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Generalized metric spaces endowed with vector-valued metrics and matrix equations by tripled fixed point theorems

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Abstract
This research focuses on proving the results of tripled fixed point and coincidence point in generalized metric spaces endowed with vector-valued metrics and matrix equations. The results from this study are illustrated by two applications.

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1 Introduction

Perov described the Banach contraction principle for contraction mappings on spaces equipped with vector-valued metrics in [11]. Later, by a different method, the results of Perov in [5] were generalized and their fixed point property of a self-mapping over generalized metric space (X,d) was studied.

In this article $M_{m,m}(\mathbb{R}^+)$ represents the set of all $m \times m$ matrices with components in $\mathbb{R}^+$, Θ represents the matrix zero and I represents the identity matrix and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $A \in M_{m,m}(\mathbb{R}^+)$, then A is called convergent to zero, if and only if $A^n \to 0$ as $n \to \infty$. We refer to [14, 15] for more details.

Let $\alpha, \beta \in \mathbb{R}^m$, where $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m)$ and $c \in \mathbb{R}$. Note that $\alpha_i \leq \beta_i$ (resp. $\alpha_i < \beta_i$) for each $1 \leq i \leq m$ and also $\alpha_i \leq c$ (resp. $\alpha_i < c$) for $1 \leq i \leq m$, respectively. We define

$$\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_m + \beta_m)$$

and

$$\alpha \cdot \beta := (\alpha_1 \cdot \beta_1, \ldots, \alpha_m \cdot \beta_m).$$

These are addition and multiplication on $\mathbb{R}^m$ (see [5, 7, 8]).

Definition 1.1 ([5]) Let X be a non-empty set. A mapping $d : X^2 \longrightarrow \mathbb{R}^m$ is called a vector-valued metric on X, if the following properties hold:

(1) $d(x^1,x^2) \geq 0$ for each $x^1,x^2 \in X$, if $d(x^1,x^2) = 0$, if and only if $x^1 = x^2$;

(2) $d(x^1,x^2) = d(x^2,x^1)$ for each $x^1,x^2 \in X$;

(3) $d(x^1,x^2) \leq d(x^1,x^3) + d(x^3,x^2)$ for each $x^1,x^2,x^3 \in X$.

If $x^1,x^2 \in \mathbb{R}^m, x^1 = (x^1_1,\dots,x^1_m)$ and $x^2 = (x^2_1,\dots,x^2_m)$, then $x^1 \leq x^2$ if and only if $x^1_i \leq x^2_i$ for $1 \leq i \leq m$. A set X is called a *generalized metric space*, equipped with a vector-valued metric d and denoted by (X,d).

Now, we need the following equivalent propositions. Their proofs are classic results in matrix analysis (see for more details [1, 12, 13]).

(1) $A \to 0$;

(2) $A^n \to 0$ as $n \to \infty$;

(3) $|\lambda| < 1$, for each $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$;

(4) the matrix $I - A$ is nonsingular and

$$(I - A)^{-1} = I + A + \cdots + A^n + \cdots;$$

(5) $A^n q \longrightarrow 0$ and $qA^n \longrightarrow 0$ as $n \longrightarrow \infty$, for each $q \in \mathbb{R}^m$.

Denote the set of all matrices $A \in M_{m,m}(\mathbb{R}^+)$ where $A^n \longrightarrow 0$ by $\mathcal{Z}M$. For the sake of simplicity, we identify the row and column vectors in $\mathbb{R}^m$.

Definition 1.2 ([3]) An element $(x^1,x^2) \in X^2$ is called a *coupled fixed point* of the mapping $F : X^2 \longrightarrow X$ if $F(x^1,x^2) = x^1, F(x^2,x^1) = x^2$.

Definition 1.3 ([10]) Suppose that $F : X^2 \longrightarrow X$ and $g : X \longrightarrow X$ are given. An element $(x^1,x^2) \in X^2$ is called a *coupled coincidence point* of the mappings F and g if $F(x^1,x^2) = gx^1$ and $F(x^2,x^1) = gx^2$. Then (gx^1,gx^2) is called a *coupled coincidence point*.

Definition 1.4 ([15]) Let $(X,d,\preceq)$ be a partially ordered complete metric space. We consider partially ordered set X. We define on X^3 the following order, for $(x^1,x^2,x^3),(u^1,u^2,u^3) \in X^3$,

$$\left(u^1,u^2,u^3\right) \preceq \left(x^1,x^2,x^3\right) \quad \Leftrightarrow \quad x^1 \succeq u^1, \qquad x^2 \preceq u^2, \qquad x^3 \succeq u^3.$$

Definition 1.5 ([2]) .Let $(X,\preceq)$ be a partially ordered set and $F : X^3 \to X$. We say that F has the *mixed monotone property* if for any $x^1,x^2,x^3 \in X$

$$\begin{aligned}
&x^1_1,x^1_2 \in X, \qquad x^1_1 \preceq x^1_2 \quad \Rightarrow \quad F\left(x^1_1,x^2,x^3\right) \preceq F\left(x^1_2,x^2,x^3\right),\\
&x^2_1,x^2_2 \in X, \qquad x^2_1 \preceq x^2_2 \quad \Rightarrow \quad F\left(x^1,x^2_1,x^3\right) \succeq F\left(x^1,x^2_2,x^3\right),\\
&x^3_1,x^3_2 \in X, \qquad x^3_1 \preceq x^3_2 \quad \Rightarrow \quad F\left(x^1,x^2,x^3_1\right) \preceq F\left(x^1,x^2,x^3_2\right),
\end{aligned}$$

that is, $F(x^1,x^2,x^3)$ is monotone non-decreasing in x^1 and x^3 and is monotone non-increasing in x^2.

Now, we present a triple fixed point of the second kind that used for mixed monotone mappings (see [9]).

Definition 1.6 ([9]) An element $(x^1,x^2,x^3) \in X^3$ is called a *triple fixed point* of the mapping $F : X^3 \longrightarrow X$ if

$$F(x^1,x^2,x^3) = x^1, \qquad F(x^2,x^1,x^2) = x^2, \qquad F(x^3,x^2,x^1) = x^3.$$

Definition 1.7 ([2]) Let (X,d) be the complete generalized metric space. The mapping $\overline{d} : X^3 \to \mathbb{R}^m$ with

$$\overline{d}\big[(x^1,x^2,x^3),(u^1,u^2,u^3)\big] = d(x^1,u^1) + d(x^2,u^2) + d(x^3,u^3)$$

defines a *metric* on X^3, which, for convenience, we denote by d, too.

