^{q(1+\frac{1}{p_2})}\bigg]\,\mathrm{d}t\,\mathrm{d}r \\
&\quad=\bigg[\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{2}{}_2F_1\bigg(q\bigg(1+\frac{1}{p_1}\bigg),2,3,1-\frac{b^{p_1}}{a^{p_1}}\bigg)\bigg] \\
&\qquad\times\bigg[\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{2}{}_2F_1\bigg(q\bigg(1+\frac{1}{p_2}\bigg),2,3,1-\frac{d^{p_2}}{c^{p_2}}\bigg)\bigg], \\
&\varphi_2^*\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\big) \\
&\quad=\int_0^1\int_0^1\bigg[(1-t)\bigg[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\bigg]^{q(1+\frac{1}{p_1})}\bigg] \\
&\qquad\times\bigg[r\bigg[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\bigg]^{q(1+\frac{1}{p_2})}\bigg]\,\mathrm{d}t\,\mathrm{d}r \\
&\quad=\bigg[\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{2}{}_2F_1\bigg(q\bigg(1+\frac{1}{p_1}\bigg),1,3,1-\frac{b^{p_1}}{a^{p_1}}\bigg)\bigg] \\
&\qquad\times\bigg[\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{2}{}_2F_1\bigg(q\bigg(1+\frac{1}{p_2}\bigg),2,3,1-\frac{d^{p_2}}{c^{p_2}}\bigg)\bigg], \\
&\varphi_3^*\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\big) \\
&\quad=\int_0^1\int_0^1\bigg[t\bigg[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\bigg]^{q(1+\frac{1}{p_1})}\bigg] \\
&\qquad\times\bigg[(1-r)\bigg[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\bigg]^{q(1+\frac{1}{p_2})}\bigg]\,\mathrm{d}t\,\mathrm{d}r \\
&\quad=\bigg[\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{2}{}_2F_1\bigg(q\bigg(1+\frac{1}{p_1}\bigg),2,3,1-\frac{b^{p_1}}{a^{p_1}}\bigg)\bigg] \\
&\qquad\times\bigg[\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{2}{}_2F_1\bigg(q\bigg(1+\frac{1}{p_2}\bigg),1,3,1-\frac{d^{p_2}}{c^{p_2}}\bigg)\bigg], \\
&\varphi_4^*\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\big) \\
&\quad=\int_0^1\int_0^1\bigg[(1-t)\bigg[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\bigg]^{q(1+\frac{1}{p_1})}\bigg]
\end{aligned}$$

$$\times\left[(1-r)\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right]\mathrm{d}t\,\mathrm{d}r$$
$$=\left[\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{2}{}_2F_1\left(q\left(1+\frac{1}{p_1}\right),1,3,1-\frac{b^{p_1}}{a^{p_1}}\right)\right]$$
$$\times\left[\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{2}{}_2F_1\left(q\left(1+\frac{1}{p_2}\right),1,3,1-\frac{d^{p_2}}{c^{p_2}}\right)\right],$$

$\varphi_5(a,b,c,d;\Omega)$ and $\varphi_6(a,b,c,d;\Omega)$ are given in Theorem 3.

II. If we take $h_1(t)=t^{s_1}$ and $h_2(r)=r^{s_2}$ in Theorem 3, then

Corollary 14 *Under the assumptions of Theorem 3, if $|\frac{\partial^2 f}{\partial r\partial t}|^q$ is a Breckner type approximately two-dimensional harmonic (p_1,s_1)-(p_2,s_2)-convex function, where $\frac{1}{p}+\frac{1}{q}=1$ and $q>1$, then*

$$\begin{aligned}&\left|\Xi\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\right)\right|\\
&\le\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}(p+1)^{\frac{2}{p}}}\left[\varphi_1^{**}\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\right|^q\right.\\
&\quad+\varphi_2^{**}\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\right|^q+\varphi_3^{**}\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\right|^q\\
&\quad+\varphi_4^{**}\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\right|^q+\varphi_5\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\\
&\quad\left.+\varphi_6\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\right]^{\frac{1}{q}},\end{aligned}$$

