

RESEARCH

Open Access



Extragradient-like method for pseudomonotone equilibrium problems on Hadamard manifolds

Junfeng Chen^{1,2*}  and Sanyang Liu¹

*Correspondence:

mailjunfeng@163.com

¹School of Mathematics and Statistics, Xidian University, 710126, Xi'an, P.R. China

²School of Mathematics and Information Science, Xianyang Normal University, 712000 Xianyang, P.R. China

Abstract

This paper presents an extragradient-like method for solving a pseudomonotone equilibrium problem with a Lipschitz-type condition on Hadamard manifolds. The algorithm only needs to know the existence of the Lipschitz-type constants of the bifunction, and the stepsize of each iteration is determined by the adjacent iterations. Convergence of the algorithm is analyzed, and its application to variational inequalities is also provided. Finally, several experiments are made to verify the effectiveness of the algorithms.

MSC: 32Q05; 47J20; 90C25; 90C33; 90C52

Keywords: Equilibrium problem; Pseudomonotone bifunction; Extragradient method; Variational inequality; Hadamard manifold

1 Introduction

The equilibrium problem (EP) provides a general setting of many problems, such as optimization problem and the complementarity problem. In the past few decades, it has been studied extensively in linear space (e.g., Hadjisavvas et al. [1], Bianchi et al. [2, 3], Blum et al. [4]).

It is necessary to extend the concept and method from linear space to Riemannian manifolds. By choosing a suitable Riemannian metric, the nonconvex optimization problem can be transformed into convex optimization problem, and the constrained optimization problem can be transformed into an unconstrained one. Some classical algorithms have been extended from linear space to Riemannian manifolds, such as by Ferreira et al. [5, 6], Li et al. [7], and Tang et al. [8]. The related work on Hadamard manifolds can be found in Kristály [9], Li et al. [10, 11], Ceng et al. [12], Zhou et al. [13] and so on.

In 2012, Colao et al. [14] studied the equilibrium problem on Hadamard manifolds. Let E be a nonempty closed convex subset on Hadamard manifolds \mathbb{M} , and $S : E \times E \rightarrow \mathbb{R}$ be a bifunction that satisfies $S(x, x) = 0, \forall x \in E$, then the form of equilibrium problem is to find $x \in E$, such that

$$S(x, y) \geq 0, \quad \forall y \in E. \quad (\text{EP})$$

© The Author(s) 2020. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

We define the solution set of equilibrium problem (EP) as $EP(S; E)$, and we always assume that $EP(S; E) \neq \emptyset$. In the special case, if $S(x, y) = \langle V(x), \exp_x^{-1} y \rangle$, where $V : E \rightarrow TM$ is a vector field satisfies $V(x) \in T_x M, \forall x \in E$, and \exp^{-1} is the inverse of exponential, then (EP) becomes the variational inequality problem. The form is: find $x \in E$, such that

$$\langle V(x), \exp_x^{-1} y \rangle \geq 0, \quad \forall y \in E. \tag{VI}$$

The solution set of variational inequality problem (VI) is denoted by $VI(V, E)$.

It is known that the KKM lemma is an important tool for studying the existence of solutions for equilibrium problems. Colao et al. [14] developed and proved Fan’s KKM lemma [15] and obtained the existence of solutions to equilibrium problem (EP) on Hadamard manifolds. For relevant conclusions, see, for instance, Yang and Pu [16], Tang et al. [17], Chen et al. [18], Batista et al. [19], Zhou et al. [20–22].

Furthermore, the existence of solutions for equilibrium problems or variational inequality problems on Riemannian manifolds has been presented by several references. In particular, Li et al. [23] established the existence and uniqueness results for variational inequality problems on Riemannian manifolds. Meanwhile Li and Yao [24] provided the existence theorems of solutions for variational inequalities for set-valued mappings on Riemannian manifolds. Very recently, Wang et al. [25] obtained the existence of solutions and the convexity properties of the solution set for the equilibrium problem on Riemannian manifolds.

Many authors have studied ideas and methods for solving equilibrium problems or variational inequality problems in linear space, for example, Korpelevich [26] first designed an extragradient method for a solution of variational inequality problem, while Censor et al. [27] proposed the subgradient extragradient method inspired by extragradient method in [26]. In 2019, Thong and Hieu [28] introduced an inertial subgradient extragradient algorithm based on the subgradient extragradient method in [27]. Then Ceng et al. [29] and Yao et al. [30] obtained the inertial algorithms for finding a common solution of the variational inequality problem and the fixed-point problem by using a subgradient approach. As for equilibrium problems, Quoc et al. [31] obtained an extragradient method for a solution of a pseudomonotone equilibrium problem, while Nguyen et al. [32] provided an iterative method for finding a common solution to an equilibrium problem and a fixed point problem based on extragradient method in [31]. Then in 2020, Yao et al. [33] improved and extended the main result in [32] to a general case.

In recent years, algorithms for solving equilibrium problem (EP) on Hadamard manifolds have received a lot of attention by some authors, such as Colao et al. [14], Salahuddin [34], and Li et al. [35]. Recently, Cruz Neto et al. [36] extended the result of Nguyen et al. [32] and obtained an extragradient method for solving the equilibrium problem on Hadamard manifolds, which is described as follows: choose $\lambda_k > 0$, compute

$$\begin{cases} y^k = \arg \min_{z \in E} \{S(x^k, z) + \frac{1}{2\lambda_k} d^2(x^k, z)\}, \\ x^{k+1} = \arg \min_{z \in E} \{S(y^k, z) + \frac{1}{2\lambda_k} d^2(x^k, z)\}, \end{cases} \tag{1}$$

where $0 < \lambda_k < \beta < \min\{\alpha_1^{-1}, \alpha_2^{-1}\}$, α_1, α_2 are constants related to Lipschitz-type constants. It should be noted that Lipschitz-type constants are unknown in general, and it is difficult to approximate them even in complex non-linear problems.

Recently, Hieu et al. [37, 38] and Yang et al. [39, 40] introduced some proximal-like algorithms in the linear setting. The stepsize of the algorithms is given by the adjacent iteration information in each iteration, so it is unnecessary to know the Lipschitz constants.

Inspired by the work above, we present a new extragradient-like method for (EP) on Hadamard manifolds. Compared with [36], our algorithm is performed without the prior knowledge of the Lipschitz-type constants. Moreover, values of the adjacent iteration points have great influence on the stepsize of the further iteration, which can effectively improve the efficiency of the iteration. We note that, if $\mathbb{M} = \mathbb{R}$, then our algorithm is an improvement of the algorithm presented in Hieu et al. [38].

The organization of the paper is as follows. In Sect. 2, we present some basic knowledge on Riemannian manifolds which will be used in this paper; for more details, see [41, 42]. In Sect. 3, we introduce the extragradient-like algorithm and analyze its convergence. Finally, in Sect. 4, we present two experiments to verify the algorithms.

2 Preliminaries

Suppose \mathbb{M} is simply connected n -dimensional Riemannian manifold, ∇ is the Levi-Civita connection, and γ is a smooth curve on \mathbb{M} . V is the unique vector field satisfies $\nabla_{\gamma'(t)} V = 0$ ($\forall t \in [a, b]$), and $V(\gamma(a)) = v$. Then the parallel transport $P_{\gamma, \gamma}(b), \gamma(a) : T_{\gamma(a)}\mathbb{M} \rightarrow T_{\gamma(b)}\mathbb{M}$ on the tangent bundle $T\mathbb{M}$ along γ is defined by

$$P_{\gamma, \gamma(b), \gamma(a)}(v) = V(\gamma(b)), \quad \forall a, b \in \mathbb{R} \text{ and } v \in T_{\gamma(a)}\mathbb{M}.$$

If γ is a minimal geodesic joining p to q , then we use $P_{q,p}$ instead of $P_{\gamma, q,p}$.

A Riemannian manifold \mathbb{M} is complete if for any $p \in \mathbb{M}$, all the geodesic $\gamma(t)$ emanating from p are defined for all $t \in \mathbb{R}$.