Definition 1.8 ([4]) Let $(X,\preceq)$ be a partially ordered set, $F : X^3 \longrightarrow X$ and $g : X \longrightarrow X$ be given. We say F has the *g-mixed monotone property* if for any $x^1,x^2,x^3 \in X$,

$$\begin{aligned}
&x_1^1,x_2^1 \in X, \qquad gx_1^1 \preceq gx_2^1 \quad \Rightarrow \quad F(x_1^1,x^2,x^3) \preceq F(x_2^1,x^2,x^3),\\
&x_1^2,x_2^2 \in X, \qquad gx_1^2 \preceq gx_2^2 \quad \Rightarrow \quad F(x^1,x_1^2,x^3) \succeq F(x^1,x_2^2,x^3),\\
&x_1^3,x_2^3 \in X, \qquad gx_1^3 \preceq gx_2^3 \quad \Rightarrow \quad F(x^1,x^2,x_1^3) \preceq F(x^1,x^2,x_2^3),
\end{aligned}$$

that is, $F(x^1,x^2,x^3)$ is monotone non-decreasing in x^1 and x^3, and monotone non-increasing in x^2.

Definition 1.9 ([15]) Let $F : X^3 \longrightarrow X$ and $g : X \longrightarrow X$ be given. F and g are called *compatible* if

$$\lim_{n\to+\infty} d\big(g(U_{123}),F(V_{123})\big) = 0,$$

where $U_{123} = F(x_n^1,x_n^2,x_n^3)$ and $V_{123} = (gx_n^1,gx_n^2,gx_n^3)$,

$$\lim_{n\to+\infty} d\big(g(U_{212}),F(V_{212})\big) = 0,$$

where $U_{212} = F(x_n^2,x_n^1,x_n^2)$ and $V_{212} = (gx_n^2,gx_n^1,gx_n^2)$,

$$\lim_{n\to+\infty} d\big(g(U_{321}),F(V_{321})\big) = 0,$$

where $U_{321} = F(x_n^3,x_n^2,x_n^1)$ and $V_{321} = (gx_n^3,gx_n^2,gx_n^1)$,
whenever $\{x_n^1\}$, $\{x_n^2\}$, and $\{x_n^3\}$ are sequences in X, such that

$$\begin{aligned}
&\lim_{n\to+\infty} U_{123} = \lim_{n\to+\infty} gx_n^1 = x^1,\\
&\lim_{n\to+\infty} U_{212} = \lim_{n\to+\infty} gx_n^2 = x^2,\\
&\lim_{n\to+\infty} U_{321} = \lim_{n\to+\infty} gx_n^3 = x^3,
\end{aligned}$$

for some $x^1,x^2,x^3 \in X$.

Definition 1.10 ([15]) Let $F : X^3 \longrightarrow X$ and $g : X \longrightarrow X$. The mappings F and g are called *weakly reciprocally continuous* if

$$\lim_{n\to+\infty} g(U_{123}) = gx^1 \quad \text{or} \quad \lim_{n\to+\infty} F(V_{123}) = F\left(x^1,x^2,x^3\right),$$
$$\lim_{n\to+\infty} g(U_{212}) = gx^2 \quad \text{or} \quad \lim_{n\to+\infty} F(V_{212}) = F\left(x^2,x^1,x^2\right),$$
$$\lim_{n\to+\infty} g(U_{321}) = gx^3 \quad \text{or} \quad \lim_{n\to+\infty} F(V_{321}) = F\left(x^3,x^2,x^1\right),$$

whenever $\{x_n^1\}$, $\{x_n^2\}$, and $\{x_n^3\}$ are sequences in X, such that

$$\lim_{n\to+\infty} U_{123} = \lim_{n\to+\infty} gx_n^1 = x^1,$$
$$\lim_{n\to+\infty} U_{212} = \lim_{n\to+\infty} gx_n^2 = x^2,$$
$$\lim_{n\to+\infty} U_{321} = \lim_{n\to+\infty} gx_n^3 = x^3,$$

for some $x^1, x^2, x^3 \in X$.

Definition 1.11 ([15]) Let $F : X^3 \longrightarrow X$ and $g : X \longrightarrow X$. The mappings F and g are called *reciprocally continuous* if

$$\lim_{n\to+\infty} g(U_{123}) = gx^1 \quad \text{and} \quad \lim_{n\to+\infty} F(V_{123}) = F\left(x^1,x^2,x^3\right),$$
$$\lim_{n\to+\infty} g(U_{212}) = gx^2 \quad \text{and} \quad \lim_{n\to+\infty} F(V_{212}) = F\left(x^2,x^1,x^2\right),$$
$$\lim_{n\to+\infty} g(U_{321}) = gx^3 \quad \text{and} \quad \lim_{n\to+\infty} F(V_{321}) = F\left(x^3,x^2,x^1\right),$$

whenever $\{x_n^1\}$, $\{x_n^2\}$, and $\{x_n^3\}$ are sequences in X, such that

$$\lim_{n\to+\infty} U_{123} = \lim_{n\to+\infty} gx_n^1 = x^1,$$
$$\lim_{n\to+\infty} U_{212} = \lim_{n\to+\infty} gx_n^2 = x^2,$$
$$\lim_{n\to+\infty} U_{321} = \lim_{n\to+\infty} gx_n^3 = x^3,$$

for some $x^1, x^2, x^3 \in X$.

Definition 1.12 ([15]) Let $(X, d, \preceq)$ be a partially ordered metric space. We say that X is regular if the following properties hold:

(i) if a non-decreasing sequence $x_n^1 \to x^1$, then $x_n^1 \preceq x^1$ for all $n \geq 0$,
(ii) if a non-increasing sequence $x_n^2 \to x^2$, then $x^2 \preceq x_n^2$ for all $n \geq 0$.

For the main result of this article, we study existence and uniqueness of triple common fixed point for a sequence of mappings $T_n : X^3 \to X$ and $g : X \to X$, where (X, d) is a complete generalized metric space.

First, we have the following two definitions from [6, 15].

Definition 1.13 ([15]) Let (X,d) be a metric space and let $T_n : X^3 \to X$ and $g : X \longrightarrow X$ are given. The sequence $\{T_n\}_{n\in\mathbb{N}_0}$ and the mapping g are said to be *compatible* if

$$\lim_{n\to+\infty} d\big(g\big(U'_{123}\big), T_n\big(V'_{123}\big)\big) = 0,$$

where $U'_{123} = T_n(x_n^1, x_n^2, x_n^3)$ and $V'_{123} = (gx_n^1, gx_n^2, gx_n^3)$

$$\lim_{n\to+\infty} d\big(g\big(U'_{212}\big), T_n\big(V'_{212}\big)\big) = 0,$$

where $U'_{212} = T_n(x_n^2, x_n^1, x_n^2)$ and $V'_{212} = (gx_n^2, gx_n^1, gx_n^2)$

$$\lim_{n\to+\infty} d\big(g\big(U'_{321}\big), T_n\big(V'_{321}\big)\big) = 0,$$

where $U'_{321} = T_n(x_n^3, x_n^2, x_n^1)$ and $V'_{321} = (gx_n^3, gx_n^2, gx_n^1)$, whenever $\{x_n^1\}$, $\{x_n^2\}$, and $\{x_n^3\}$ are sequences in X, such that

$$\lim_{n\to+\infty} U'_{123} = \lim_{n\to+\infty} gx_{n+1}^1 = x^1,$$
$$\lim_{n\to+\infty} U'_{212} = \lim_{n\to+\infty} gx_{n+1}^2 = x^2,$$
$$\lim_{n\to+\infty} U'_{321} = \lim_{n\to+\infty} gx_{n+1}^3 = x^3$$

for some $x^1, x^2, x^3 \in X$.

