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Isoperimetric inequalities of the fourth order Neumann eigenvalues

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Abstract
In this paper, we obtain some isoperimetric inequalities for the first $(n-1)$ eigenvalues of the fourth order Neumann Laplacian on bounded domains in an n-dimensional Euclidean space. Our result supports strongly the conjecture of Chasman.

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Keywords: Eigenvalues; Neumann problem; Isoperimetric inequality

1 Introduction

Letting Ω be a bounded domain with a smooth boundary $\partial\Omega$ in the Euclidean space $\mathbb{R}^n$, we consider the Neumann problem of the Laplacian Δ as follows:

$$\begin{cases} \Delta u = \mu u, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν is the outward unit normal to the boundary. It is well known that the free membrane problem (1.1) has a discrete spectrum consisting of a sequence

$$0 = \mu_0 < \mu_1 \le \mu_2 \le \cdots \to +\infty.$$

When Ω is a bounded domain in $\mathbb{R}^2$, Szegö [6] proved the following classical isoperimetric inequality:

$$\mu_1(\Omega) \le \mu_1(B_\Omega), \tag{1.2}$$

where B_Ω is the ball of same volume as Ω. Weinberger [11] generalized this result to n-dimensions. Ashbaugh and Benguria [2] extended the Szegö–Weinberger inequality (1.2) to the bounded domains in hyperbolic space and a hemisphere. On the other hand, Ashbaugh and Benguria [1] conjectured that

$$\sum_{i=1}^{n} \frac{1}{\mu_i(\Omega)} \ge \frac{n}{\mu_1(B_\Omega)}, \quad \text{with equality if and only if } \Omega \text{ is a ball}, \tag{1.3}$$

where $\mu_i(\Omega)$ is the ith Neumann eigenvalue on Ω, $\mu_1(B_\Omega)$ is the first nonzero Neumann eigenvalue on B_Ω. In [10], Wang and Xia proved an isoperimetric inequality for the sums of the reciprocals of the first $(n-1)$ nonzero eigenvalues of the Neumann Laplacian on bounded domains in $\mathbb{R}^n$ as follows:

$$\sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \geq \frac{n-1}{\mu_1(B_\Omega)}, \quad \text{with equality if and only if } \Omega \text{ is a ball,} \tag{1.4}$$

which means the Ashbaugh–Benguria's conjecture is true for the first $(n-1)$ nonzero eigenvalues of the Neumann Laplacian on bounded domains in $\mathbb{R}^n$. So (1.4) supports the above conjectures of Ashbaugh and Benguria. On the other hand, Benguria, et al. [3] proved a result which is similar to (1.4) for the first $(n-1)$ nontrivial Neumann eigenvalues on domains in a hemisphere of S^n. Moreover, some works on eigenvalues are related to the spectra of matrix operators and can be seen in [7–9].

Let Δ and $\overline{\Delta}$ be the Laplace–Beltrami operators on Ω and $\partial\Omega$, respectively. Let ∇ and $\overline{\nabla}$ be the gradient operators on Ω and $\partial\Omega$, respectively. Consider the following Neumann eigenvalue problem of the bi-harmonic operator:

$$\begin{cases} \Delta^2 u - \tau\Delta u = \Lambda u & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \tau\frac{\partial u}{\partial \nu} - \operatorname{div}_{\partial\Omega}(\nabla^2 u(\nu)) - \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\tau \geq 0$ and σ are two constants, $\operatorname{div}_{\partial\Omega}$ denotes the tangential divergence operator on $\partial\Omega$, and $\nabla^2 u$ is the Hessian of u, ν is the outward unit normal to the boundary. In this setting, problem (1.5) has a discrete spectrum, and all eigenvalues in the discrete spectrum can be listed nondecreasingly as follows:

$$0 = \Lambda_0 < \Lambda_1 \leq \Lambda_2 \leq \cdots \uparrow +\infty.$$

By the Rayleigh–Ritz characterization, the $(k+1)$th eigenvalue of (1.5) can be given as follows (see, e.g., [5]):

$$\Lambda_{k+1} = \inf_{u\in H^2(\Omega)} \left\{ Q[u] = \frac{\int_\Omega [|\nabla^2 u|^2 + \tau|\nabla u|^2]\,dx}{\int_\Omega u^2\,dx} \,\middle|\, \int_\Omega u u_j = 0, j = 1, \ldots, k \right\}. \tag{1.6}$$