Suppose \mathbb{M} is complete, and $\gamma(\cdot) = \gamma_v(\cdot, p)$ is the geodesic, the exponential map $\exp_p : T_p\mathbb{M} \rightarrow \mathbb{M}$ at p is defined by $\exp_p v = \gamma_v(1, p)$, $\forall v \in T_p\mathbb{M}$, then $\exp_p tv = \gamma_v(t, p)$, $\forall t \in \mathbb{R}$. We note here that $\forall p \in \mathbb{M}$, \exp_p is differentiable on $T_p\mathbb{M}$, and $\exp_p : T_p\mathbb{M} \rightarrow \mathbb{M}$ is a diffeomorphism.

A complete, simply connected Riemannian manifold of nonpositive sectional curvature is named a Hadamard manifold. In this paper, we assume that \mathbb{M} is an n dimensional Hadamard manifold.

Proposition 2.1 ([43]) *Let $p \in \mathbb{M}$, then $\exp_p : T_p\mathbb{M} \rightarrow \mathbb{M}$ is a diffeomorphism, and for any $p, q \in \mathbb{M}$, there exists a unique normalized geodesic $\gamma_{q,p}$ joining p to q .*

A geodesic triangle $\Delta(p_1, p_2, p_3)$ of a Riemannian manifold is a set consisting of three points p_1, p_2, p_3 , and three minimal geodesic joining these points.

Proposition 2.2 ([42]) *Let $\Delta(p_1, p_2, p_3)$ be a geodesic triangle on Hadamard manifolds \mathbb{M} . Then*

$$d^2(p_1, p_2) + d^2(p_2, p_3) - 2\langle \exp_{p_2}^{-1} p_1, \exp_{p_2}^{-1} p_3 \rangle \leq d^2(p_3, p_1), \tag{2}$$

where $\exp_{p_2}^{-1}$ is the inverse of \exp_{p_2} .

Proposition 2.3 ([44]) *Let $\Delta(p_1, p_2, p_3)$ be a geodesic triangle on \mathbb{M} , Then there exist three points (i.e. p'_1, p'_2 , and p'_3) in \mathbb{R}^2 , such that*

$$d(p_1, p_2) = \|p'_1 - p'_2\|, \quad d(p_2, p_3) = \|p'_2 - p'_3\|, \quad d(p_3, p_1) = \|p'_3 - p'_1\|.$$

Lemma 2.4 ([7]) *Let $\Delta(p_1, p_2, p_3)$ be a geodesic triangle on \mathbb{M} and the comparison triangle be $\Delta(p'_1, p'_2, p'_3)$.*

(1) *Let α, β, γ be the angles of $\Delta(p_1, p_2, p_3)$ at the vertices p_1, p_2, p_3 , and α', β', γ' be the angles of $\Delta(p'_1, p'_2, p'_3)$ at the vertices p'_1, p'_2, p'_3 . Then*

$$\alpha' \geq \alpha, \quad \beta' \geq \beta, \quad \gamma' \geq \gamma.$$

(2) *Let z be a point in the geodesic joining p_1 to p_2 , and $z' \in [p'_1, p'_2]$ is the comparison point, if $d(z, p_1) = \|z' - p'_1\|, d(z, p_2) = \|z' - p'_2\|$, then*

$$d(z, p_3) \leq \|z' - p'_3\|.$$

Lemma 2.5 ([45]) *Let $x_0 \in \mathbb{M}, \{x_n\} \subset \mathbb{M}$, and $x_n \rightarrow x_0$. Then, for $\forall y \in \mathbb{M}$,*

$$\exp_{x_n}^{-1} y \rightarrow \exp_{x_0}^{-1} y, \quad \exp_{y}^{-1} x_n \rightarrow \exp_{y}^{-1} x_0.$$

Definition 2.6 ([46]) *A subset $E \subset \mathbb{M}$ is said to be convex if for any $p, q \in E$, the geodesic connecting p and q is still in E .*

Definition 2.7 ([41]) *Let ω be a real-valued function on \mathbb{M} , ω is said to be convex if for any geodesic γ on \mathbb{M} , the composition function $\omega \circ \gamma : [a, b] \rightarrow \mathbb{R}$ is convex.*

Definition 2.8 ([34]) *Let $\omega : \mathbb{M} \rightarrow \mathbb{R}$ be a convex and $z \in \mathbb{M}$. A vector $u \in T_z\mathbb{M}$ is called a subgradient of ω at z , iff*

$$\omega(y) \geq \omega(z) + \langle u, \exp_z^{-1} y \rangle, \quad \forall y \in \mathbb{M}.$$

The set of all subgradients of ω is named the subdifferential of ω at z , which is represented by $\partial\omega(z)$, and the domain of $\partial\omega$ is $\mathcal{D}(\partial\omega) = \{z \in \mathbb{M} | \partial\omega(z) \neq \emptyset\}$, and $\partial\omega(z)$ is a closed and convex set.

Proposition 2.9 ([6]) *Let $\omega : [a, b] \rightarrow \mathbb{R}$ be a convex, for any point p in Hadamard manifolds \mathbb{M} , we have $\mathcal{D}(\partial\omega) = \mathbb{M}$.*

Definition 2.10 ([45]) *Let \mathbb{M} be a Hadamard manifolds, ω be a lower semicontinuous, proper and convex function, $\omega \subset \mathbb{M}$ and $\mathcal{D}(\omega) = \mathbb{M}$, then the proximal mapping $\text{prox}_{\lambda\omega} : \mathbb{M} \rightarrow \mathbb{M}$ is defined as*

$$\text{prox}_{\lambda\omega}(z) := \underset{y \in \mathbb{M}}{\text{argmin}} \left\{ \omega(y) + \frac{1}{2\lambda} d^2(z, y) \right\}, \quad \forall z \in \mathbb{M}, \lambda > 0.$$

From [6, Lemma 4.2], $\text{prox}_{\lambda\omega}(\cdot)$ is a single-valued and $\mathcal{D}(\text{prox}_{\lambda\omega}) = \mathbb{M}$, and for each $z \in \mathbb{M}$, there exists a unique point $p = \text{prox}_{\lambda\omega}(z)$, which is characterized by

$$\exp_p^{-1} z \in \lambda \partial\omega(p).$$

Combining this and Definition 2.8, we have the following.

Lemma 2.11 *Let ω be a lower semicontinuous, proper and convex function on Hadamard manifold \mathbb{M} , and $z, p \in \mathbb{M}$, $\lambda > 0$. If $p = \text{prox}_{\lambda\omega}(z)$, $\forall y \in \mathbb{M}$, Then*

$$\langle \exp_p^{-1} y, \exp_p^{-1} z \rangle \leq \lambda(\omega(y) - \omega(p)).$$

Remark 1 From Lemma 2.11, if $z = \text{prox}_{\lambda\omega}(z)$, then

$$z \in \text{Argmin}\{\omega(y) : y \in E\} := \left\{ z \in E : \omega(z) = \min_{y \in E} \omega(y) \right\}.$$

For a closed and convex $E \subseteq \mathbb{M}$, the projection $P_E : \mathbb{M} \rightarrow E$ is defined for all $z \in \mathbb{M}$, such that $P_E(z) = \text{argmin}\{d(z, y), \forall y \in E\}$.

Definition 2.12 ([47]) For a bifunction $S : E \times E \rightarrow \mathbb{R}, \forall (z, y) \in E \times E$:

- (1) If $S(z, y) + S(y, z) \leq 0$, then S is called monotone.
- (2) If $S(z, y) \geq 0 \Rightarrow S(y, z) \leq 0$, then S is called pseudomonotone.

Definition 2.13 ([48]) Let \mathbb{M} be a Hadamard manifolds, $E \subset \mathbb{M}$, and $S : E \times E \rightarrow \mathbb{R}$. S satisfies a Lipschitz-type condition, if there exist $k_1, k_2 > 0$ such that

$$S(x, y) + S(y, z) \geq S(x, z) - k_1 d^2(x, y) - k_2 d^2(y, z), \quad \forall x, y, z \in E.$$

Lemma 2.14 ([49]) *Let $\{a_n\}_{n \in \mathbb{N}}$ ($a_n > 0$), $\{b_n\}_{n \in \mathbb{N}}$ ($b_n > 0$) be two real sequences and there exists $N > 0$, for all $n > N$. such that $a_{n+1} \leq a_n - b_n$. Then $\{a_n\}_{n \in \mathbb{N}}$ is convergent and $\lim_{n \rightarrow \infty} b_n = 0$.*

In addition, compared with Definitions 2.12 and 2.13, for the variational inequality (VI), we have the following definitions. Let V is a single-valued vector field, and $\mathcal{D}(V)$ be the domain of V .