Definition 1.14 ([15]) Let (X,d) be a metric space and let $T_n : X^3 \to X$ and $g : X \longrightarrow X$ are given. $\{T_n\}_{n\in\mathbb{N}_0}$ and g are called *weakly reciprocally continuous* if

$$\lim_{n\to+\infty} g\big(U'_{123}\big) = gx^1,$$
$$\lim_{n\to+\infty} g\big(U'_{212}\big) = gx^2,$$
$$\lim_{n\to+\infty} g\big(U'_{321}\big) = gx^3,$$

whenever $\{x_n^1\}$, $\{x_n^2\}$, and $\{x_n^3\}$ are sequences in X, such that

$$\lim_{n\to+\infty} U'_{123} = \lim_{n\to+\infty} gx_{n+1}^1 = x^1,$$
$$\lim_{n\to+\infty} U'_{212} = \lim_{n\to+\infty} gx_{n+1}^2 = x^2,$$
$$\lim_{n\to+\infty} U'_{321} = \lim_{n\to+\infty} gx_{n+1}^3 = x^3$$

for some $x^1, x^2, x^3 \in X$.

2 Main results

We start with the following statement, which we will use in the main theorem. Inspired by Definition 1.8 we have the following definition.

Definition 2.1 Let $(X,\preceq)$ be a partially ordered set, $T_n : X^3 \to X, n \in \mathbb{N}_0$, and $g : X \to X$. We say that $\{T_n\}_{n\in\mathbb{N}_0}$ has the *g-mixed monotone property* if for any $x^1,x^2,x^3,x'^1,x'^2,x'^3 \in X$,

$$gx^1 \preceq gx'^1, \qquad gx'^2 \preceq gx^2 \quad \textit{and} \quad gx^3 \preceq gx'^3, \tag{2.1}$$

imply that

$$\begin{aligned} &T_n\bigl(x^1,x^2,x^3\bigr) \preceq T_{n+1}\bigl(x'^1,x'^2,x'^3\bigr), \qquad T_{n+1}\bigl(x'^2,x'^1,x'^2\bigr) \preceq T_n\bigl(x^2,x^1,x^2\bigr) \quad \textit{and} \\ &T_{n+1}\bigl(x^3,x^2,x^1\bigr) \preceq T_n\bigl(x'^3,x'^2,x'^1\bigr). \end{aligned} \tag{2.2}$$

Definition 2.2 Suppose that $T_i : X^3 \to X$ and $g : X \to X$ are given. We say $\{T_i\}_{i\in\mathbb{N}_0}$ and g satisfy the (K) property if

$$\begin{aligned} d(T_i\bigl(x^1,x^2,x^3\bigr), T_j\bigl(u^1,u^2,u^3\bigr) \preceq{}& A\bigl[d\bigl(g\bigl(x^1\bigr), T_i\bigl(x^1,x^2,x^3\bigr)\bigr) \\ &+ d\bigl(gu^1, T_j\bigl(u^1,u^2,u^3\bigr)\bigr)\bigr] \\ &+ B\bigl(d\bigl(gu^1,gx^1\bigr)\bigr) \end{aligned} \tag{2.3}$$

for $x^1,x^2,x^3,u^1,u^2,u^3 \in X$ with $gx^1 \succeq gu^1, gu^2 \succeq gx^2, gx^3 \succeq gu^3$ or $gx^1 \preceq gu^1, gu^2 \preceq gx^2, gx^3 \preceq gu^3$, $I \neq A = (a_{ij}), I \neq B = (b_{ij}) \in M_{m,m}(\mathbb{R}^+)$, $(A+B)(I-A)^{-1} \in \mathcal{Z}M$.

Definition 2.3 If T_0 and g have a non-decreasing transcendence point in x_0^1, x_0^3 and a non-increasing transcendence point in x_0^2, then we say T_0 and g have a *mixed triple transcendence point*, if there exist $x_0^1, x_0^2, x_0^3 \in X$ such that

$$T_0\bigl(x_0^1,x_0^2,x_0^3\bigr) \succeq gx_0^1, \qquad T_0\bigl(x_0^2,x_0^1,x_0^2\bigr) \preceq gx_0^2 \quad \textit{and} \quad T_0\bigl(x_0^3,x_0^1,x_0^2\bigr) \succeq gx_0^3. \tag{2.4}$$

Lemma 2.4 *Let $(X,d,\preceq)$ be a partially ordered complete generalized metric space. Let g be a self-mapping on X and $\{T_i\}_{i\in\mathbb{N}_0}$ be a sequence of mappings from X^3 into X and having a g-mixed monotone property with $T_i(X^3) \subseteq g(X)$. If T_0 and g have a mixed triple transcendence point, then*

(a) *there are sequences $\{x_n^1\}, \{x_n^2\}$ and $\{x_n^3\}$ in X such that*

$$\begin{aligned} &gx_n^1 = T_{n-1}\bigl(x_{n-1}^1,x_{n-1}^2,x_{n-1}^3\bigr), \qquad gx_n^2 = T_{n-1}\bigl(x_{n-1}^2,x_{n-1}^1,x_{n-1}^2\bigr) \quad \textit{and} \\ &gx_n^3 = T_{n-1}\bigl(x_{n-1}^3,x_{n-1}^1,x_{n-1}^2\bigr); \end{aligned}$$

(b) *$\{gx_n^1\}, \{gx_n^3\}$ are non-decreasing sequences and $\{gx_n^2\}$ is a non-increasing sequence.*

Proof (a) By hypothesis, let for $x_0^1, x_0^2, x_0^3 \in X$ the condition (2.4) hold. Since $T_0(X^3) \subseteq g(X)$, we can define $x_1^1, x_1^2, x_1^3 \in X$ such that

$$\begin{aligned} &gx_1^1 = T_0\bigl(x_0^1,x_0^2,x_0^3\bigr), \qquad gx_1^2 = T_0\bigl(x_0^2,x_0^1,x_0^2\bigr) \quad \textit{and} \\ &gx_1^3 = T_0\bigl(x_0^3,x_0^2,x_0^1\bigr). \end{aligned}$$