Letting B_Ω be the ball of same volume as Ω, Chasman [5] proved the following isoperimetric inequality:

$$\Lambda_1(\Omega) \leq \Lambda_1(B_\Omega), \quad \text{with equality if and only if } \Omega \text{ is a ball.}$$

Chasman [5] also conjectured that

$$\sum_{i=1}^{n} \frac{1}{\Lambda_i(\Omega)} \geq \frac{n}{\Lambda_1(B_\Omega)}, \quad \text{with equality if and only if } \Omega \text{ is a ball.} \tag{1.7}$$

In this paper, we prove an isoperimetric inequality for the sums of the reciprocals of the first $(n-1)$ nonzero eigenvalues of the fourth Neumann Laplacian which supports the

Chasman's conjecture, actually, we get

$$\sum_{i=1}^{n-1}\frac{1}{\Lambda_i(\Omega)}\geq\frac{n-1}{\Lambda_1(B_\Omega)},\quad \text{with equality if and only if } \Omega \text{ is a ball.} \tag{1.8}$$

In [4], Buoso et al. proved a quantitative isoperimetric inequality for the fundamental tone of problem (1.5) as follows:

$$\Lambda_1(\Omega)\leq\bigl(1-\eta_{n,\tau,|\Omega|}A^2(\Omega)\bigr)\Lambda_1(B_\Omega), \tag{1.9}$$

where $\eta_{n,\tau,|\Omega|}>0$, and $A(\Omega)$ is the so-called Fraenkel asymmetry of the domain $\Omega\in\mathbb{R}^n$, which is defined by:

$$A(\Omega):=\inf\left\{\frac{|\Omega\Delta B_\Omega|}{|\Omega|}\right\},$$

where B_Ω is the ball of same volume as Ω and $\Omega\Delta B_\Omega$ is the symmetric difference of Ω and B_Ω. In what follows, we generalize (1.9) to the sum of the first $(n-1)$ eigenvalues, and we get

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\Lambda_i(\Omega)\leq\bigl(1-\eta_{n,\tau,|\Omega|}A^2(\Omega)\bigr)\Lambda_1(B_\Omega). \tag{1.10}$$

2 Preliminaries

In this section, we recall some notations and results, more details can be seen in [4, 5].

Let j_1, i_1 be the ultraspherical and modified ultraspherical Bessel functions of the first kind and order 1, respectively; j_1, i_1 can be expressed by the standard Bessel and modified Bessel functions of the first kind J_ν, I_ν as follows:

$$j_1(z)=z^{1-n/2}J_{n/2}(z),\qquad i_1(z)=z^{1-n/2}I_{n/2}(z).$$

Let B be the unit ball in $\mathbb{R}^n$ centered at the origin and ω_n be the Lebesgue measure $|B|$ of B, and let $\Lambda_1(B)$ be the first eigenvalue of problem (1.5) on unit ball B. For $\tau>0$, a, b are positive constants satisfying the conditions $a^2b^2=\lambda_1(B)$ and $b^2-a^2=\tau$. Set

$$R(r)=j_1(ar)+\gamma i_1(br),\quad \gamma=\frac{-a^2j_1''(a)}{b^2i_1''(b)}>0.$$

Then we define the function $\rho:[0,+\infty)\to[0,+\infty)$ as

$$\rho(r)=\begin{cases}R(r), & r\in[0,1),\\ R(1)+(r-1)R'(1), & r\in[1,+\infty).\end{cases}$$