where

$$\begin{aligned}&\varphi_1^{**}\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\right)\\
&=\int_0^1\int_0^1\left[t^{s_1}\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\right]\left[r^{s_2}\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right]\mathrm{d}t\,\mathrm{d}r\\
&=\left[\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{s_1+1}{}_2F_1\left(q\left(1+\frac{1}{p_1}\right),s_1+1,s_1+2,1-\frac{b^{p_1}}{a^{p_1}}\right)\right]\\
&\quad\times\left[\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{s_2+1}{}_2F_1\left(q\left(1+\frac{1}{p_2}\right),s_2+1,s_2+2,1-\frac{d^{p_2}}{c^{p_2}}\right)\right],\\
&\varphi_2^{**}\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\right)\\
&=\int_0^1\int_0^1\left[(1-t)^{s_1}\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\right]\\
&\quad\times\left[r^{s_2}\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right]\mathrm{d}t\,\mathrm{d}r\\
&=\left[\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{s_1+1}{}_2F_1\left(q\left(1+\frac{1}{p_1}\right),1,s_1+2,1-\frac{b^{p_1}}{a^{p_1}}\right)\right]\end{aligned}$$

$$\times\left[\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{s_2+1}{}_2F_1\left(q\left(1+\frac{1}{p_2}\right),s_2+1,s_2+2,1-\frac{d^{p_2}}{c^{p_2}}\right)\right],$$

$$\varphi_3^{**}\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\right)$$
$$=\int_0^1\int_0^1\left[t^{s_1}\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\right]$$
$$\times\left[(1-r)^{s_2}\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right]\mathrm{d}t\,\mathrm{d}r$$
$$=\left[\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{s_1+1}{}_2F_1\left(q\left(1+\frac{1}{p_1}\right),s_1+1,s_1+2,1-\frac{b^{p_1}}{a^{p_1}}\right)\right]$$
$$\times\left[\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{s_2+1}{}_2F_1\left(q\left(1+\frac{1}{p_2}\right),1,s_2+2,1-\frac{d^{p_2}}{c^{p_2}}\right)\right],$$

$$\varphi_4^{**}\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\right)$$
$$=\int_0^1\int_0^1\left[(1-t)^{s_1}\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\right]$$
$$\times\left[(1-r)^{s_2}\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right]\mathrm{d}t\,\mathrm{d}r$$
$$=\left[\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{s_1+1}{}_2F_1\left(q\left(1+\frac{1}{p_1}\right),1,s_1+2,1-\frac{b^{p_1}}{a^{p_1}}\right)\right]$$
$$\times\left[\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{s_2+1}{}_2F_1\left(q\left(1+\frac{1}{p_2}\right),1,s_2+2,1-\frac{d^{p_2}}{c^{p_2}}\right)\right],$$

$\varphi_5(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega)$ and $\varphi_6(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega)$ are given in Theorem 3.

III. If we take $h_1(t)=t^{-s_1}$ and $h_2(r)=r^{-s_2}$ in Theorem 3, then

Corollary 15 *Under the assumptions of Theorem 3, if $|\frac{\partial^2 f}{\partial r\,\partial t}|^q$ is a Godunova–Levin type approximately two-dimensional harmonic (p_1,s_1)-(p_2,s_2)-convex function, where $\frac{1}{p}+\frac{1}{q}=1$ and $q>1$, we have*

$$\left|\Xi\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\right)\right|$$
$$\leq\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}(p+1)^{\frac{2}{p}}}\left[\varphi_1^{***}\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\right|^q\right.$$
$$+\varphi_2^{***}\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\right|^q+\varphi_3^{***}\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\right|^q$$
$$+\varphi_4^{***}\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\right|^q+\varphi_5\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)$$
$$\left.+\varphi_6\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\right)\right]^{\frac{1}{q}},$$

where

$$\varphi_1^{***}\left(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\right)$$