Again since $T_0(X^3) \subseteq g(X)$, there exist $x_2^1, x_2^2, x_2^3 \in X$ such that

$$gx_2^1 = T_1\bigl(x_1^1,x_1^2,x_1^3\bigr), \qquad gx_2^2 = T_1\bigl(x_1^2,x_1^1,x_1^2\bigr) \quad \textit{and}$$

$$gx_2^3 = T_1\big(x_1^3, x_1^2, x_1^1\big).$$

Continuing this technique, we get

$$\begin{aligned} &gx_n^1 = T_{n-1}\big(x_{n-1}^1, x_{n-1}^2, x_{n-1}^3\big), \qquad gx_n^2 = T_n\big(x_{n-1}^2, x_{n-1}^1, x_{n-1}^2\big) \quad \textit{and} \\ &gx_n^3 = T_n\big(x_{n-1}^3, x_{n-1}^2, x_{n-1}^1\big), \quad \textit{for all } n \geq 0. \end{aligned} \tag{2.5}$$

(b) Now, by mathematical induction, we show that

$$gx_n^1 \preceq gx_{n+1}^1, \qquad gx_n^2 \succeq gx_{n+1}^2 \quad \textit{and} \quad gx_n^3 \preceq gx_{n+1}^3, \tag{2.6}$$

for all $n \geq 0$. To this end, since (2.4) holds, in the light of

$$gx_1^1 = T_0\big(x_0^1, x_0^2, x_0^3\big), \qquad gx_1^2 = T_0\big(x_0^2, x_0^1, x_0^3\big) \quad \textit{and} \quad gx_1^3 = T_0\big(x_0^3, x_0^2, x_0^1\big),$$

we have

$$gx_0^1 \preceq gx_1^1, \qquad gx_0^2 \succeq gx_1^2, \qquad gx_0^3 \preceq gx_1^3,$$

that is, (2.6) holds for $n = 0$. We assume that (2.6) holds for some $n > 0$. Now, by (2.5) and (2.6), the result is achieved. Thus, we are done. □

Before expressing the main theorems, we first give the following examples.

Example 2.5

1. $A = \frac{1}{4}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $B = \frac{1}{6}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ are matrices in $\mathcal{Z}M$. It is easy to see that $(A + B)(I - A)^{-1} \in \mathcal{Z}M$.
2. $A = \left(\begin{smallmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{smallmatrix}\right)$ are matrices in $\mathcal{Z}M$. It is easy to see that $(A + B)(I - A)^{-1} \in \mathcal{Z}M$.
3. Let $A = \alpha I$ and $B = ((I - \alpha)^3 - \alpha)I$ be matrices in $\mathcal{Z}M$. Then for $\alpha = \frac{1}{4}, \frac{1}{5}, \frac{1}{7}, \frac{1}{8}$ it is clear that $(A + B)(I - A)^{-1} \in \mathcal{Z}M$.

Theorem 2.6 *In addition to the conditions of Lemma 2.4, let $g(X) \subseteq X$ be complete, $\{T_i\}_{i\in\mathbb{N}_0}$ and g be compatible, weakly reciprocally continuous, where g is monotonic non-decreasing, continuous, and satisfies the condition (K). If $g(X)$ is regular and A, B are nonzero matrices in $\mathcal{Z}M$, then $\{T_i\}_{i\in\mathbb{N}_0}$ and g have a triple coincidence point.*

Proof Let $\{x_n^1\}, \{x_n^2\}$ *and* $\{x_n^3\}$ be the same sequences which are constructed in Lemma 2.4. By (2.3), we get

$$\begin{aligned} d\big(gx_n^1, gx_{n+1}^1\big) &= d\big(T_{n-1}\big(x_{n-1}^1, x_{n-1}^2, x_{n-1}^3\big), T_n\big(x_n^1, x_n^2, x_n^3\big)\big) \\ &\leq A\big[d\big(gx_{n-1}^1, T_{n-1}\big(x_{n-1}^1, x_{n-1}^2, x_{n-1}^3\big)\big) \\ &\quad + d\big(gx_n^1, T_n\big(x_n^1, x_n^2, x_n^3\big)\big)\big] + B(d\big(gx_n^1, gx_{n-1}^1\big) \\ &= A\big[d\big(gx_{n-1}^1, gx_n^1\big) + d\big(gx_n^1, gx_{n+1}^1\big)\big] \\ &\quad + B(d\big(gx_n^1, gx_{n-1}^1\big). \end{aligned}$$

It follows that

$$d(gx_n^1, gx_{n+1}^1) \preceq (A+B)(I-A)^{-1} d(gx_{n-1}^1, gx_n^1) \tag{2.7}$$

and similarly

$$d(gx_n^2, gx_{n+1}^2) \preceq (A+B)(I-A)^{-1} d(gx_{n-1}^2, gx_n^2) \tag{2.8}$$

and

$$d(gx_n^3, gx_{n+1}^3) \preceq (A+B)(I-A)^{-1} d(gx_{n-1}^3, gx_n^3). \tag{2.9}$$

Adding (2.7)–(2.9), we have

$$\begin{aligned}\delta_n &:= d(gx_n^1, gx_{n+1}^1) + d(gx_n^2, gx_{n+1}^2) + d(gx_n^3, gx_{n+1}^3)\\ &\preceq (A+B)(I-A)^{-1}\big[d(gx_{n-1}^1, gx_n^1) + d(gx_{n-1}^2, gx_n^2)\\ &\quad + d(gx_{n-1}^3, gx_n^3)\big]\\ &= \big((A+B)(I-A)^{-1}\big)\delta_{n-1}.\end{aligned}$$

We set $C = (A+B)(I-A)^{-1}$, for all $n \in \mathbb{N}$, then

$$\Theta \preceq \delta_n \preceq C\delta_{n-1} \preceq C^2\delta_{n-2} \preceq \cdots \preceq C^n\delta_0.$$

Moreover, with repeated use of the triangle inequality and for $p > \Theta$, we get

$$\begin{aligned}&d(gx_n^1, gx_{n+p}^1) + d(gx_n^2, gx_{n+p}^2) + d(gx_n^3, gx_{n+p}^3)\\ &\quad\preceq d(gx_n^1, gx_{n+1}^1) + d(gx_n^2, gx_{n+1}^2) + d(gx_n^3, gx_{n+1}^3)\\ &\qquad + d(gx_{n+1}^1, gx_{n+2}^1) + d(gx_{n+1}^2, gx_{n+2}^2) + d(gx_{n+1}^3, gx_{n+2}^3)\\ &\qquad + \cdots + d(gx_{n+p-1}^1, gx_{n+p}^1) + d(gx_{n+p-1}^2, gx_{n+p}^2)\\ &\qquad + d(gx_{n+p-1}^3, gx_{n+p}^3)\\ &\quad= \delta_n + \delta_{n+1} + \cdots + \delta_{n+p-1}\\ &\quad\preceq (C^n + C^{n+1} + \cdots + C^{n+p-1})\delta_0\\ &\quad\preceq C^n(I + C + \cdots + C^{p-1} + \cdots)\delta_0\\ &\quad= C^n(I-C)^{-1}\delta_0.\end{aligned}$$