Let $u_i:\mathbb{R}^n\to\mathbb{R}$ be defined by

$$u_i(x):=\rho\bigl(|x|\bigr)\frac{x_i}{|x|},\quad \text{for } i=1,\dots,n. \tag{2.1}$$

The functions $u_i|_B$ are, in fact, the eigenfunctions associated with the eigenvalues $\lambda_1(B)$ of problem (1.5) on unit ball B. We know that $\lambda_1(B)$ has multiplicity and u_i satisfy

$$\sum_{i=1}^{n} |u_i|^2 = \rho^2\bigl(|x|\bigr), \tag{2.2}$$

$$\sum_{i=1}^{n} |\nabla u_i|^2 = \frac{n-1}{|x|^2}\rho\bigl(|x|\bigr)^2 + \bigl(\rho'\bigl(|x|\bigr)\bigr)^2, \tag{2.3}$$

$$\sum_{i=1}^{n} \bigl|\nabla^2 u_i\bigr|^2 = \bigl(\rho''\bigl(|x|\bigr)\bigr)^2 + \frac{3(n-1)}{|x|^4}\bigl[\rho\bigl(|x|\bigr) - |x|\rho'\bigl(|x|\bigr)\bigr]^2. \tag{2.4}$$

Define $N[\rho] = \sum_{i=1}^{n}(|\nabla^2 u_i|^2 + \tau|\nabla u_i|^2)$. Then ρ and $N[\rho]$ satisfy the following properties which given in [4, 5].

Lemma 2.1 *Function ρ and $N[\rho]$ satisfy the following properties*:

(1) $\rho''(r) < 0$ *for all* $r \geq 0$, *therefore* ρ' *is nonincreasing.*
(2) $\rho(r) - r\rho'(r) \geq 0$, *with equality holding only for* $r = 0$.
(3) *The function* $\rho^2(r)$ *is strictly increasing.*
(4) *The function* $\rho^2(r)/r^2$ *is decreasing.*
(5) *The function* $3(\rho(r) - r\rho'(r))^2/r^4 + \tau\rho^2(r)/r^2$ *is decreasing.*
(6) $N[\rho(r_1)] > N[\rho(r_2)]$ *for any* $r_1 \in [0,1)$, $r_2 \in [1,+\infty)$.
(7) *For all* $r \geq 0$, *we have*

$$N\bigl[\rho(r)\bigr] = \bigl(\rho''(r)\bigr)^2 + \frac{3(n-1)(\rho(r) - r\rho'(r))^2}{r^4} + \tau(n-1)\frac{\rho^2(r)}{r^2} + \tau\bigl(\rho'(r)\bigr)^2.$$

(8) *For all* $r \geq 1$, $N[\rho(r)]$ *is decreasing.*

We introduce the notation of a partially monotonic function. A function F is *partially monotonic* on Ω if it satisfies

$$F(x) > F(y), \quad \text{for all } x \in \Omega \text{ and } y \notin \Omega. \tag{2.5}$$

It is seen that $N[\rho(r)]$ is a partially monotonic function from Lemma 2.1.

Lemma 2.2 *For any radial function $F(r(x))$ that satisfies the partially monotonicity condition on B_Ω,*

$$\int_\Omega F\,dx \leq \int_{B_\Omega} F\,dx \tag{2.6}$$

with equality if and only if $\Omega = B_\Omega$. For any strictly increasing radial function $F(r(x))$,

$$\int_\Omega F\,dx \geq \int_{B_\Omega} F\,dx \tag{2.7}$$

with equality if and only if $\Omega = B_\Omega$.

Lemma 2.3 *For all $s > 0$, we have*

$$\Lambda_i(\tau,\Omega) = s^4 \Lambda_i\big(s^{-2}\tau, s\Omega\big), \quad i = 1,\dots,n, \tag{2.8}$$

where $s\Omega = \{x \in \mathbb{R}^n : x/s \in \Omega\}$ for $s > 0$.