Definition 2.15 ([50]) If there exists a constant $L > 0$ such that

$$\|P_{y,x} V(x) - V(y)\| \leq Ld(x, y), \quad \forall x, y \in \mathbb{M},$$

then V is called Lipschitz continuous.

Definition 2.16 ([43]) For all $x, y \in \mathcal{D}(V)$,

$$\langle V(x), \exp_x^{-1} y \rangle \geq 0 \quad \Rightarrow \quad \langle V(y), \exp_y^{-1} x \rangle \leq 0,$$

V is called pseudomonotone.

3 Main result

In this section, inspired by the algorithms in Hieu et al. [37, 38] and Yang et al. [39, 40], we introduce an extragradient-like algorithm for solving equilibrium problems (EP), and analyze the convergence of sequences generated by the algorithm. Finally, we apply the algorithm to solving the variational inequality problem (VI) as a particular case.

Unless explicitly stated otherwise, the subset E is a nonempty closed convex subset on \mathbb{M} , and the bifunction S satisfies the following conditions:

- (A1) For each $z \in E$, S is pseudomonotone on E , i.e., $S(z, y) \geq 0 \Rightarrow S(y, z) \leq 0$;
- (A2) S satisfies the Lipschitz-type condition on E , i.e., $S(x, y) + S(y, z) \geq S(x, z) - k_1 d^2(x, y) - k_2 d^2(y, z)$;
- (A3) $S(x, \cdot)$ is convex and subdifferentiable on E , \forall fixed $x \in E$;
- (A4) $S(\cdot, y)$ is upper semicontinuous, $\forall y \in E$.

In order to describe the new algorithm more conveniently, we note that $[a]_+ = \max\{0, a\}$ and adopt the convention $\frac{0}{0} = +\infty$, and $\frac{1}{0} = +\infty$.

Algorithm 3.1 (Extragradient-like algorithm for solving (EP))

Initialization: Choose $x_0, \bar{x}_0, \bar{x}_1 \in E$, $\lambda_1 > 0$, $\delta \in (0, 1)$, $\theta \in (0, 1]$, $\alpha \in (0, 1)$, $\varphi \in (1 - \frac{1-\theta}{2-\theta}\alpha, 1)$.

Iterative Steps: Suppose $x_{n-1}, \bar{x}_{n-1}, \bar{x}_n$ are obtained.

Step 1 Calculate

$$\begin{cases} x_n = \gamma_{x_{n-1}, \bar{x}_n}(\varphi), \\ \bar{x}_{n+1} = \text{prox}_{\lambda_n S(\bar{x}_n, \cdot)}(x_n). \end{cases}$$

If $\bar{x}_{n+1} = x_n = \bar{x}_n$, then stop: \bar{x}_n is a solution. Otherwise,

Step 2 Compute

$$\lambda_{n+1} = \min \left\{ \lambda_n, \frac{\alpha \delta \theta}{4\varphi \Lambda} (d^2(\bar{x}_n, \bar{x}_{n-1}) + d^2(\bar{x}_{n+1}, \bar{x}_n)) \right\},$$

where $\Lambda = [S(\bar{x}_{n-1}, \bar{x}_{n+1}) - S(\bar{x}_{n-1}, \bar{x}_n) - S(\bar{x}_n, \bar{x}_{n+1})]_+$. Set $n := n + 1$ and return.

Remark 2 Under the conditions (A1)–(A4) and $\bar{x}_{n+1} = x_n = \bar{x}_n$, then by Lemma 2.11 we obtain

$$S(\bar{x}_n, y) \geq S(\bar{x}_n, \bar{x}_{n+1}) + \frac{1}{\lambda_n} (\exp_{\bar{x}_{n+1}}^{-1} x_n, \exp_{\bar{x}_{n+1}}^{-1} y) \geq 0, \quad \forall y \in E.$$

So $\bar{x}_{n+1} \in \text{EP}(S; E)$.

Remark 3 By Definition 2.13, if the hypothesis (A2) holds, then there exist $k_1 > 0$, $k_2 > 0$ such that

$$\begin{aligned} S(\bar{x}_{n-1}, \bar{x}_{n+1}) - S(\bar{x}_{n-1}, \bar{x}_n) - S(\bar{x}_n, \bar{x}_{n+1}) &\leq k_1 d^2(\bar{x}_n, \bar{x}_{n-1}) + k_2 d^2(\bar{x}_{n+1}, \bar{x}_n) \\ &\leq \max\{k_1, k_2\} (d^2(\bar{x}_n, \bar{x}_{n-1}) + d^2(\bar{x}_{n+1}, \bar{x}_n)), \end{aligned}$$

then $\{\lambda_n\}$ is bounded from below by $\{\lambda_1, \frac{\alpha \delta \theta}{4\varphi \max\{c_1, c_2\}}\}$. Moreover, $\{\lambda_n\}$ is a monotonically decreasing sequence. Thus, $\lim_{n \rightarrow \infty} \lambda_n$ exists (i.e. $\lim_{n \rightarrow \infty} \lambda_n = \lambda > 0$). It should be noted that, if $S(\bar{x}_{n-1}, \bar{x}_{n+1}) - S(\bar{x}_{n-1}, \bar{x}_n) - S(\bar{x}_n, \bar{x}_{n+1}) \leq 0$, then $\lambda_{n+1} := \lambda_n$.

Remark 4 From $x_n = \gamma_{x_{n-1}, \bar{x}_n}(\varphi)$, we have $x_n = \exp_{\bar{x}_n}^{-1} \varphi \exp_{\bar{x}_n}^{-1} x_{n-1}$, it implies that x_{n-1}, \bar{x}_n, x_n lies in the same geodesic. From [51], we have

$$\exp_{x_{n-1}}^{-1} \bar{x}_n = \frac{1}{1 - \varphi} \exp_{x_{n-1}}^{-1} x_n, \tag{3}$$

$$\exp_{\bar{x}_n}^{-1} x_n = \exp_{\bar{x}_n}^{-1} (\exp_{\bar{x}_n}^{-1} \varphi \exp_{\bar{x}_n}^{-1} x_{n-1}) = \varphi \exp_{\bar{x}_n}^{-1} x_{n-1}, \tag{4}$$

$$\exp_{x_n}^{-1} \bar{x}_n = \frac{-\varphi}{1 - \varphi} \exp_{x_n}^{-1} x_{n-1}. \tag{5}$$

By the definition of \bar{x}_{n+1} and Remark 2, we know that, if Algorithm 3.1 terminates after finite iterations, then $\bar{x}_{n+1} \in \text{EP}(S; E)$. Otherwise, we have Lemma 3.1 and Theorem 3.2.

Lemma 3.1 *Suppose (A1)–(A4) hold, and $\text{EP}(S; E) \neq \emptyset$, let $\{x_n\}$ be sequences generated by Algorithm 3.1. Then $\{x_n\}$ is bounded.*

Proof Since $\bar{x}_{n+1} = \text{prox}_{\lambda_n S(\bar{x}_n, \cdot)}(x_n)$, by Lemma 2.11, $\forall z \in E$, we obtain

$$\langle \exp_{\bar{x}_{n+1}}^{-1} x_n, \exp_{\bar{x}_{n+1}}^{-1} z \rangle \leq \lambda_n (S(\bar{x}_n, z) - S(\bar{x}_n, \bar{x}_{n+1})), \tag{6}$$

$$\langle \exp_{x_n}^{-1} x_{n-1}, \exp_{x_n}^{-1} z \rangle \leq \lambda_{n-1} (S(\bar{x}_{n-1}, z) - S(\bar{x}_{n-1}, \bar{x}_n)). \tag{7}$$

Let $s \in \text{EP}(S; E)$, substituting $z := s$ into (6) and $z := \bar{x}_{n+1}$ into (7), we have

$$\langle \exp_{\bar{x}_{n+1}}^{-1} x_n, \exp_{\bar{x}_{n+1}}^{-1} s \rangle \leq \lambda_n (S(\bar{x}_n, s) - S(\bar{x}_n, \bar{x}_{n+1})), \tag{8}$$

$$\langle \exp_{x_n}^{-1} x_{n-1}, \exp_{x_n}^{-1} \bar{x}_{n+1} \rangle \leq \lambda_{n-1} (S(\bar{x}_{n-1}, \bar{x}_{n+1}) - S(\bar{x}_{n-1}, \bar{x}_n)). \tag{9}$$