$$
\begin{aligned}
&= \int_0^1 \int_0^1 \left[t^{-s_1} \left[\frac{a^{p_1} b^{p_1}}{(tb^{p_1} + (1-t)a^{p_1})} \right]^{q(1+\frac{1}{p_1})} \right] \left[r^{-s_2} \left[\frac{c^{p_2} d^{p_2}}{(rd^{p_2} + (1-r)c^{p_2})} \right]^{q(1+\frac{1}{p_2})} \right] \mathrm{d}t\,\mathrm{d}r \\
&= \left[\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{1-s_1} {}_2F_1\left(q\left(1+\frac{1}{p_1}\right), 1-s_1, 2-s_1, 1-\frac{b^{p_1}}{a^{p_1}} \right) \right] \\
&\quad \times \left[\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{1-s_2} {}_2F_1\left(q\left(1+\frac{1}{p_2}\right), 1-s_2, 2-s_2, 1-\frac{d^{p_2}}{c^{p_2}} \right) \right],
\end{aligned}
$$

$$
\begin{aligned}
&\varphi_2^{***}\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2}; \Omega\right) \\
&= \int_0^1 \int_0^1 \left[(1-t)^{-s_1} \left[\frac{a^{p_1} b^{p_1}}{(tb^{p_1} + (1-t)a^{p_1})} \right]^{q(1+\frac{1}{p_1})} \right] \\
&\quad \times \left[r^{-s_2} \left[\frac{c^{p_2} d^{p_2}}{(rd^{p_2} + (1-r)c^{p_2})} \right]^{q(1+\frac{1}{p_2})} \right] \mathrm{d}t\,\mathrm{d}r \\
&= \left[\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{1-s_1} {}_2F_1\left(q\left(1+\frac{1}{p_1}\right), 1, 2-s_1, 1-\frac{b^{p_1}}{a^{p_1}} \right) \right] \\
&\quad \times \left[\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{1-s_2} {}_2F_1\left(q\left(1+\frac{1}{p_2}\right), 1-s_2, 2-s_2, 1-\frac{d^{p_2}}{c^{p_2}} \right) \right],
\end{aligned}
$$

$$
\begin{aligned}
&\varphi_3^{***}\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2}; \Omega\right) \\
&= \int_0^1 \int_0^1 \left[t^{-s_1} \left[\frac{a^{p_1} b^{p_1}}{(tb^{p_1} + (1-t)a^{p_1})} \right]^{q(1+\frac{1}{p_1})} \right] \\
&\quad \times \left[(1-r)^{-s_2} \left[\frac{c^{p_2} d^{p_2}}{(rd^{p_2} + (1-r)c^{p_2})} \right]^{q(1+\frac{1}{p_2})} \right] \mathrm{d}t\,\mathrm{d}r \\
&= \left[\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{1-s_1} {}_2F_1\left(q\left(1+\frac{1}{p_1}\right), 1-s_1, 2-s_1, 1-\frac{b^{p_1}}{a^{p_1}} \right) \right] \\
&\quad \times \left[\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{1-s_2} {}_2F_1\left(q\left(1+\frac{1}{p_2}\right), 1, 2-s_2, 1-\frac{d^{p_2}}{c^{p_2}} \right) \right],
\end{aligned}
$$

$$
\begin{aligned}
&\varphi_4^{***}\left(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2}; \Omega\right) \\
&= \int_0^1 \int_0^1 \left[(1-t)^{-s_1} \left[\frac{a^{p_1} b^{p_1}}{(tb^{p_1} + (1-t)a^{p_1})} \right]^{q(1+\frac{1}{p_1})} \right] \\
&\quad \times \left[(1-r)^{-s_2} \left[\frac{c^{p_2} d^{p_2}}{(rd^{p_2} + (1-r)c^{p_2})} \right]^{q(1+\frac{1}{p_2})} \right] \mathrm{d}t\,\mathrm{d}r \\
&= \left[\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{1-s_1} {}_2F_1\left(q\left(1+\frac{1}{p_1}\right), 1, 2-s_1, 1-\frac{b^{p_1}}{a^{p_1}} \right) \right] \\
&\quad \times \left[\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{1-s_2} {}_2F_1\left(q\left(1+\frac{1}{p_2}\right), 1, 2-s_2, 1-\frac{d^{p_2}}{c^{p_2}} \right) \right],
\end{aligned}
$$

$\varphi_5(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2}; \Omega)$ *and* $\varphi_6(a^{p_1}, b^{p_1}, c^{p_2}, d^{p_2}; \Omega)$ *are given in Theorem* 3.