Proof For any $u \in H^2(\Omega)$ with

$$u \neq 0 \quad \text{and} \quad \int_\Omega u\,dx = \int_\Omega uu_1\,dx = \cdots = \int_\Omega uu_{i-1}\,dx = 0, \quad i = 1,\dots,n,$$

let $\tilde{u}(x) = u(x/s)$, then $\tilde{u}$ is a valid trial function on $s\Omega$ and so

$$\begin{aligned} Q_{s^{-2}\tau,s\Omega}[\tilde{u}] &= \frac{\int_{s\Omega}(|\nabla^2\tilde{u}|^2 + s^{-2}\tau|\nabla\tilde{u}|^2)\,dx}{\int_{s\Omega} u^2\,dx} \\ &= \frac{\int_{s\Omega}(|s^{-2}(\nabla^2 u)(x/s)|^2 + s^{-2}\tau|s^{-1}(\nabla u)(x/s)|^2)\,dx}{\int_{s\Omega} u(x/s)^2\,dx} \\ &= \frac{s^{-4+n}\int_\Omega(|(\nabla^2 u)|^2 + \tau|(\nabla u)|^2)\,dy}{s^n\int_\Omega u^2\,dy} \quad (\text{substituting } y = x/s) \\ &= s^{-4} Q_{\tau,\Omega}[u]. \end{aligned} \tag{2.9}$$

The lemma follows from (1.6). □

3 Proofs of the main results

In this section, we give the proofs of the main results of this paper.

Theorem 3.1 *Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$ and let B_Ω be the ball of same volume as Ω, then the first $(n-1)$ eigenvalues of (1.5) in Ω satisfy*

$$\sum_{i=1}^{n-1} \frac{1}{\Lambda_i(\Omega)} \geq \frac{n-1}{\Lambda_1(B_\Omega)}, \tag{3.1}$$

with equality if and only if Ω is a ball.

Proof Assume that the volume of Ω is equal to that of the unit ball B. Letting $\varphi_i = \frac{\rho(r)x_i}{r}$, we know that

$$\int_\Omega \varphi_i(r)\,dx = 0, \quad \text{for } i = 1,\dots,n,$$

which means φ_i is perpendicular to $u_0 = 1/\sqrt{|\Omega|}$, which is the first eigenfunction of (1.5). Letting $\{u_j\}_{j=0}^\infty$ be an orthonormal set of eigenfunctions of (1.5) on Ω, next we will show that there exists new coordinate functions $\{x_i'\}_{i=1}^n$ such that

$$\int_\Omega \frac{\rho(r)x_i'}{r} u_j\,dx = 0, \tag{3.2}$$

for $j = 1,\dots,i-1$ and $i = 2,\dots,n$. To see this, we define an $n \times n$ matrix $A = (a_{ij})$, where $a_{ji} = \int_\Omega \varphi_i u_j\,dx = \int_\Omega \frac{\rho(r)}{r} x_i u_j\,dx$, for $i,j = 1,2,\dots,n$. Using the orthogonalization of Gram and Schmidt (QR-factorization theorem), we know that there exist an upper-triangular matrix $T = (T_{ji})$ and an orthogonal matrix $B = (b_{ji})$ such that $T = UQ$, i.e.,

$$T_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj} = \int_\Omega \sum_{k=1}^{n} \frac{\rho(r)}{r} b_{ik}x_k u_j\,dx = 0, \quad 1 \le j < i \le n.$$

Letting $x_i' = \sum_{k=1}^n b_{ik}x_k$, $i = 1,\dots,n$, we get (3.2). Since $B = (b_{ji})$ is an orthogonal matrix, $\{x_i'\}_{i=1}^n$ is also a set of coordinate functions. Therefore, denoting x_i', $i = 1\dots,n$ still by x_i, $i = 1\dots,n$, and $\varphi_i = \frac{\rho(r)}{r}x_i$, we have

$$\varphi_i \neq 0 \quad \text{and} \quad \int_\Omega \varphi_i\,dx = \int_\Omega \varphi_i u_1\,dx = \cdots = \int_\Omega \varphi_i u_{i-1}\,dx = 0, \quad i = 1,\dots,n.$$