We have

$$\begin{aligned}&d(gx_n^1, gx_{n+p}^1) + d(gx_n^2, gx_{n+p}^2) + d(gx_n^3, gx_{n+p}^3)\\ &\quad\preceq \big((A+B)(I-A)^{-1}\big)^n\big(I - (A+B)(I-A)^{-1}\big)^{-1}\delta_0.\end{aligned}$$

Now, taking the limit as $n \to +\infty$, we conclude

$$\lim_{n\to+\infty} d(gx_n^1, gx_{n+p}^1) + d(gx_n^2, gx_{n+p}^2) + d(gx_n^3, gx_{n+p}^3) = 0.$$

This implies that

$$\lim_{n\to+\infty} d(gx_n^1, gx_{n+p}^1) = \lim_{n\to+\infty} d(gx_n^2, gx_{n+p}^2) = \lim_{n\to+\infty} d(gx_n^3, gx_{n+p}^3) = 0.$$

Thus, $\{gx_n^1\}$, $\{gx_n^2\}$ and $\{gx_n^3\}$ are Cauchy sequences in X. Since $g(X)$ is complete, there exists $(x'^1, x'^2, x'^3) \in X^3$, with

$$\lim_{n\to+\infty} \{gx_n^1\} = gx'^1 := x^1, \qquad \lim_{n\to+\infty} \{gy_n\} = gx'^2 := x^2 \quad \textit{and}$$
$$\lim_{n\to+\infty} \{gz_n\} = gx'^3 := x^3.$$

By construction, we have

$$\lim_{n\to+\infty} gx_{n+1}^1 = \lim_{n\to+\infty} T_n(x_n^1, x_n^2, x_n^3) = x^1,$$
$$\lim_{n\to+\infty} gx_{n+1}^2 = \lim_{n\to+\infty} T_n(x_n^2, x_n^1, x_n^2) = x^2,$$

and

$$\lim_{n\to+\infty} gx_{n+1}^3 = \lim_{n\to+\infty} T_n(x_n^3, x_n^2, x_n^1) = x^3.$$

Since $\{T_i\}_{i\in\mathbb{N}_0}$ and g are weakly reciprocally continuous and compatible, we have

$$\lim_{n\to+\infty} T_n(gx_n^1, gx_n^2, gx_n^3) = gx^1,$$
$$\lim_{n\to+\infty} T_n(gx_n^2, gx_n^1, gx_n^2) = gx^2,$$

and

$$\lim_{n\to+\infty} T_n(gx_n^3, gx_n^2, gx_n^1) = gx^3.$$

Since $\{gx_n^1\}$ and $\{gx_n^3\}$ are non-decreasing and $\{gx_n^2\}$ is non-increasing, using the regularity of X, we have $gx_n^1 \preceq x^1, x^2 \preceq gx_n^2$ and $gx_n^3 \preceq x^3$ for all $n \geq 0$. So by (2.3), we get

$$\begin{aligned} d(T_i(x^1,x^2,x^3), T_n(gx_n^1, gx_n^2, gx_n^3)) \leq A\bigl[&d(gx^1, T_i(x^1,x^2,x^3)) \\ &+ d(g(gx_n^1, T_n(gx_n^1, gx_n^2, gx_n^3))\bigr] \\ &+ B(d(g(gx_n^1, gx^1)). \end{aligned}$$

Taking the limit as $n \to +\infty$, we obtain $gx^1 = T_i(x^1,x^2,x^3)$. Similarly, it can be proved that $gx^2 = T_i(x^2,x^1,x^2)$ and $gx^3 = T_i(x^3,x^2,x^1)$. Thus, (x^1,x^2,x^3) is a triple coincidence point of $\{T_i\}_{i\in\mathbb{N}}$ and g. □

If in Theorem 2.6 g is the identity mapping, then we have the following corollary.

Corollary 2.7 *Let $(X, d, \preceq)$ be a partially ordered complete generalized metric space. Let $\{T_i\}_{i\in\mathbb{N}\cup\{0\}}$ be a mixed monotone sequence of mappings from X^3 into X, where $\{T_m\}$ and $\mathrm{Id}: X \to X$ satisfy the (K) property. Also T_0 and Id have a mixed transcendence point. If $g(X)$ is regular, then there exists $(x^1, x^2, x^3) \in X^3$, such that $x^1 = T_i(x^1, x^2, x^3), x^2 = T_i(x^2, x^1, x^2)$, and $x^3 = T_i(x^3, x^2, x^1)$ for $i \in \mathbb{N}_0$.*

Definition 2.8 We say that (x^1, x^2, x^3) is a triple comparable with (u^1, u^2, u^3) if and only if

$$\begin{array}{llll} x^1 \succeq u^1, & x^2 \preceq u^2, & x^3 \succeq u^3 & \textit{or} \\ x^1 \preceq u^1, & x^2 \succeq u^2, & x^3 \preceq u^3 & \textit{or} \\ x^1 \succeq u^2, & x^2 \preceq u^3, & x^3 \succeq u^1 & \textit{or} \\ x^1 \preceq u^2, & x^2 \succeq u^3, & x^3 \preceq u^1 & \textit{or} \\ x^1 \succeq u^3, & x^2 \preceq u^1, & x^3 \succeq u^2 & \textit{or} \\ x^1 \preceq u^3, & x^2 \succeq u^1, & x^3 \preceq u^2. & \end{array}$$

If in the above definition we replace (x^1, x^2, x^3) and (u^1, u^2, u^3) with (gx^1, gx^2, gx^3) and (gu^1, gu^2, gu^3), we call (x^1, x^2, x^3) a triple comparable with (u^1, u^2, u^3) with respect to g.