IV. If we take $h_1(t) = 1$ and $h_2(r) = 1$ in Theorem 3, then

Corollary 16 *Under the assumptions of Theorem 3, if* $|\frac{\partial^2 f}{\partial r\,\partial t}|^q$ *is an approximately two-dimensional harmonic* (p_1,P)*-*(p_2,P)*-convex function, where* $\frac{1}{p}+\frac{1}{q}=1$ *and* $q>1$*, then*

$$\begin{aligned}&\bigl|\Xi\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\bigr)\bigr|\\&\quad\le\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}(p+1)^{\frac{2}{p}}}\bigl[\varphi\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr)\bigr]^{\frac{1}{q}}\biggl[\biggl|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\biggr|^q\\&\qquad+\biggl|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\biggr|^q+\biggl|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\biggr|^q+\biggl|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\biggr|^q+\Delta(a,b)+\Delta(c,d)\biggr]^{\frac{1}{q}},\end{aligned}$$

where

$$\begin{aligned}&\varphi\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\bigr)\\&\quad=\int_0^1\int_0^1\biggl[\biggl[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\biggr]^{q(1+\frac{1}{p_1})}\biggr]\biggl[\biggl[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\biggr]^{q(1+\frac{1}{p_2})}\biggr]\,\mathrm{d}t\,\mathrm{d}r\\&\quad=\biggl(\biggl[\bigl(b^{p_1}\bigr)^{q(1+\frac{1}{p_1})}{}_2F_1\biggl(q\biggl(1+\frac{1}{p_1}\biggr),1,2,1-\frac{b^{p_1}}{a^{p_1}}\biggr)\biggr]\\&\qquad\times\biggl[\bigl(d^{p_2}\bigr)^{q(1+\frac{1}{p_2})}{}_2F_1\biggl(q\biggl(1+\frac{1}{p_2}\biggr),1,2,1-\frac{d^{p_2}}{c^{p_2}}\biggr)\biggr]\biggr).\end{aligned}$$

V. If we take $\Delta(a,b)=-\mu(t^\sigma(1-t)+t(1-t)^\sigma)(\|\frac{1}{b^{p_1}}-\frac{1}{a^{p_1}}\|)^\sigma$ and $\Delta(c,d)=-\mu(r^\sigma(1-r)+r(1-r)^\sigma)(\|\frac{1}{d^{p_2}}-\frac{1}{c^{p_2}}\|)^\sigma$ for some $\mu>0$ in Theorem 3, then

Corollary 17 *Under the assumptions of Theorem 3, if* $|\frac{\partial^2 f}{\partial r\,\partial t}|^q$ *is a two-dimensional harmonically strong* (p_1,h_1)*-*(p_2,h_2)*-convex function of higher order, where* $\frac{1}{p}+\frac{1}{q}=1$ *and* $q>1$*, then*

$$\begin{aligned}&\bigl|\Xi\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega\bigr)\bigr|\\&\quad\le\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}(p+1)^{\frac{2}{p}}}\biggl[\varphi_1\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr)\biggl|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\biggr|^q\\&\qquad+\varphi_2\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr)\biggl|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\biggr|^q+\varphi_3\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr)\biggl|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\biggr|^q\\&\qquad+\varphi_4\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr)\biggl|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\biggr|^q+\varphi_5^*\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr)\\&\qquad+\varphi_6^*\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega\bigr)\biggr]^{\frac{1}{q}},\end{aligned}$$

where $\varphi_1(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega),\varphi_2(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega),\varphi_3(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega),\varphi_4(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)$ *are given in Theorem 3, and*