It follows from the Rayleigh–Ritz inequality that

$$\Lambda_i(\Omega)\int_\Omega \varphi_i^2\,dx \le \int_\Omega \left(\left|\nabla^2\varphi_i\right|^2 + \tau|\nabla\varphi_i|^2\right)dx, \quad i = 1,\dots,n, \tag{3.3}$$

which implies that

$$\int_\Omega \varphi_i^2\,dx \le \frac{1}{\Lambda_i(\Omega)}\int_\Omega \left(\left|\nabla^2\varphi_i\right|^2 + \tau|\nabla\varphi_i|^2\right)dx, \quad i = 1,\dots,n. \tag{3.4}$$

Summing over i from 1 to n, we have

$$\sum_{i=1}^{n}\int_\Omega \varphi_i^2\,dx \le \sum_{i=1}^{n}\frac{1}{\Lambda_i(\Omega)}\int_\Omega \left(\left|\nabla^2\varphi_i\right|^2 + \tau|\nabla\varphi_i|^2\right)dx. \tag{3.5}$$

Since $\sum_{i=1}^n |\nabla^2\varphi_i|^2 = (\rho'')^2 + \frac{3(n-1)}{r^4}(\rho - r\rho')^2$, for any point $p \in \Omega$, by a transformation of coordinates if necessary, we have $|\nabla^2\varphi_i|^2 \le \frac{(\rho'')^2}{n-1} + \frac{3}{r^4}(\rho - r\rho')^2$, $i = 1,\dots,n$. Then

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{1}{\Lambda_i(\Omega)}\left|\nabla^2\varphi_i\right|^2 \\
&= \sum_{i=1}^{n-1}\frac{1}{\Lambda_i(\Omega)}\left|\nabla^2\varphi_i\right|^2 + \frac{1}{\Lambda_n(\Omega)}\left|\nabla^2\varphi_i\right|^2 \\
&= \sum_{i=1}^{n-1}\frac{1}{\Lambda_i(\Omega)}\left|\nabla^2\varphi_i\right|^2 + \frac{1}{\Lambda_n(\Omega)}\left(\left(\rho''\right)^2 + \frac{3(n-1)}{r^4}\left(\rho - r\rho'\right)^2 - \sum_{j=1}^{n-1}\left|\nabla^2\varphi_j\right|^2\right) \\
&\le \sum_{i=1}^{n-1}\frac{1}{\Lambda_i(\Omega)}\left|\nabla^2\varphi_i\right|^2 + \sum_{j=1}^{n-1}\frac{1}{\Lambda_j(\Omega)}\left(\frac{(\rho'')^2 + \frac{3(n-1)}{r^4}(\rho - r\rho')^2}{n-1} - \left|\nabla^2\varphi_j\right|^2\right) \\
&= \frac{1}{n-1}\left(\left(\rho''\right)^2 + \frac{3(n-1)}{r^4}\left(\rho - r\rho'\right)^2\right)\sum_{i=1}^{n-1}\frac{1}{\Lambda_i(\Omega)}.
\end{aligned} \tag{3.6}$$