Since S is pseudomonotone, and $s \in \text{EP}(S; E)$, we obtain $S(s, \bar{x}_n) \geq 0$, so $S(\bar{x}_n, s) \leq 0$. From (8) and $\lambda_n > 0$, it follows that

$$\langle \exp_{\bar{x}_{n+1}}^{-1} x_n, \exp_{\bar{x}_{n+1}}^{-1} s \rangle \leq -\lambda_n S(\bar{x}_n, \bar{x}_{n+1}). \tag{10}$$

Combining (9) and (4), we obtain for $\lambda_n > 0$ that

$$\frac{\lambda_n}{\lambda_{n-1} \varphi} \langle \exp_{x_n}^{-1} x_{n-1}, \exp_{x_n}^{-1} \bar{x}_{n+1} \rangle \leq \lambda_n (S(\bar{x}_{n-1}, \bar{x}_{n+1}) - S(\bar{x}_{n-1}, \bar{x}_n)). \tag{11}$$

On the other hand, applying inequality (2) in Proposition 2.2, gives

$$2 \langle \exp_{x_n}^{-1} x_{n-1}, \exp_{x_n}^{-1} \bar{x}_{n+1} \rangle \geq d^2(\bar{x}_n, x_n) + d^2(\bar{x}_n, \bar{x}_{n+1}) - d^2(x_n, \bar{x}_{n+1}), \tag{12}$$

$$2 \langle \exp_{\bar{x}_{n+1}}^{-1} x_n, \exp_{\bar{x}_{n+1}}^{-1} s \rangle \geq d^2(\bar{x}_{n+1}, x_n) + d^2(\bar{x}_{n+1}, s) - d^2(x_n, s). \tag{13}$$

By multiplying the both sides of inequality (12) by $\frac{\lambda_n}{\lambda_{n-1} \varphi} > 0$, and then adding the both sides of the resulting equation to inequality (13), we get

$$\begin{aligned} & 2 \langle \exp_{\bar{x}_{n+1}}^{-1} x_n, \exp_{\bar{x}_{n+1}}^{-1} s \rangle + 2 \frac{\lambda_n}{\lambda_{n-1} \varphi} \langle \exp_{x_n}^{-1} x_{n-1}, \exp_{x_n}^{-1} \bar{x}_{n+1} \rangle \\ & \geq d^2(x_n, \bar{x}_{n+1}) + d^2(\bar{x}_{n+1}, s) - d^2(x_n, s) \end{aligned}$$

$$\begin{aligned}
 & + \frac{\lambda_n}{\lambda_{n-1}} \frac{1}{\varphi} (d^2(x_n, \bar{x}_n) + d^2(\bar{x}_{n+1}, \bar{x}_n) - d^2(x_n, \bar{x}_{n+1})) \\
 & = \left(1 - \frac{\lambda_n}{\lambda_{n-1}} \frac{1}{\varphi}\right) d^2(x_n, \bar{x}_{n+1}) - d^2(x_n, s) \\
 & \quad + \frac{\lambda_n}{\lambda_{n-1}} \frac{1}{\varphi} (d^2(x_n, \bar{x}_n) + d^2(\bar{x}_{n+1}, \bar{x}_n)) + d^2(\bar{x}_{n+1}, s). \tag{14}
 \end{aligned}$$

Combining Eqs. (14), (10) and (11), we get for $\lambda_n > 0$

$$\begin{aligned}
 d^2(\bar{x}_{n+1}, s) & \leq \left(\frac{\lambda_n}{\lambda_{n-1}} \frac{1}{\varphi} - 1\right) d^2(x_n, \bar{x}_{n+1}) + d^2(x_n, s) \\
 & \quad + 2\lambda_{n+1} \frac{\lambda_n}{\lambda_{n+1}} (S(\bar{x}_{n-1}, \bar{x}_{n+1}) - S(\bar{x}_{n-1}, \bar{x}_n) - S(\bar{x}_n, \bar{x}_{n+1})) \\
 & \quad - \frac{\lambda_n}{\lambda_{n-1}} \frac{1}{\varphi} (d^2(x_n, \bar{x}_n) + d^2(\bar{x}_{n+1}, \bar{x}_n)). \tag{15}
 \end{aligned}$$

By the definition of λ_n and (15), we obtain

$$\begin{aligned}
 d^2(\bar{x}_{n+1}, s) & \leq \left(\frac{\lambda_n}{\lambda_{n-1}} \frac{1}{\varphi} - 1\right) d^2(x_n, \bar{x}_{n+1}) + d^2(x_n, s) \\
 & \quad + \frac{1}{2} \delta \frac{\lambda_n}{\lambda_{n+1}} \frac{1}{\varphi} \alpha \theta (d^2(\bar{x}_n, \bar{x}_{n-1}) + d^2(\bar{x}_n, \bar{x}_{n+1})) \\
 & \quad - \frac{\lambda_n}{\lambda_{n-1}} \frac{1}{\varphi} (d^2(x_n, \bar{x}_n) + d^2(\bar{x}_{n+1}, \bar{x}_n)). \tag{16}
 \end{aligned}$$

From Remark 3 $\lambda_n \rightarrow \lambda > 0$, and $0 < \delta < 1$. Hence, there exists $N \geq 0$, such that, for all $n \geq N$, $0 < \lambda_n \frac{\delta}{\lambda_{n+1}} < 1$, and $\frac{\lambda_n}{\lambda_{n-1}} \frac{1}{\varphi} - 1 \leq \frac{\lambda_{n-1}}{\lambda_{n-1}} \frac{1}{\varphi} - 1 = \frac{1}{\varphi} - 1$. Thus, from (16), we have

$$\begin{aligned}
 d^2(\bar{x}_{n+1}, s) & \leq \left(\frac{1}{\varphi} - 1\right) d^2(x_n, \bar{x}_{n+1}) + d^2(x_n, s) - \frac{\alpha}{\varphi} (d^2(x_n, \bar{x}_n) + d^2(\bar{x}_{n+1}, \bar{x}_n)) \\
 & \quad + \frac{1}{2} \delta \frac{1}{\varphi} \alpha \theta (d^2(\bar{x}_n, \bar{x}_{n-1}) + d^2(\bar{x}_n, \bar{x}_{n+1})), \quad \forall n \geq N. \tag{17}
 \end{aligned}$$

Now, we estimate the term $d^2(\bar{x}_{n+1}, s)$ in (17). Fix $n \geq 0$, set $p = \bar{x}_{n+1}$, $q = x_n$ in geodesic triangle $\Delta(s, p, q)$. Then using Lemma 2.3 in the comparison triangle $\Delta(s', p', q')$, we have

$$d(s, \bar{x}_{n+1}) = d(s, p) = \|p' - s'\|, \quad d(s, x_n) = d(s, q) = \|q' - s'\|.$$

Recall from Algorithm 3.1 that $x_{n+1} = \exp_{\bar{x}_{n+1}}^{-1} \varphi \exp_{\bar{x}_{n+1}}^{-1} x_n$. The comparison point of x_{n+1} is $x'_{n+1} = (1 - \varphi)p' + \varphi q'$. Let β and β' denote the angles at s and s' , respectively. From Lemma 2.4(1), we have $\beta \leq \beta'$, thus $\cos \beta' \leq \cos \beta$. Then, from Lemma 2.4(2) we have

$$\begin{aligned}
 d^2(x_{n+1}, s) & \leq \|(1 - \varphi)p' + \varphi q' - s'\|^2 \\
 & = \|(1 - \varphi)(p' - s') + \varphi(q' - s')\|^2 \\
 & = (1 - \varphi)^2 \|p' - s'\|^2 + \varphi^2 \|q' - s'\|^2 + 2\varphi(1 - \varphi) \|p' - s'\| \|q' - s'\| \cos \beta' \\
 & \leq (1 - \varphi)^2 d^2(p, s) + \varphi^2 d^2(q, s) + 2\varphi(1 - \varphi) d(p, s) d(q, s) \cos \beta \\
 & = (1 - \varphi)^2 d^2(\bar{x}_{n+1}, s) + \varphi^2 d^2(x_n, s) + 2\varphi(1 - \varphi) \langle \exp_s^{-1} \bar{x}_{n+1}, \exp_s^{-1} x_n \rangle. \tag{18}
 \end{aligned}$$

Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
 2\langle \exp_s^{-1} x_n, \exp_s^{-1} \bar{x}_{n+1} \rangle &\leq 2\|\exp_s^{-1} x_n\| \|\exp_s^{-1} \bar{x}_{n+1}\| \\
 &\leq d^2(s, x_n) + d^2(s, \bar{x}_{n+1}).
 \end{aligned}
 \tag{19}$$

By substituting (19) into (18), we get

$$d^2(\bar{x}_{n+1}, s) \geq -\frac{\varphi}{1-\varphi}d^2(x_n, s) + \frac{1}{1-\varphi}d^2(x_{n+1}, s).
 \tag{20}$$

Consequently, combining (17) and (20), we have for all $n \geq N$

$$\begin{aligned}
 &\frac{1}{1-\varphi}d^2(x_{n+1}, s) - \frac{\varphi}{1-\varphi}d^2(x_n, s) \\
 &\leq d^2(x_n, s) + \left(\frac{1}{\varphi} - 1\right)d^2(x_n, \bar{x}_{n+1}) - \frac{\alpha}{\varphi}(d^2(x_n, \bar{x}_n) + d^2(\bar{x}_{n+1}, \bar{x}_n)) \\
 &\quad + \frac{\alpha}{2\varphi}\theta(d^2(\bar{x}_n, \bar{x}_{n-1}) + d^2(\bar{x}_n, \bar{x}_{n+1})).
 \end{aligned}
 \tag{21}$$

Next, we need Lemma 2.14 to complete the proof. From (21) and (2), we get

$$\begin{aligned}
 &\frac{1}{1-\varphi}d^2(x_{n+1}, s) + \frac{\alpha\theta}{2\varphi}(d^2(\bar{x}_{n+1}, \bar{x}_n)) \\
 &\leq \frac{\varphi}{1-\varphi}d^2(x_n, s) + \frac{\alpha\theta}{2\varphi}(d^2(\bar{x}_{n+1}, \bar{x}_n) + d^2(x_n, s) + \left(\frac{1}{\varphi} - 1\right)d^2(x_n, \bar{x}_{n+1})) \\
 &\quad - \frac{\alpha}{\varphi}(d^2(x_n, \bar{x}_n) + d^2(\bar{x}_{n+1}, \bar{x}_n)) + \frac{\alpha\theta}{2\varphi}(d^2(\bar{x}_n, \bar{x}_{n-1}) + d^2(\bar{x}_n, \bar{x}_{n+1})) \\
 &= \frac{1}{1-\varphi}d^2(x_n, s) + \frac{\alpha\theta}{2\varphi}d^2(\bar{x}_n, \bar{x}_{n-1}) + \frac{(\theta-1)\alpha}{\varphi}d^2(\bar{x}_{n+1}, \bar{x}_n) \\
 &\quad + \left(\frac{1}{\varphi} - 1\right)d^2(\bar{x}_{n+1}, x_n) - \frac{\alpha}{\varphi}d^2(x_n, \bar{x}_n) \\
 &\leq \frac{1}{1-\varphi}d^2(x_n, s) + \frac{\alpha\theta}{2\varphi}d^2(\bar{x}_n, \bar{x}_{n-1}) + \frac{(\theta-1)\alpha}{\varphi}d^2(\bar{x}_{n+1}, \bar{x}_n) + \left(\frac{1}{\varphi} - 1\right)d^2(\bar{x}_{n+1}, x_n) \\
 &\quad - \frac{\alpha}{\varphi}(d^2(x_n, \bar{x}_{n+1}) + d^2(\bar{x}_n, \bar{x}_{n+1})) - 2\langle \exp_{\bar{x}_{n+1}}^{-1} x_n, \exp_{\bar{x}_{n+1}}^{-1} \bar{x}_n \rangle.
 \end{aligned}
 \tag{22}$$

Moreover, note that $\theta \in (0, 1]$, then $2 - \theta > 0$, and it follows from the Cauchy–Schwarz inequality that

$$\begin{aligned}
 2\langle \exp_{\bar{x}_{n+1}}^{-1} x_n, \exp_{\bar{x}_{n+1}}^{-1} \bar{x}_n \rangle &\leq 2\|\exp_{\bar{x}_{n+1}}^{-1} x_n\| \|\exp_{\bar{x}_{n+1}}^{-1} \bar{x}_n\| \\
 &\leq \frac{1}{2-\theta}\|\exp_{\bar{x}_{n+1}}^{-1} x_n\|^2 + (2-\theta)\|\exp_{\bar{x}_{n+1}}^{-1} \bar{x}_n\|^2 \\
 &= \frac{1}{2-\theta}d^2(\bar{x}_{n+1}, x_n) + (2-\theta)d^2(\bar{x}_{n+1}, \bar{x}_n).
 \end{aligned}
 \tag{23}$$

Equations (23) and (22) imply

$$\begin{aligned} & \frac{1}{1-\varphi}d^2(x_{n+1},s) + \frac{\alpha\theta}{2\varphi}d^2(\bar{x}_{n+1},\bar{x}_n) \\ & \leq \frac{1}{1-\varphi}d^2(x_n,s) + \frac{\alpha\theta}{2\varphi}d^2(\bar{x}_n,\bar{x}_{n-1}) + \left(\frac{1}{\varphi} - 1 - \frac{\alpha}{\varphi} + \frac{\alpha}{\varphi(2-\theta)}\right)d^2(\bar{x}_{n+1},x_n). \end{aligned} \tag{24}$$

Now we set

$$\begin{aligned} a_n &= \frac{1}{1-\varphi}d^2(x_n,s) + \frac{\alpha\theta}{2\varphi}d^2(\bar{x}_n,\bar{x}_{n-1}), \\ b_n &= -\left(\frac{1}{\varphi} - 1 - \frac{\alpha}{\varphi} + \frac{\alpha}{\varphi(2-\theta)}\right)d^2(\bar{x}_{n+1},x_n). \end{aligned}$$

It follows from $\varphi \in (1 - \frac{1-\theta}{2}\alpha, 1)$ that $b_n > 0$, then from (24), we have, for all $n \geq N$, $a_{n+1} \leq a_n - b_n$. Then we get the conclusion from Lemma 2.14 that $\{a_n\}$ is bounded, $\lim_{n \rightarrow \infty} a_n$ exists, $\lim_{n \rightarrow \infty} b_n = 0$, and $\lim_{n \rightarrow \infty} d(\bar{x}_{n+1}, x_n) = 0$.

Moreover, by using the triangle inequality, it follows that

$$\begin{aligned} d(\bar{x}_n, x_n) + d(x_n, \bar{x}_{n-1}) &\geq d(\bar{x}_n, \bar{x}_{n-1}), \\ d(x_n, x_{n-1}) + d(x_{n-1}, \bar{x}_{n-1}) &\geq d(x_n, \bar{x}_{n-1}). \end{aligned}$$

Combining this and Eqs. (4) and (5), we can obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} d(\bar{x}_n, x_n) &= \lim_{n \rightarrow \infty} d(x_n, x_{n-1}) = 0, \\ \lim_{n \rightarrow \infty} d(\bar{x}_n, \bar{x}_{n-1}) &= \lim_{n \rightarrow \infty} d(\bar{x}_{n+1}, x_n) = 0, \\ \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \left(\frac{1}{1-\varphi}d^2(x_n,s) + \frac{\alpha\theta}{2\varphi}d^2(\bar{x}_n,\bar{x}_{n-1})\right) > 0. \end{aligned} \tag{25}$$

Thus, we see that $\{x_n\}$ and $\{\bar{x}_n\}$ are bounded. □

Theorem 3.2 *Assume that (A1)–(A4) hold, and $EP(S; E) \neq \emptyset$, then the sequences $\{x_n\}$ generated by Algorithm 3.1 converge to a solution of the equilibrium problem (EP).*

Proof By Lemma 3.1, we know that $\{x_n\}$ and $\{\bar{x}_n\}$ are bounded, and there exists a subsequence $\{x_l\}$ of $\{x_n\}$ that converges to $x^* \in E$. It follows from (25) that

$$\lim_{k \rightarrow \infty} d(\bar{x}_l, x_l) = \lim_{k \rightarrow \infty} d(x_l, \bar{x}_{l+1}) = \lim_{k \rightarrow \infty} d(\bar{x}_l, \bar{x}_{l-1}) = 0. \tag{26}$$