$$\begin{aligned}&\varphi_5^*\bigl(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\bigr)\\&\quad=-\mu\biggl(\biggl\|\frac{1}{b^{p_1}}-\frac{1}{a^{p_1}}\biggr\|\biggr)^\sigma\biggl(\int_0^1\int_0^1\biggl[\bigl(t^\sigma(1-t)+t(1-t)^\sigma\bigr)\biggl[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\biggr]^{q(1+\frac{1}{p_1})}\biggr]\end{aligned}$$

$$\times\left[\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right]\mathrm{d}t\,\mathrm{d}r\Big)$$

$$= -\mu\left(\left\|\frac{1}{b^{p_1}}-\frac{1}{a^{p_1}}\right\|\right)^{\sigma}\left(\left[\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{(\sigma+1)(\sigma+2)}{}_2F_1\left(q\left(1+\frac{1}{p_1}\right),\sigma+1,\sigma+3,1-\frac{b^{p_1}}{a^{p_1}}\right)\right.\right.$$

$$\left.+\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{(\sigma+2)(\sigma+1)}{}_2F_1\left(q\left(1+\frac{1}{p_1}\right),2,\sigma+3,1-\frac{b^{p_1}}{a^{p_1}}\right)\right]$$

$$\left.\times\left[(d^{p_2})^{q(1+\frac{1}{p_2})}{}_2F_1\left(q\left(1+\frac{1}{p_2}\right),1,2,1-\frac{d^{p_2}}{c^{p_2}}\right)\right]\right),$$

$$\varphi_6^{*}(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega)$$

$$= -\mu\left(\left\|\frac{1}{d^{p_2}}-\frac{1}{c^{p_2}}\right\|\right)^{\sigma}\left(\int_0^1\int_0^1\left[\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\right]\right.$$

$$\left.\times\left[\left(r^{\sigma}(1-r)+r(1-r)^{\sigma}\right)\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right]\mathrm{d}t\,\mathrm{d}r\right)$$

$$= -\mu\left(\left\|\frac{1}{d^{p_2}}-\frac{1}{c^{p_2}}\right\|\right)^{\sigma}\left(\left[(b^{p_1})^{q(1+\frac{1}{p_1})}{}_2F_1\left(q\left(1+\frac{1}{p_1}\right),1,2,1-\frac{b^{p_1}}{a^{p_1}}\right)\right]\right.$$

$$\times\left[\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{(\sigma+1)(\sigma+2)}{}_2F_1\left(q\left(1+\frac{1}{p_2}\right),\sigma+1,\sigma+3,1-\frac{d^{p_2}}{c^{p_2}}\right)\right.$$

$$\left.\left.+\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{(\sigma+2)(\sigma+1)}{}_2F_1\left(q\left(1+\frac{1}{p_2}\right),2,\sigma+3,1-\frac{d^{p_2}}{c^{p_2}}\right)\right]\right).$$

VI. If we take $\sigma=2$ in Corollary 17, then

Corollary 18 *Under the assumptions of Corollary 17, if* $|\frac{\partial^2 f}{\partial r\,\partial t}|^q$ *is a two-dimensional harmonically strong* (p_1,h_1)-(p_2,h_2)*-convex function, where* $\frac{1}{p}+\frac{1}{q}=1$ *and* $q>1$*, then*

$$\left|\Xi(a^{p_1},b^{p_1},c^{p_2},d^{p_2},x,y:\Omega)\right|$$

$$\leq\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2a^{p_1}b^{p_1}c^{p_2}d^{p_2}(p+1)^{\frac{2}{p}}}\left[\varphi_1(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,c)\right|^q\right.$$

$$+\varphi_2(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,c)\right|^q+\varphi_3(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)\left|\frac{\partial^2 f}{\partial r\,\partial t}(a,d)\right|^q$$

$$+\varphi_4(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)\left|\frac{\partial^2 f}{\partial r\,\partial t}(b,d)\right|^q+\varphi_5^{**}(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)$$

$$\left.+\varphi_6^{**}(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)\right]^{\frac{1}{q}},$$

where $\varphi_1(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega),\varphi_2(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega),\varphi_3(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega),\varphi_4(a^{p_1},b^{p_1},c^{p_2},d^{p_2}:\Omega)$ *are given in Theorem 3, and*