Theorem 2.9 *Let $(X, d, \preceq)$ be a partially ordered complete generalized metric space. Let g be a self-mapping on X and $\{T_i\}_{i\in\mathbb{N}_0}$ be a sequence of mappings from X^3 into X. Let $\{T_i\}_{i\in\mathbb{N}_0}$ and g satisfy the condition (K) and have triple coincidence points comparable with respect to g, then $\{T_i\}_{i\in\mathbb{N}_0}$ and g have a unique triple common fixed point.*

Proof According to Theorem 2.6, the set of tripled coincidence points is non-empty. First, we show that, if (x^1, x^2, x^3) and (x'^1, x'^2, x'^3) are triple coincidence points, that is, if

$$\begin{aligned} &gx^1 = T_i(x^1, x^2, x^3), \qquad gx^2 = T_i(x^2, x^1, x^2), \qquad gx^3 = T_i(x^3, x^2, x^1), \\ &gx'^1 = T_i(x'^1, x'^2, x'^3), \qquad gx'^2 = T_i(x'^2, x'^1, x'^2), \qquad gx'^3 = T_i(x'^3, x'^2, x'^1), \end{aligned}$$

then $gx^1 = gx'^1, gx^2 = gx'^2$ and $gx^3 = gx'^3$. Since the set of triple coincidence points is a triple comparable, applying condition (2.3) implies

$$\begin{aligned} d(gx^1, gx'^1) &= d\bigl(T_i(x^1, x^2, x^3), T_j(x'^1, x'^2, x'^3)\bigr) \\ &\preceq A\bigl[d\bigl(gx^1, T_i(x^1, x^2, x^3)\bigr) + d\bigl(gx'^1, T_j(x'^1, x'^2, x'^3)\bigr)\bigr] \\ &\quad + Bd(gx'^1, gx^1). \end{aligned}$$

Therefore, as $I \neq B \in \mathcal{Z}M$, $d(gx^1, gx'^1) = \Theta$, that is, $gx^1 = gx'^1$. Similarly, it can be proved that $gx^2 = gx'^2$ and $gx^3 = gx'^3$. So $gx^1 = gx^2 = gx^3 = gx'^1 = gx'^2 = gx'^3$.

Therefore, $\{T_i\}_{i\in\mathbb{N}}$ and g have a unique triple coincidence point (gx^1, gx^1). Since two compatible mappings commute at their coincidence points, thus, clearly, $\{T_i\}_{i\in\mathbb{N}}$ and g have a unique tripled common fixed point whenever $\{T_i\}_{i\in\mathbb{N}}$ and g are weakly compatible. □

Example 2.10 Let $X = [0,1]$. Define

$$d(x^1,x^2) = \begin{pmatrix} |x^1 - x^2| \\ |x^1 - x^2| \end{pmatrix}.$$

Then (X,d) is a partially ordered complete generalized metric space. Define

$$A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}.$$

Because A and B are nonzero matrices in $\mathcal{Z}M$ and considering the mapping $T_i : X^3 \to X$ and $g : X \to X$ with

$$T_i(x^1,x^2,x^3) = \frac{x^1 + x^2 + x^3}{3^i}, \qquad g(x^1) = 9x^1,$$

it can be easily verified by mathematical induction that the inequality (2.3) holds for all $x^1, x^2, x^3 \in X$, that is, we see that the greatest value of the first side happens when $i = 1, j \to \infty$, in this case for $i = 1$ we have

$$\begin{aligned} &\begin{pmatrix} |\frac{x^1+x^2+x^3}{3} - \frac{u^1+u^2+u^3}{3^j}| \\ |\frac{x^1+x^2+x^3}{3} - \frac{u^1+u^2+u^3}{3^j}| \end{pmatrix} \\ &\quad\le \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} |9x^1 - \frac{x^1+x^2+x^3}{3}| + |9u^1 - \frac{u^1+u^2+u^3}{3^j}| \\ |9x^1 - \frac{x^1+x^2+x^3}{3}| + |9u^1 - \frac{u^1+u^2+u^3}{3^j}| \end{pmatrix} \\ &\qquad + \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} |9(u^1 - x^1)| \\ |9(u^1 - x^1)| \end{pmatrix}. \end{aligned}$$

Now for $j = j + 1$ we have

$$\begin{aligned} \alpha &:= \begin{pmatrix} |\frac{x^1+x^2+x^3}{3} - \frac{1}{3}\frac{u^1+u^2+u^3}{3^j}| \\ |\frac{x^1+x^2+x^3}{3} - \frac{1}{3}\frac{u^1+u^2+u^3}{3^j}| \end{pmatrix} \\ &\le \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} |9x^1 - \frac{x^1+x^2+x^3}{3}| + |3u^1 - \frac{1}{3}\frac{u^1+u^2+u^3}{3^j}| \\ |9x^1 - \frac{x^1+x^2+x^3}{3}| + |3u^1 - \frac{1}{3}\frac{u^1+u^2+u^3}{3^j}| \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} |(\frac{u^1}{3} - x^1)| \\ |(\frac{u^1}{3} - x^1)| \end{pmatrix} := \beta. \end{aligned}$$

So

$$\begin{aligned} \alpha &\le \frac{1}{3} \begin{pmatrix} |\frac{x^1+x^2+x^3}{3} - \frac{u^1+u^2+u^3}{3^j}| \\ |\frac{x^1+x^2+x^3}{3} - \frac{u^1+u^2+u^3}{3^j}| \end{pmatrix} + \frac{2}{3} \begin{pmatrix} |\frac{x^1+x^2+x^3}{3}| \\ |\frac{x^1+x^2+x^3}{3}| \end{pmatrix} \\ &\le \frac{1}{3} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} |9x^1 - \frac{x^1+x^2+x^3}{3}| + |9u - \frac{u^1+u^2+u^3}{3^j}| \\ |9x^1 - \frac{x^1+x^2+x^3}{3}| + |9u - \frac{u^1+u^2+u^3}{3^j}| \end{pmatrix} \\ &\quad + \frac{1}{3} \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} |u^1 - x^1| \\ |u^1 - x^1| \end{pmatrix} + \frac{2}{3} \begin{pmatrix} |\frac{x^1+x^2+x^3}{3}| \\ |\frac{x^1+x^2+x^3}{3}| \end{pmatrix} \le \beta. \end{aligned}$$

Thus all the hypotheses of Theorem 2.6 are satisfied and $(0,0,0)$ is the triple coincident point of g and $\{T_i\}_{i\in\mathbb{N}_0}$. Moreover, using the same $\{T_i\}_{i\in\mathbb{N}_0}$ and g in Theorem 2.9, $(0,0,0)$ is the unique triple common fixed point of g and $\{T_i\}_{i\in\mathbb{N}_0}$.

Before explaining the application, it is necessary to provide the following definition, which we will use in Theorem 3.1.

Definition 2.11 Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices in $\mathcal{Z}M$. Then

$$A \le B \quad \Leftrightarrow \quad a_{ij} \le b_{ij}, \quad 1 \le i,j \le m$$
$$\max\{A,B\} = C = (c_{ij}) \quad \textit{where } c_{ij} = \max\{a_{ij}, b_{ij}\}.$$

Clearly if $A \le B$ then $\max\{A,B\} = B$.