It follows from inequality (6) that

$$\lambda_l S(\bar{x}_l, z) \geq \lambda_l S(\bar{x}_l, \bar{x}_{l+1}) + \langle \exp_{\bar{x}_{l+1}}^{-1} x_l, \exp_{\bar{x}_{l+1}}^{-1} \bar{x} \rangle, \quad \forall \bar{x} \in E. \tag{27}$$

On the other hand, since S satisfies the Lipschitz-type condition, we have

$$\begin{aligned} \lambda_l S(\bar{x}_l, \bar{x}_{l+1}) &\geq \lambda_l (S(\bar{x}_{l-1}, \bar{x}_{l+1}) - f(\bar{x}_{l-1}, \bar{x}_l)) \\ &\quad - \lambda_l c_1 d^2(\bar{x}_l, \bar{x}_{l-1}) - \lambda_l c_2 d^2(\bar{x}_l, \bar{x}_{l+1}). \end{aligned} \tag{28}$$

From Eqs. (11) and (28), it follows that

$$\begin{aligned} \lambda_l S(\bar{x}_l, \bar{x}_{l+1}) &\geq \frac{\lambda_l}{\lambda_{l-1}} \frac{1}{\varphi} \left(\exp_{\bar{x}_l}^{-1} x_l, \exp_{\bar{x}_l}^{-1} \bar{x}_{l+1} \right) \\ &\quad - \lambda_l c_1 d^2(\bar{x}_l, \bar{x}_{l-1}) - \lambda_l c_2 d^2(\bar{x}_l, \bar{x}_{l+1}). \end{aligned} \tag{29}$$

Now, combining (27) and (29), we get, for $\forall \bar{x} \in E$,

$$\begin{aligned} S(\bar{x}_l, z) &\geq \frac{1}{\lambda_{l-1}} \frac{1}{\varphi} \left(\exp_{\bar{x}_l}^{-1} x_l, \exp_{\bar{x}_l}^{-1} \bar{x}_{l+1} \right) + \frac{1}{\lambda_l} \left(\exp_{\bar{x}_{l+1}}^{-1} x_l, \exp_{\bar{x}_{l+1}}^{-1} \bar{x} \right) \\ &\quad - c_1 d^2(\bar{x}_l, \bar{x}_{l-1}) - c_2 d^2(\bar{x}_l, \bar{x}_{l+1}). \end{aligned} \tag{30}$$

From Lemma 2.5, (26), (30), the boundedness of $\{x_n\}$, and $\lim_{n \rightarrow \infty} \lambda_n = \lambda > 0$, we obtain

$$S(x^*, z) \geq 0, \quad \forall y \in E. \tag{31}$$

So we obtain $x^* \in \text{EP}(S; E)$.

Next, we will prove that $\{x_n\}_{n \in \mathbb{N}}$ has a unique cluster point. Suppose that $\{x_n\}_{n \in \mathbb{N}}$ has at least two cluster points $\bar{x}_1, \bar{x}_2 \in \text{EP}(S; E)$. Let $\{x_{n_i}\}$ be a sequence such that $x_{n_i} \rightarrow \bar{x}_1, x_{n_j} \rightarrow \bar{x}_2$, as $i \rightarrow \infty$. By Lemma 2.2, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} d^2(x_n, \bar{x}_2) &= \lim_{i \rightarrow \infty} d^2(x_{n_i}, \bar{x}_2) \\ &\geq \lim_{i \rightarrow \infty} \left(d^2(x_{n_i}, \bar{x}_1) + d^2(\bar{x}_1, \bar{x}_2) - 2 \left(\exp_{\bar{x}_1}^{-1} x_{n_i}, \exp_{\bar{x}_1}^{-1} \bar{x}_2 \right) \right) \\ &= \lim_{n \rightarrow \infty} d^2(x_n, \bar{x}_1) + d^2(\bar{x}_1, \bar{x}_2) \end{aligned} \tag{32}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} d^2(x_n, \bar{x}_1) &= \lim_{j \rightarrow \infty} d^2(x_{n_j}, \bar{x}_1) \\ &\geq \lim_{j \rightarrow \infty} \left(d^2(x_{n_j}, \bar{x}_2) + d^2(\bar{x}_2, \bar{x}_1) - 2 \left(\exp_{\bar{x}_2}^{-1} x_{n_j}, \exp_{\bar{x}_2}^{-1} \bar{x}_1 \right) \right) \\ &= \lim_{n \rightarrow \infty} d^2(x_n, \bar{x}_2) + d^2(\bar{x}_2, \bar{x}_1). \end{aligned} \tag{33}$$

By summing (32) and (33), we have $\bar{x}_1 = \bar{x}_2$. So $\{x_n\}_{n \in \mathbb{N}}$ has a unique cluster point. □

Remark 5 From Algorithm 3.1, we can obtain a new method for solving the pseudomonotone variational inequality (VI). If a vector field V is Lipschitz-continuous and pseudomonotone, then the conditions (A1)–(A4) hold for S with $k_1 = k_2 = \frac{L}{2}$. So, we can get the following algorithm for solving (VI).

Algorithm 3.2 (Extragradient-like algorithm for solving (VI))

Initialization: Choose $x_0, \bar{x}_0, \bar{x}_1 \in E, \lambda_1 > 0, \delta \in (0, 1), \theta \in (0, 1], \alpha \in (0, 1), \varphi \in (1 - \frac{1-\theta}{2-\theta}\alpha, 1)$.

Iterative Steps: Suppose $x_{n-1}, \bar{x}_{n-1}, \bar{x}_n$ are obtained.

Step 1 Calculate

$$\begin{cases} x_n = \gamma_{x_{n-1}, \bar{x}_n}(\varphi), \\ \bar{x}_{n+1} = P_E(\exp_{x_n} - \lambda_n V(\bar{x}_n)). \end{cases}$$

If $\bar{x}_{n+1} = x_n = \bar{x}_n$, then stop: \bar{x}_n is a solution. Otherwise,

Step 2 Compute

$$\lambda_{n+1} = \min \left\{ \lambda_n, \frac{\alpha \delta \theta}{4\varphi \Lambda} (d^2(\bar{x}_n, \bar{x}_{n-1}) + d^2(\bar{x}_{n+1}, \bar{x}_n)) \right\},$$

where $\Lambda = [(P_{\bar{x}_n, \bar{x}_{n-1}} V(\bar{x}_{n-1}) - V(\bar{x}_n), \exp_{\bar{x}_n}^{-1} \bar{x}_{n+1})_+]$. Set $n := n + 1$ and return.

As for the convergence of Algorithm 3.2, if Algorithm 3.2 terminates after finite iterations, we have $\bar{x}_{n+1} = x_n = \bar{x}_n$, it follows that $\bar{x}_n = P_E(\bar{x}_n - \lambda V(\bar{x}_n))$, thus $\bar{x}_n \in \text{VI}(V, E)$ follows directly from [43]; otherwise, we can find a sequence $\{x_n\}$ generated by Algorithm 3.2 converging to some $x^* \in \text{VI}(V, E)$, as $n \rightarrow \infty$. The analysis process is completely similar to that of Theorem 3.2, which we omit here.

4 Numerical experiments

In this section, we perform two experiments to show the numerical behaviors of proposed algorithms in this paper. We take $\mathbb{M} = \mathbb{R}_{++}^m = \{x \in \mathbb{R} : x > 0\}$, and involve two experiments named Test 1 and Test 2 to verify the effectiveness of Algorithms 3.1 and 3.2, respectively.