$$\varphi_5^{**}(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega)$$

$$= -\mu\left(\left\|\frac{1}{b^{p_1}}-\frac{1}{a^{p_1}}\right\|\right)^{2}\left(\int_0^1\int_0^1\left[t(1-t)\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\right]\right.$$

$$\times \left[\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right] \mathrm{d}t\,\mathrm{d}r\Bigg)$$

$$= -\mu\left(\left\|\frac{1}{b^{p_1}}-\frac{1}{a^{p_1}}\right\|\right)^2\left(\left[\frac{(b^{p_1})^{q(1+\frac{1}{p_1})}}{6}\,{}_2F_1\left(q\left(1+\frac{1}{p_1}\right),2,4,1-\frac{b^{p_1}}{a^{p_1}}\right)\right.\right.$$

$$\left.\left.\times\left[(d^{p_2})^{q(1+\frac{1}{p_2})}\,{}_2F_1\left(q\left(1+\frac{1}{p_2}\right),1,2,1-\frac{d^{p_2}}{c^{p_2}}\right)\right]\right]\right),$$

$$\varphi_6^*\big(a^{p_1},b^{p_1},c^{p_2},d^{p_2};\Omega\big)$$

$$= -\mu\left(\left\|\frac{1}{d^{p_2}}-\frac{1}{c^{p_2}}\right\|\right)^2\left(\int_0^1\int_0^1\left[\left[\frac{a^{p_1}b^{p_1}}{(tb^{p_1}+(1-t)a^{p_1})}\right]^{q(1+\frac{1}{p_1})}\right]\right.$$

$$\left.\times\left[r(1-r)\left[\frac{c^{p_2}d^{p_2}}{(rd^{p_2}+(1-r)c^{p_2})}\right]^{q(1+\frac{1}{p_2})}\right]\mathrm{d}t\,\mathrm{d}r\right)$$

$$= -\mu\left(\left\|\frac{1}{d^{p_2}}-\frac{1}{c^{p_2}}\right\|\right)^2\left(\left[(b^{p_1})^{q(1+\frac{1}{p_1})}\,{}_2F_1\left(q\left(1+\frac{1}{p_1}\right),1,2,1-\frac{b^{p_1}}{a^{p_1}}\right)\right]\right.$$

$$\left.\times\left[\frac{(d^{p_2})^{q(1+\frac{1}{p_2})}}{6}\,{}_2F_1\left(q\left(1+\frac{1}{p_2}\right),2,4,1-\frac{d^{p_2}}{c^{p_2}}\right)\right]\right]\Bigg).$$

4 Conclusion

In this paper, we defined a new interesting class of functions, two-dimensional approximately harmonic (p_1,h_1)-(p_2,h_2)-convex function, and some Hermite–Hadamard type integral inequalities are provided as well on the coordinates. As a particular cases for p_1,h_1,p_2,h_2, we get several new classes of functions. For instance, $p_1 = 1 = p_2$ and $h_1 = h = h_2$, we get the results of the paper [20] for approximately harmonic h-convex function. These results can be applied in convex analysis, optimization, and different areas of pure and applied sciences. The authors hope that these results will serve as a motivation for future work in this fascinating area.

Acknowledgements
We thank the editor and referees for their careful reading and valuable suggestions to make the article reader-friendly.

Funding
There is no funding available for the publication of this paper.

Availability of data and materials
There is no additional data required for finding the results of this paper.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have equal contribution in this article. All authors read and approved the final manuscript.

Author details
[1] COMSATS University Islamabad, Lahore Campus, 54000, Lahore, Pakistan. [2] Department of Mathematics, Faculty of Technical Science, University Ismail Qemali, Vlora, Albania. [3] Department of Natural Sciences and Humanities, University of Engineering and Technology, Lahore, Pakistan (RCET) 54000, Pakistan. [4] School of Information Science and Technology, Yunnan Normal University, 650500, Kunming, China.