Similarly, we have

$$\begin{aligned}\sum_{i=1}^{n}\frac{1}{\Lambda_i(\Omega)}|\nabla\varphi_i|^2 &= \sum_{i=1}^{n-1}\frac{1}{\Lambda_i(\Omega)}|\nabla\varphi_i|^2+\frac{1}{\Lambda_n(\Omega)}|\nabla\varphi_i|^2\\ &= \sum_{i=1}^{n-1}\frac{1}{\Lambda_i(\Omega)}|\nabla\varphi_i|^2+\frac{1}{\Lambda_n(\Omega)}\left(\frac{n-1}{r^2}\rho^2+\left(\rho'\right)^2-\sum_{j=1}^{n-1}|\nabla\varphi_j|^2\right)\\ &\le \sum_{i=1}^{n-1}\frac{1}{\Lambda_i(\Omega)}|\nabla\varphi_i|^2+\sum_{j=1}^{n-1}\frac{1}{\Lambda_j(\Omega)}\left(\frac{\frac{n-1}{r^2}\rho^2+(\rho')^2}{n-1}-|\nabla\varphi_j|^2\right)\\ &= \frac{1}{n-1}\left(\frac{n-1}{r^2}\rho^2+\left(\rho'\right)^2\right)\sum_{i=1}^{n-1}\frac{1}{\Lambda_i(\Omega)}.\end{aligned}\tag{3.7}$$

On the other hand,

$$\sum_{i=1}^{n}|\varphi_i|^2=\rho^2.\tag{3.8}$$

Substituting (3.6)–(3.8) into (3.5), we have

$$\begin{aligned}\frac{1}{n-1}\sum_{i=1}^{n-1}\frac{1}{\Lambda_i(\Omega)} &\ge \frac{\int_\Omega \rho^2\,dx}{\int_\Omega((\rho'')^2+\frac{3(n-1)}{r^4}(\rho-r\rho')^2+\tau(\frac{n-1}{r^2}\rho^2+(\rho')^2))\,dx}\\ &= \frac{\int_\Omega \rho^2\,dx}{\int_\Omega N[\rho]\,dx}\ge\frac{\int_{B_\Omega}\rho^2\,dx}{\int_{B_\Omega}N[\rho]\,dx}=\frac{1}{\Lambda_1(B_\Omega)},\end{aligned}\tag{3.9}$$

the last step is deduced by Lemma 2.2. If the equality holds, then equality holds in (3.9), which implies Ω must be a unit ball. By Lemma 2.3, for any domain Ω in $\mathbb{R}^n$, we get

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\frac{1}{\Lambda_i(\Omega)}\ge\frac{1}{\Lambda_1(B_\Omega)}.\tag{3.10}$$

This completes the proof of Theorem 3.1. □

Theorem 3.2 *Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$ and let B_Ω be the ball of same volume as Ω, then the first $(n-1)$ eigenvalues of (1.5) in Ω satisfy*

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\Lambda_i(\Omega)\le\left(1-\eta_{n,\tau,|\Omega|}\mathcal{A}^2(\Omega)\right)\Lambda_1(B_\Omega).\tag{3.11}$$

Proof Case 1. Ω is a bounded domain in $\mathbb{R}^n$ of class C^1 with the same measure as the unit ball B. By a similar argument as in the proof of Theorem 3.1, we have

$$\Lambda_i(\Omega)\int_\Omega\varphi_i^2\,dx\le\int_\Omega\left(\left|\nabla^2\varphi_i\right|^2+\tau|\nabla\varphi_i|^2\right)dx,\quad i=1,\dots,n.\tag{3.12}$$

Summing over i from 1 to n, we have

$$\sum_{i=1}^{n} \Lambda_i(\Omega) \int_\Omega \varphi_i^2\,dx \le \sum_{i=1}^{n} \int_\Omega \left(\left|\nabla^2 \varphi_i\right|^2 + \tau |\nabla \varphi_i|^2 \right) dx = \int_\Omega N[\rho]\,dx. \tag{3.13}$$

Since $\sum_{i=1}^{n} \varphi_i^2 = \rho^2$, for any point $p \in \Omega$, by a transformation of coordinates if necessary, we have $\varphi_i^2 \le \frac{\rho^2}{n-1}$, $i = 1, \ldots, n$. Then

$$\begin{aligned} \sum_{i=1}^{n} \Lambda_i(\Omega)\varphi_i^2 &= \sum_{i=1}^{n-1} \Lambda_i(\Omega)\varphi_i^2 + \Lambda_n(\Omega)\varphi_n^2 \\ &= \sum_{i=1}^{n-1} \Lambda_i(\Omega)\varphi_i^2 + \Lambda_n(\Omega)\left(\rho^2 - \sum_{j=1}^{n-1} \varphi_j^2 \right) \\ &\ge \sum_{i=1}^{n-1} \Lambda_i(\Omega)\varphi_i^2 + \sum_{j=1}^{n-1} \Lambda_j \left(\frac{\rho^2}{n-1} - \varphi_j^2 \right) \\ &= \sum_{i=1}^{n-1} \Lambda_i \frac{\rho^2}{n-1}. \end{aligned} \tag{3.14}$$