We choose $\alpha = 0.95, \delta = 0.90, \theta = 0.5, 0.75, 0.90$, and $\varphi \in (1 - \frac{1-\theta}{2-\theta}\alpha, 1)$ is a random number, and $\bar{x}_1, x_0, \bar{x}_0$ by *Matlab* code `10*rand(m,1)`. The termination criterion is

$$\varepsilon \geq d^2(\bar{x}_{n+1}, x_n) + d^2(\bar{x}_n, x_n).$$

Example 4.1 Let $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{M}_1 = (\mathbb{R}_{++}, \langle \cdot, \cdot \rangle)$ be the Riemannian manifold with $\langle x, y \rangle := xy, \forall x, y \in \mathbb{R}_{++}$. It can be seen from Ref. [52] that the section curvature of \mathbb{M}_1 is zero, thus \mathbb{M}_1 is a Hadamard manifold. Suppose that $x, y \in \mathbb{M}_1$ and $u \in T_x \mathbb{M}_1$ with $\|v\|_2 = 1$, then

$$\begin{cases} d(x, y) := |\ln(\frac{x}{y})|, \\ \exp_x tv = xe^{(v/x)t}, \quad t \in (0, +\infty), \\ \exp_x^{-1} y = x \ln(\frac{y}{x}). \end{cases} \tag{34}$$

Let \mathbb{R}_{++}^m be the product space of \mathbb{R}_{++} , that is, $\mathbb{R}_{++}^m = \{(x_1, x_2, \dots, x_m)^T : x_i \in \mathbb{R}_{++}, i = 1, 2, \dots, m\}$. Let $\mathbb{M} = (\mathbb{R}_{++}^m, \langle \cdot, \cdot \rangle)$ be the m -dimensional Hadamard manifolds with metric $\langle u, v \rangle := u^T v$, and $d(x, y) := |\ln(x/y)| = |\ln(\sum_{i=1}^m (x_i/y_i))|$, where $x, y \in \mathbb{M}, x = (x_i), y = (y_i), i = 1, 2, \dots, m$.

Test 1 *In this test, we verify the effectiveness of Algorithm 3.1 in $\mathbb{M} = (\mathbb{R}_{++}^m, \langle \cdot, \cdot \rangle)$. We consider an extension of Nash equilibrium model, which was introduced in [53, 54]. The form is as follows:*

$$S(x, y) = \langle P_1 x + P_2 y + p, y - x \rangle,$$

Table 1 Performance of algorithm 3.1 for the number of iterations (Iter.) and the computing time (Time) measured in seconds with $m = 20, 300, 500$.

m	$\theta = 0.50$		$\theta = 0.75$		$\theta = 0.90$	
	Iter.	Time	Iter.	Time	Iter.	Time
20	42	0.2156	84	0.4119	315	1.4444
	50	0.2327	102	0.4924	323	1.5290
300	92	1.9221	91	1.8240	336	6.0876
	113	2.1877	117	2.3066	407	7.6408
500	127	6.8464	107	4.9920	394	17.7671
	154	7.5554	172	8.2371	447	20.6179

the feasible set $E \subset \mathbb{M}$ given by

$$E := \{x = (x_1, x_2, \dots, x_m)^T : 1 \leq x_i \leq 100, i = 1, \dots, m\},$$

$x, y \in E, p = (p_1, p_2, \dots, p_m)^T \in \mathbb{R}^m$ is chosen randomly with its elements in $[1, m]$, and the matrices P_1 and P_2 are two square matrices of order m such that P_2 is symmetric positive semidefinite and $P_2 - P_1$ is negative semidefinite.

From [54], we know that S is pseudomonotone. Moreover, from [31, Lemma 6.2], we obtain a bifunction S that satisfies (A2) with the Lipschitz-type constants $k_1 = k_2 = \frac{\|P_2 - P_1\|}{2}$. Assumptions (A3), (A4) are automatically fulfilled and so Algorithm 3.1 can be applied in this case.

For numerical experiment, we take $\lambda_1 = \frac{1}{\|P_2 - P_1\|}$, and $m = 20, 300, 500$. For each m , we have generated two random samples with different choice of P_1, P_2 and p . The number of iterations (Iter.) and the computing time (Time) measured in seconds are described in Table 1.

Test 2 We consider the performance of Algorithm 3.2 in $\mathbb{M} = (\mathbb{R}_{++}^m, \langle \cdot, \cdot \rangle)$. Let the feasible set $E := \{x = (x_1, x_2, \dots, x_m)^T : 1 \leq x_i \leq 10, i = 1, \dots, m\}$ be a closed convex subset of \mathbb{R}_{++}^m and $V : E \rightarrow \mathbb{R}$ be a single-valued vector field defined by

$$V(x) := \sum_{i=1}^m (x_i \ln x_i), \quad \forall x \in E.$$

According to [55, Example 1], V is monotone and Lipschitz continuous. Therefore, the conditions (A1) and (A2) are valid, assuming that (A3) and (A4) are automatically verified, then Algorithm 3.2 can be applied in this case.

For the numerical experiment, we take $\lambda_1 = 0.4, m = 200, 300, 500$, and generate three random samples with different choice of initial points. The number of iterations (Iter.) and the computing time (Time) measured in seconds are described in Table 2.

5 Conclusions

In this paper, a new algorithm for solving the equilibrium problem on Hadamard manifolds is presented, in which the bifunctions satisfy the Lipschitz type extension and are pseudomonotone. Compared with the existing algorithm, the advantage of this algorithm is that the Lipschitz constants can be unknown.

Table 2 Performance of algorithm 3.2 for the number of iterations (Iter.) and the computing time (Time) measured in seconds with $m = 20, 300, 500$

m	$\theta = 0.50$		$\theta = 0.75$		$\theta = 0.90$	
	Iter.	Time	Iter.	Time	Iter.	Time
20	1464	0.0808	2026	0.0372	3129	0.0726
	1652	0.0346	3780	0.0861	4400	0.0810
	2270	0.0656	4071	0.0885	5386	0.0912
300	3074	0.3259	3067	0.3437	5087	0.4620
	3661	0.4124	3229	0.3325	5645	0.4973
	3821	0.3667	4071	0.3831	5676	0.5060
500	2775	0.4022	2544	0.3629	5966	0.7700
	3286	0.5029	3803	0.4874	6663	0.8707
	3892	0.5162	4461	0.5949	7199	0.8826

Funding

The project was supported by the National Natural Science Foundation of China under Grant 61877046, 11602184.

Availability of data and materials

All data analyzed during this research are included in this paper.

Competing interests

The authors have no competing interests in this paper.

Authors' contributions

The authors contributed equally to this paper. All authors read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 27 April 2020 Accepted: 3 August 2020 Published online: 08 August 2020

References

- Hadjisavvas, N., Schaible, S.: Quasimonotone variational inequalities in Banach spaces. *J. Optim. Theory Appl.* **90**(1), 95–111 (1996)
- Bianchi, M., Schaible, S.: Generalized monotone bifunctions and equilibrium problems. *J. Optim. Theory Appl.* **90**(1), 31–43 (1996)
- Bianchi, M., Schaible, S.: Equilibrium problems under generalized convexity and generalized monotonicity. *J. Glob. Optim.* **30**(2–3), 121–134 (2004)
- Blum, E., Oettli, W.: Variational principles for equilibrium problems. In: *Parametric Optimization and Related Topics, III*, Güstrow, 1991. *Approx. Optim.*, vol. 3, pp. 79–88. Peter Lang, Frankfurt am Main (1993)
- Ferreira, O.P., Pérez, L.R.L., Németh, S.Z.: Singularities of monotone vector fields and an extragradient-type algorithm. *J. Glob. Optim.* **31**(1), 133–151 (2005)
- Ferreira, O.P., Oliveira, P.R.: Proximal point algorithm on Riemannian manifolds. *Optimization* **51**(2), 257–270 (2002)
- Li, C., López, G., Martín-Márquez, V.: Iterative algorithms for nonexpansive mappings on Hadamard manifolds. *Taiwan. J. Math.* **14**(2), 541–559 (2010)
- Tang, G.J., Zhou, L.W., Huang, N.J.: The proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds. *Optim. Lett.* **7**(4), 779–790 (2013)
- Kristály, A.: Location of Nash equilibria: a Riemannian geometrical approach. *Proc. Am. Math. Soc.* **138**(5), 1803–1810 (2010)
- Li, S.L., Li, C., Liou, Y.C., Yao, J.C.: Existence of solutions for variational inequalities on Riemannian manifolds. *Nonlinear Anal.* **71**(11), 5695–5706 (2009)
- Li, C., López, G., Martín-Márquez, V., Wang, J.H.: Resolvents of set-valued monotone vector fields in Hadamard manifolds. *Set-Valued Var. Anal.* **19**(3), 361–383 (2011)
- Ceng, L.C., Li, X., Qin, X.: Parallel proximal point methods for systems of vector optimization problems on Hadamard manifolds without convexity. *Optimization* **69**(2), 357–383 (2020)
- Zhou, L.-W., Huang, N.-J.: Existence of solutions for vector optimization on Hadamard manifolds. *J. Optim. Theory Appl.* **157**(1), 44–53 (2013)
- Colao, V., López, G., Marino, G., Martín-Márquez, V.: Equilibrium problems in Hadamard manifolds. *J. Math. Anal. Appl.* **388**(1), 61–77 (2012)
- Fan, K.: A generalization of Tychonoff's fixed point theorem. *Math. Ann.* **142**, 305–310 (1960/61)
- Yang, Z., Pu, Y.J.: Existence and stability of solutions for maximal element theorem on Hadamard manifolds with applications. *Nonlinear Anal.* **75**(2), 516–525 (2012)
- Tang, G.J., Zhou, L.W., Huang, N.J.: Existence results for a class of hemivariational inequality problems on Hadamard manifolds. *Optimization* **65**(7), 1451–1461 (2016)
- Chen, S.L., Huang, N.J.: Vector variational inequalities and vector optimization problems on Hadamard manifolds. *Optim. Lett.* **10**(4), 753–767 (2016)