3 Application 1

In this part, we will use the results of Sect. 2 to extract some results for the existence and uniqueness of solutions of the integral equations system. Consider the following integral equations system:

$$\begin{aligned} x^1(t) &= \int_0^T \bigl(f\bigl(t,s,x^1(s)\bigr) + g\bigl(t,s,x^2(s)\bigr) + h\bigl(t,s,x^3(s)\bigr)\,ds + v(t), \\ x^2(t) &= \int_0^T \bigl(f\bigl(t,s,x^2(s)\bigr) + g\bigl(t,s,x^3(s)\bigr) + h\bigl(t,s,x^1(s)\bigr)\,ds + v(t), \\ x^3(t) &= \int_0^T \bigl(f\bigl(t,s,x^3(s)\bigr) + g\bigl(t,s,x^1(s)\bigr) + h\bigl(t,s,x^2(s)\bigr)\,ds + v(t), \end{aligned} \tag{3.1}$$

for all $t,s \in [0,T]$, for some $T > 0$.

Let $X = C([0,T],\mathbb{R})$ be continuous real functions, defined on the interval $[0,T]$, endowed with a metric

$$d\bigl(x^1,x^2\bigr) = \begin{pmatrix} \max_{0\le t\le T} |x^1(t) - x^2(t)| \\ \max_{0\le t\le T} |x^1(t) - x^2(t)| \end{pmatrix}.$$

We define the partial order "$\preceq$" on X as follows:

for $x^1,x^2 \in X, x^1 \preceq x^2 \Leftrightarrow x^1(t) \preceq x^2(t)$ for any $t \in [0,T]$.

Thus, $(X,d,\preceq)$ is a partially ordered complete generalized metric space. For (3.1) we consider the following hypotheses:

(i) $f,g,h \in [0,T] \times [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}^2$ are continuous;

(ii) $v : [0,T] \longrightarrow \mathbb{R}$ is continuous;

(iii) there exists $\rho : [0,T] \longrightarrow M_{2\times 2}([0,T])$, such that, for all $x^1,x^2 \in X$,

$$\begin{aligned} 0 &\le \bigl|f\bigl(t,s,x^1(s)\bigr) - f\bigl(t,s,x^2(s)\bigr)\bigr| \le \rho_1(t)d\bigl(x^1,x^2\bigr), \\ 0 &\le \bigl|g\bigl(t,s,x^2(s)\bigr) - g\bigl(t,s,x^1(s)\bigr)\bigr| \le \rho_2(t)d\bigl(x^1,x^2\bigr), \\ 0 &\le \bigl|h\bigl(t,s,x^1(s)\bigr) - h\bigl(t,s,x^2(s)\bigr)\bigr| \le \rho_3(t)d\bigl(x^1,x^2\bigr), \end{aligned} \tag{3.2}$$

for all $s,t \in [0,T]$ with $\rho(t) \le A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ and $\rho(t) \le B = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$. Because A and B are nonzero matrices in $\mathcal{Z}M$;

(iv) we suppose that $\rho_1(t) + \rho_2(t) + \rho_3(t) < 1$ and

$$\rho(t) = \max\{\rho_1(t), \rho_2(t), \rho_3(t)\};$$

(v) there are functions $\alpha, \beta, \gamma : [0,T] \longrightarrow \mathbb{R}$ which are continuous, such that

$$\alpha \le \int_0^T (f(t,s,\alpha(s)) + g(t,s,\beta(s)) + h(t,s,\gamma(s))\,ds + \nu(t),$$
$$\beta \ge \int_0^T (f(t,s,\beta(s)) + g(t,s,\alpha(s)) + h(t,s,\beta(s))\,ds + \nu(t),$$
$$\gamma \le \int_0^T (f(t,s,\gamma(s)) + g(t,s,\beta(s)) + h(t,s,\alpha(s))\,ds + \nu(t).$$

Theorem 3.1 *Under hypotheses* (i)–(v), (3.1) *has a unique solution in* X.

Proof We consider the operator defined by $T_i : X^3 \longrightarrow X$, with

$$\begin{aligned} T(x^1,x^2,x^3) &= T_i(x^1,x^2,x^3) \\ &= \int_0^T (f(t,s,x^1(s)) + g(t,s,x^2(s)) + h(t,s,x^3(s))\,ds + \nu(t), \end{aligned}$$

for any $x^1, x^2, x^3 \in X$ and $t,s \in [0,T]$.

We prove that the operator $\{T_i\}_{i\in\mathbb{N}}$ fulfills the conditions of Corollary 2.7. First, we show that $\{T_i\}_{i\in\mathbb{N}}$ has the mixed monotone property. Let $x^1, u^1 \in X$ with $x^1 \le u^1$ and $t,s \in [0,T]$, then we have

$$T_i(u^1,x^2,x^3)(t) - T_i(x^1,x^2,x^3)(t) = \int_0^T (f(t,s,u^1(s)) - f(t,s,x^1(s))\,ds.$$

Given that $x^1(t) \le u^1(t)$ for all $t \in [0,T]$ and based on our assumption (3.2), we have

$$T_i(u^1,x^2,x^3)(t) - T_i(x^1,x^2,x^3)(t) \ge 0,$$

that is, $T_i(u^1,x^2,x^3)(t) \ge T_i(x^1,x^2,x^3)(t)$. For $x^2, u^2 \in X$ with $x^2 \le u^2$ and $t,s \in [0,T]$, then we have

$$T_i(x^1,x^2,x^3)(t) - T_i(x^1,u^2,x^3)(t) = \int_0^T (f(t,s,x^2(s)) - f(t,s,u^2(s))\,ds.$$

Given that $x^2(t) \le u^2(t)$ for all $t \in [0,T]$ and based on our assumption (3.2), we have

$$T_i(x^1,x^2,x^3)(t) - T_i(x^1,u^2,x^3)(t) \le 0,$$

that is, $T_i(x^1,x^2,x^3)(t) \ge T_i(x^1,u^2,x^3)(t)$. Similarly, we have

$$T_i(x^1,x^2,u^3)(t) - T_i(x^1,u^2,x^3)(t) \ge 0,$$

that is, $T_i(x^1,x^2,x^3)(t) \leq T_i(x^1,x^2,u^3)(t)$. So, $\{T_i\}_{i\in\mathbb{N}}$ has the mixed monotone property. Now, we estimate $d(T_i(x^1,x^2,x^3),T_j(u^1,u^2,u^3))$ for $x^1 \preceq u^1, u^2 \preceq x^2, x^3 \preceq u^3$ or $x^1 \succeq u^1, u^2 \succeq x^2, x^3 \succeq u^3$ and with $\{T_i\}_{i\in\mathbb{N}}$ having the mixed monotone property, we get

$$\begin{aligned} &d\bigl(T_i\bigl(x^1,x^2,x^3\bigr),T_j\bigl(u^1,u^2,u^3\bigr)\bigr) \\ &\quad = \begin{pmatrix} \max_{0\leq t\leq T} |T_i(x^1,x^2,x^3)(t) - T_j(u^1,u^2,u^3)(t)| \\ \max_{0\leq t\leq T} |T_i(x^1,x^2,x^3)(t) - T_j(u^1,u^2,u^3)(t)| \end{pmatrix}. \end{aligned}$$