Substituting (3.13) into (3.14), we have

$$\frac{1}{n-1} \sum_{i=1}^{n-1} \Lambda_i(\Omega) \le \frac{\int_\Omega N[\rho]\,dx}{\int_\Omega \rho^2\,dx}. \tag{3.15}$$

On the other hand, we have

$$\Lambda_1(B) = \frac{\int_B N[\rho]\,dx}{\int_B \rho^2\,dx}. \tag{3.16}$$

Combining (3.15) and (3.16), we have

$$\Lambda_1(B) \int_B \rho^2\,dx - \frac{1}{n-1} \sum_{i=1}^{n-1} \Lambda_i(\Omega) \int_\Omega \rho^2\,dx \ge \int_B N[\rho]\,dx - \int_\Omega N[\rho]\,dx. \tag{3.17}$$

From equation (16) in [4], we know that

$$\Lambda_1(B) \int_B \rho^2\,dx - \Lambda_1(\Omega) \int_\Omega \rho^2\,dx \le C_{n,\tau}^{(1)} \left(\Lambda_1(B) - \Lambda_1(\Omega) \right),$$

where $C_{n,\tau}^{(1)} = n\omega_n \int_0^1 \rho^2(r) r^{n-1}\,dr$. Then we have

$$\Lambda_1(B) \int_B \rho^2\,dx - \frac{1}{n-1} \sum_{i=1}^{n-1} \Lambda_i(\Omega) \int_\Omega \rho^2\,dx \le C_{n,\tau}^{(1)} \left(\Lambda_1(B) - \frac{1}{n-1} \sum_{i=1}^{n-1} \Lambda_i(\Omega) \right). \tag{3.18}$$

From (15) and (20) in [4], we know that

$$\Lambda_1(B) \int_B \rho^2\,dx - \Lambda_1(\Omega) \int_\Omega \rho^2\,dx \ge \int_{B/B_1} N(\rho)\,dx - \int_{B_2/B} N(\rho)\,dx,$$

and

$$\int_{B/B_1} N(\rho)\,dx - \int_{B_2/B} N(\rho)\,dx = C^{(2)}_{n,\tau}\alpha^2,$$

where B_1 and B_2 are two balls centered at the origin with radii r_1, r_2 such that $|\Omega \cap B| = |B_1| = \omega_n r_1^n$ and $|\Omega/B| = |B_2/B| = \omega_n(r_2^n - 1)$. Then we have

$$\int_B N[\rho]\,dx - \int_\Omega N[\rho]\,dx \geq C^{(2)}_{n,\tau}\frac{|\Omega \Delta B|}{|\Omega|}, \tag{3.19}$$

where $C^{(2)}_{n,\tau} = n\omega_n((3+\tau)(R(1)-R'(1))^2 + 2\tau R'(1)(R(1)-R'(1)))c_n$.