19. Batista, E.E.A., Bento, G.C., Ferreira, O.P.: An existence result for the generalized vector equilibrium problem on Hadamard manifolds. *J. Optim. Theory Appl.* **167**(2), 550–557 (2015)
20. Zhou, L.W., Huang, N.J.: A revision on geodesic pseudo-convex combination and Knaster–Kuratowski–Mazurkiewicz theorem on Hadamard manifolds. *J. Optim. Theory Appl.* **182**(3), 1186–1198 (2019)
21. Zhou, L.W., Xiao, Y.B., Huang, N.J.: New characterization of geodesic convexity on Hadamard manifolds with applications. *J. Optim. Theory Appl.* **172**(3), 824–844 (2017)
22. Zhou, L.W., Huang, N.J.: Existence of solutions for vector optimization on Hadamard manifolds. *J. Optim. Theory Appl.* **157**(1), 44–53 (2013)
23. Li, S.L., Li, C., Liou, Y.C., Yao, J.C.: Existence of solutions for variational inequalities on Riemannian manifolds. *Nonlinear Anal.* **71**(11), 5695–5706 (2009)
24. Li, C., Yao, J.C.: Variational inequalities for set-valued vector fields on Riemannian manifolds: convexity of the solution set and the proximal point algorithm. *SIAM J. Control Optim.* **50**(4), 2486–2514 (2012)
25. Wang, X., López, G., Li, C., Yao, J.C.: Equilibrium problems on Riemannian manifolds with applications. *J. Math. Anal. Appl.* **473**(2), 866–891 (2019)
26. Korpelevič, G.M.: An extragradient method for finding saddle points and for other problems. *Ėkon. Mat. Metody* **12**(4), 747–756 (1976)
27. Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. *J. Optim. Theory Appl.* **148**(2), 318–335 (2011)
28. Thong, D.V., Hieu, D.V.: Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems. *Numer. Algorithms* **80**(4), 1283–1307 (2019)
29. Ceng, L.C., Qin, X., Shehu, Y., Yao, J.C.: Mildly inertial subgradient extragradient method for variational inequalities involving an asymptotically nonexpansive and finitely many nonexpansive mappings. *Mathematics* **7**, 881 (2019)
30. Yao, Y.H., Postolache, M., Yao, J.C.: Convergence of an extragradient algorithm for fixed point and variational inequality problems. *J. Nonlinear Convex Anal.* **20**(12), 2623–2631 (2019)
31. Tran, D.Q., Dung, M.L., Nguyen, V.H.: Extragradient algorithms extended to equilibrium problems. *Optimization* **57**(6), 749–776 (2008)
32. Nguyen, T.T.V., Strodiot, J.J., Nguyen, V.H.: Hybrid methods for solving simultaneously an equilibrium problem and countably many fixed point problems in a Hilbert space. *J. Optim. Theory Appl.* **160**(3), 809–831 (2014)
33. Yao, Y., Shahzad, N., Yao, J.C.: Projected subgradient algorithms for pseudomonotone equilibrium problems and fixed points of pseudocontractive operators. *Mathematics* **8**, 461 (2020)
34. Salahuddin: The existence of solution for equilibrium problems in Hadamard manifolds. *Trans. A. Razmadze Math. Inst.* **171**(3), 381–388 (2017)
35. Li, X.B., Zhou, L.W., Huang, N.J.: Gap functions and descent methods for equilibrium problems on Hadamard manifolds. *J. Nonlinear Convex Anal.* **17**(4), 807–826 (2016)
36. Cruz Neto, J.X., Santos, P.S.M., Soares, P.A. Jr.: An extragradient method for equilibrium problems on Hadamard manifolds. *Optim. Lett.* **10**(6), 1327–1336 (2016)
37. Hieu, D., Strodiot, J.J., Muu, L.: Modified golden ratio algorithms for solving equilibrium problems (2019). <https://arxiv.org/abs/1907.04013>.
38. Hieu, D.V.: New inertial algorithm for a class of equilibrium problems. *Numer. Algorithms* **80**(4), 1413–1436 (2019)
39. Yang, J., Liu, H., Liu, Z.: Modified subgradient extragradient algorithms for solving monotone variational inequalities. *Optimization* **67**(12), 2247–2258 (2018)
40. Yang, J., Liu, H.: A self-adaptive method for pseudomonotone equilibrium problems and variational inequalities. *Comput. Optim. Appl.* **75**(2), 423–440 (2020)
41. do Carmo, M.: *Riemannian Geometry*. *Mathematics: Theory & Applications*. 300 pp. Birkhäuser Boston, Boston (1992)
42. Sakai, T.: *Riemannian Geometry*. *Translations of Mathematical Monographs*, vol. 149. 358 pp. Am. Math. Soc., Providence (1996)
43. Tang, G.J., Huang, N.J.: Korpelevich’s method for variational inequality problems on Hadamard manifolds. *J. Glob. Optim.* **54**(3), 493–509 (2012)
44. Bridson, M.R., Haefliger, A.: *Metric Spaces of Non-positive Curvature*. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 319. 643 pp. Springer, Berlin (1999)
45. Li, X.B., Zhou, L.W., Huang, N.J.: Gap functions and global error bounds for generalized mixed variational inequalities on Hadamard manifolds. *J. Optim. Theory Appl.* **168**(3), 830–849 (2016)
46. Udriște, C.: *Convex Functions and Optimization Methods on Riemannian Manifolds*. *Mathematics and Its Applications*, vol. 297. 348 pp. Kluwer Academic, Dordrecht (1994)
47. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. *Math. Stud.* **63**(1–4), 123–145 (1994)
48. Mastroeni, G.: On auxiliary principle for equilibrium problems. In: *Equilibrium Problems and Variational Models*, Erice, 2000. *Nonconvex Optim. Appl.*, vol. 68, pp. 289–298. Kluwer Academic, Norwell (2003)
49. Malitsky, Y.: Proximal extrapolated gradient methods for variational inequalities. *Optim. Methods Softw.* **33**(1), 140–164 (2018)
50. Wang, J.H., López, G., Martín-Márquez, V., Li, C.: Monotone and accretive vector fields on Riemannian manifolds. *J. Optim. Theory Appl.* **146**(3), 691–708 (2010)
51. Ansari, Q.H., Islam, M., Yao, J.-C.: Nonsmooth variational inequalities on Hadamard manifolds. *Appl. Anal.* **99**(2), 340–358 (2020)
52. Ansari, Q.H., Babu, F., Yao, J.C.: Regularization of proximal point algorithms in Hadamard manifolds. *J. Fixed Point Theory Appl.* **21**(1), 25 (2019)
53. Facchinei, F., Pang, J.S.: *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Vol. II. Springer Series in Operations Research. 693 pp. Springer, Berlin (2003)
54. Khammahawong, K., Kumam, P., Chaipunya, P., Yao, J.C., Wen, C.F., Jirakitpuwapat, W.: An extragradient algorithm for strongly pseudomonotone equilibrium problems on Hadamard manifolds. *Thai J. Math.* **18**(1), 350–371 (2020)
55. Ansari, Q.H., Babu, F.: Proximal point algorithm for inclusion problems in Hadamard manifolds with applications. *Optim. Lett.* (2019). <https://doi.org/10.1007/s11590-019-01483-0>