Now, for all $t \in [0,T]$ by using (3.2), we have

$$\begin{aligned} &\bigl|T_i\bigl(x^1,x^2,x^3\bigr)(t) - T_j\bigl(u^1,u^2,u^3\bigr)(t)\bigr| \\ &\quad = \biggl|\int_0^T (f\bigl(t,s,x^1(s)\bigr) + g\bigl(t,s,x^2(s)\bigr) + h\bigl(t,s,x^3(s)\bigr)ds \\ &\qquad - \int_0^T (f\bigl(t,s,u^1(s)\bigr) + g\bigl(t,s,u^2(s)\bigr) + h\bigl(t,s,u^3(s)\bigr)\,ds\biggr| \\ &\quad \leq \int_0^T \bigl|(f\bigl(t,s,x^1(s)\bigr) - f\bigl(t,s,u^1(s)\bigr)\bigr|\,ds \\ &\qquad + \int_0^T \bigl|(g\bigl(t,s,x^2(s)\bigr) - g\bigl(t,s,u^2(s)\bigr)\bigr|\,ds \\ &\qquad + \int_0^T \bigl|(h\bigl(t,s,x^3(s)\bigr) - h\bigl(t,s,u^3(s)\bigr)\bigr|\,ds \\ &\quad \leq \rho_1(t)d\bigl(x^1,u^1\bigr) + \rho_2(t)d\bigl(x^2,u^2\bigr) + \rho_3(t)d\bigl(x^3,u^3\bigr) \\ &\quad \leq \rho(t)\bigl(d\bigl(x^1,u^1\bigr) + d\bigl(x^2,u^2\bigr) + d\bigl(x^3,u^3\bigr)\bigr) \\ &\quad \leq B\bigl(d\bigl(x^1,u^1\bigr) + d\bigl(x^2,u^2\bigr) + d\bigl(x^3,u^3\bigr)\bigr). \end{aligned}$$

Consequently,

$$\begin{aligned} d\bigl(T_i\bigl(x^1,x^2,x^3\bigr),T_j\bigl(u^1,u^2,u^3\bigr)\bigr) &\leq B\begin{pmatrix} d(x^1,u^1) + d(x^2,u^2) + d(x^3,u^3) \\ d(x^1,u^1) + d(x^2,u^2) + d(x^3,u^3) \end{pmatrix} \\ &\leq A\bigl[d\bigl(x^1,T_i\bigl(x^1,x^2,x^3\bigr)\bigr) + d\bigl(u^1,T_j\bigl(u^1,u^2,u^3\bigr)\bigr)\bigr] \\ &\quad + Bd\bigl(u^1,x^1\bigr). \end{aligned}$$

Let α,β,γ be the same as (v); then we have

$$\alpha \leq T_i(\alpha,\beta,\gamma), \qquad \beta \geq T_i(\beta,\alpha,\beta), \qquad \gamma \leq T_i(\gamma,\beta,\alpha).$$

If $x_0^1 = \alpha, x_0^2 = \beta, x_0^3 = \gamma$, then all assumptions of Corollary 2.7 are fulfilled. So, there exists a triple fixed point (x^1,x^2,x^3) for the operator $\{T_i\}_{i\in\mathbb{N}}$; that is, $T_i(x^1,x^2,x^3) = x^1$, $T_i(x^2,x^1,x^2) = x^2$, and $T_i(x^3,x^2,x^1) = x^3$ for $i \in \mathbb{N}$. □

4 Application 2

Now if we consider the sequence of the integral equations system below, in which

$$
\begin{aligned}
x^1(t) &= \int_0^T \big(f_i\big(t,s,x^1(s)\big) + g_i\big(t,s,x^2(s)\big) + h_i\big(t,s,x^3(s)\big)\,ds + \nu(t),\\
x^2(t) &= \int_0^T \big(f_i\big(t,s,x^2(s)\big) + g_i\big(t,s,x^3(s)\big) + h_i\big(t,s,x^1(s)\big)\,ds + \nu(t),\\
x^3(t) &= \int_0^T \big(f_i\big(t,s,x^3(s)\big) + g_i\big(t,s,x^1(s)\big) + h_i\big(t,s,x^2(s)\big)\,ds + \nu(t),
\end{aligned}
\tag{4.1}
$$

for all $t, s \in [0, T]$, for some $T > 0$, then, similar to Theorem 3.1, this sequence of the integral equations system with the conditions given below will have a simultaneous solution.

Let $X = C([0, T], \mathbb{R})$ be equipped with metric defined in Sect. 3 and "$\preceq$" be the partial order on X. Thus, $(X, d, \preceq)$ is a partially ordered complete generalized metric space. For (4.1) we consider the following hypotheses:

(i) $f_i, g_i, h_i \in [0, T] \times [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}^2$ are continuous;

(ii) $\nu : [0, T] \longrightarrow \mathbb{R}$ is continuous;

(iii) there exists $\rho : [0, T] \longrightarrow M_{2\times 2}([0, T])$, such that, for all $x^1, x^2 \in X$, we have

$$
\begin{aligned}
&0 \le \big|f_i\big(t,s,x^1(s)\big) - f_i\big(t,s,x^2(s)\big)\big| \le \rho_1(t) d\big(x^1, x^2\big),\\
&0 \le \big|g_i\big(t,s,x^2(s)\big) - g_i\big(t,s,x^1(s)\big)\big| \le \rho_2(t) d\big(x^1, x^2\big),\\
&0 \le \big|h_i\big(t,s,x^1(s)\big) - h_i\big(t,s,x^2(s)\big)\big| \le \rho_3(t) d\big(x^1, x^2\big),
\end{aligned}
\tag{4.2}
$$

for all $s, t \in [0, T]$ with $\rho(t) \le A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ and $\rho(t) \le B = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$;

(iv) we suppose that $\rho_1(t) + \rho_2(t) + \rho_3(t) < 1$ and

$$
\rho(t) = \max\big\{\rho_1(t), \rho_2(t), \rho_3(t)\big\};
$$

(v) there are functions $\alpha, \beta, \gamma : [0, T] \longrightarrow \mathbb{R}$ which are continuous, such that

$$
\begin{aligned}
\alpha &\le \int_0^T \big(f_i\big(t,s,\alpha(s)\big) + g_i\big(t,s,\beta(s)\big) + h_i\big(t,s,\gamma(s)\big)\,ds + \nu(t),\\
\beta &\ge \int_0^T \big(f_i\big(t,s,\beta(s)\big) + g_i\big(t,s,\alpha(s)\big) + h_i\big(t,s,\beta(s)\big)\,ds + \nu(t),\\
\gamma &\le \int_0^T \big(f_i\big(t,s,\gamma(s)\big) + g_i\big(t,s,\beta(s)\big) + h_i\big(t,s,\alpha(s)\big)\,ds + \nu(t).
\end{aligned}
$$

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