Publisher's Note

Received: 22 March 2020 Accepted: 25 September 2020 Published online: 09 October 2020

References

1. Işcan, İ.: Hermite–Hadamard type inequalities for harmonically convex functions. Hacet. J. Math. Stat. **43**(6), 935–942 (2014)
2. Noor, M.A., Noor, K.I., Awan, M.U., Costache, S.: Some integral inequalities for harmonically *h*-convex functions. UPB Sci. Bull., Ser. A, Appl. Math. Phys. **77**(1), 1–12 (2015)
3. Noor, M.A., Noor, K.I., Iftikhar, S.: Integral inequalities for differentiable p-harmonic convex functions. Filomat **31**(20), 6575–6584 (2017)
4. Noor, M.A., Noor, K.I., Awan, M.U.: Integral inequalities for coordinated harmonically convex functions. Complex Var. Elliptic Equ. **60**, 776–786 (2015)
5. Awan, M.U., Noor, M.A., Mihai, M.V., Noor, K.I., Akhtar, N.: On approximately harmonic *h*-convex functions depending on a given function. Filomat **33**(12), 3783–3793 (2019)
6. Awan, M.U., Noor, M.A., Mihai, M.V., Noor, K.I.: Some fractional extensions of trapezium inequalities via coordinated harmonic convex functions. J. Nonlinear Sci. Appl. **10**, 1714–1730 (2017)
7. Awan, M.U., Noor, M.A., Mihai, M.V., Noor, K.I., Khan, A.G.: Some new bounds for Simpson's rule involving special functions via harmonic *h*-convexity. J. Nonlinear Sci. Appl. **10**, 1755–1766 (2017)
8. Burai, P., Hazy, A.: On approximately *h*-convex functions. J. Convex Anal. **18**(2), 447–454 (2011)
9. Cristescu, G., Noor, M.A., Awan, M.U.: Bounds of the second degree cumulative frontier gaps of functions with generalized convexity. Carpath. J. Math. **31**(2), 173–180 (2015)
10. Cristescu, G., Lupsa, L.: Non-connected Convexities and Applications. Kluwer Academic, Dordrecht (2002)
11. Dragomir, S.S., Pearce, C.E.M.: Selected Topics on Hermite–Hadamard Inequalities and Applications. Victoria University, Australia (2000)
12. Niculescu, C.P., Persson, L.-E.: Convex Functions and Their Applications. A Contemporary Approach, 2nd edn. CMS Books in Mathematics, vol. 23. Springer, New York (2018)
13. Mihai, M.V., Noor, M.A., Noor, K.I., Awan, M.U.: Some integral inequalities for harmonic *h*-convex functions involving hypergeometric functions. Appl. Math. Comput. **252**, 257–262 (2015)
14. Varosanec, S.: On *h*-convexity. J. Math. Anal. Appl. **326**, 303–311 (2007)
15. Zhang, Y., Du, T.S., Wang, H., Shen, Y.-J.: Different types of quantum integral inequalities via (α, m)-convexity. J. Inequal. Appl. **2018** 264 (2018)
16. Liao, J.G., Wu, S.H., Du, T.S.: The Sugeno integral with respect to α-preinvex functions. Fuzzy Sets Syst. **379**, 102–114 (2020)
17. Du, T.S., Awan, M.U., Kashuri, A., Zhao, S.S.: Some k-fractional extensions of the trapezium inequalities through generalized relative semi-(m; h)-preinvexity. Appl. Anal. **2019**, 1–21 (2019). https://doi.org/10.1080/00036811.2019.1616083
18. Du, T.S., Wang, H., Khan, M.A., Zhang, Y.: Certain integral inequalities considering generalized m-convexity on fractal sets and their applications. Fractals **27**(7), 1950117 (2019).
19. Set, E.: New inequalities of Ostrowski type for mappings whose derivatives are *s*-convex in the second sense via fractional integrals. Comput. Math. Appl. **63**, 1147–1154 (2012)
20. Akhtar, N., Awan, M.U., Kashuri, A., Mihai, M.V., Noor, M.A., Noor, K.I.: Approximately two dimensional harmonic *h*-convex functions and related integral inequalities. (Submitted)