Combining (3.18) and (3.19), we have

$$\Lambda_1(B) - \frac{1}{n-1}\sum_{i=1}^{n-1}\Lambda_i(\Omega) \geq \frac{C^{(2)}_{n,\tau}}{C^{(1)}_{n,\tau}}A^2(\Omega),$$

which implies that

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\Lambda_i(\Omega) \leq \Lambda_1(B)\left(1 - \frac{C^{(2)}_{n,\tau}}{\Lambda_1(B)C^{(1)}_{n,\tau}}A^2(\Omega)\right). \tag{3.20}$$

Case 2. Ω is the generic domain in $\mathbb{R}^n$ of class C^1. Since

$$\Lambda_i(\tau,\Omega) = s^4\Lambda_i\bigl(s^{-2}\tau, s\Omega\bigr), \quad i = 1,\dots,n, \tag{3.21}$$

for all $s > 0$. Taking $s = (\omega_n/|\Omega|)^{\frac{1}{n}}$, for any domain Ω in $\mathbb{R}^n$ of class C^1, we infer from (3.21) that

$$\begin{aligned}\frac{1}{n-1}\sum_{i=1}^{n-1}\Lambda_i(\tau,\Omega) &= s^4\frac{1}{n-1}\sum_{i=1}^{n-1}\Lambda_i\bigl(s^{-2}\tau, s\Omega\bigr)\\ &\leq s^4\Lambda_1\bigl(s^{-2}\tau, B\bigr)\left(1 - \frac{C^{(2)}_{n,s^{-2}\tau}}{\Lambda_1(s^{-2}\tau, B)C^{(1)}_{n,s^{-2}\tau}}A^2(s\Omega)\right)\\ &= \Lambda_1\bigl(s^{-2}\tau, B\bigr)\left(1 - \frac{C^{(2)}_{n,s^{-2}\tau}}{\Lambda_1(s^{-2}\tau, B)C^{(1)}_{n,s^{-2}\tau}}A^2(\Omega)\right).\end{aligned}$$

Setting $\eta_{n,\tau,|\Omega|} = \frac{C^{(2)}_{n,s^{-2}\tau}}{\Lambda_1(s^{-2}\tau,B)C^{(1)}_{n,s^{-2}\tau}}$, we have (1.10). This completes the proof of Theorem 3.2. □

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References

1. Ashbaugh, M., Benguria, R.: Universal bounds for the low eigenvalues of Neumann Laplacians in N dimensions. SIAM J. Math. Anal. **24**, 557–570 (1993)
2. Ashbaugh, M., Benguria, R.: Sharp upper bound to the first nonzero Neumann eigenvalue for bounded domains in spaces of constant curvature. J. Lond. Math. Soc. **52**(2), 402–416 (1995)
3. Benguria, R., Brandolini, B., Chiacchio, F.: A sharp estimate for Neumann eigenvalues of the Laplace–Beltrami operator for domains in a hemisphere. arXiv preprint (2018). arXiv:1809.05695v1
4. Buoso, D., Chasman, L.M., Provenzano, L.: On the stability of some isoperimetric inequalities for the fundamental tones of free plates. J. Spectr. Theory **8**(3), 843–869 (2018)
5. Chasman, L.M.: An isoperimetric inequality for fundamental tones of free plates. Commun. Math. Phys. **303**(2), 421–449 (2011)
6. Szegö, G.: Inequalities for certain eigenvalues of a membrane of given area. J. Ration. Mech. Anal. **3**, 343–356 (1954)
7. Tripathy, B.C., Paul, A.: The spectrum of the operator $D(r,0,s,0,t)$ over the sequence spaces c_0 and c. J. Math. **2013**, Article ID 430965 (2013)
8. Tripathy, B.C., Paul, A.: The spectrum of the operator $D(r,0,0,s)$ over the sequence space c_0 and c. Kyungpook Math. J. **53**(2), 247–256 (2013)
9. Tripathy, B.C., Saikia, P.: On the spectrum of the Cesàro operator C_1 on $\overline{bv} \cap 1_\infty$. Math. Slovaca **63**(3), 563–572 (2013)
10. Wang, Q., Xia, C.: On a conjecture of Ashbaugh and Benguria about lower eigenvalues of the Neumann Laplacian. arXiv preprint (2018). arXiv:1808.09520v3
11. Weinberger, H.F.: An isoperimetric inequality for the N-dimensional free membrane problem. J. Ration. Mech. Anal. **5**, 633–636 (1956)