q-Analogues and properties of the Laplace-type integral operator in the quantum calculus theory

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Abstract

In this paper, we discuss the q-Laplace-type integral operator on certain class of special functions. We propose q-analogues and obtain results involving polynomials of even orders and functions of q-trigonometric types. Moreover, we establish results related to q-hyperbolic functions and certain q-differential operators. Relying on the given q-differentiation formulas, we finally derive the nth derivative of the q-Laplace-type integral and attain formulas including q-convolution products.

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1 Introduction

The subject of quantum calculus is known as the calculus without limits; it has earned an eminent reputation and publicity due to its vast applications in mathematics, statistics and physics. Much of the theory of the quantum calculus relies on the Jackson q-derivatives and q-integrals, which replace the classical derivative by a difference operator, which allows one to deal with sets of non-differentiable functions. Recently, this area has been stimulated to grow rapidly by many researchers and a variety of new results can be found in Refs. [1–16] and the references cited therein. By fixing a real number q such that $0 < q < 1$, the q-derivative of a differentiable function ϑ is defined by [1]

$$D_q \vartheta(x) = \frac{\vartheta(x) - \vartheta(qx)}{(1-q)x} \quad (x \neq 0).$$

The q-integrals from 0 to y and from 0 to ∞ are, respectively, defined by [1]

$$\int_0^y \vartheta(x)\, d_q x = (1-q)y \sum_{n=0}^{\infty} \vartheta(yq^n) q^n \tag{1}$$

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and

$$\int_0^\infty \vartheta(x)\,d_q x = (1-q) \sum_{n=-\infty}^\infty \vartheta\left(q^n\right) q^n, \tag{2}$$

provided their respective series converge absolutely. The q-integration by parts is given by [16]

$$\int_a^b g(x) D_q \vartheta(x)\,d_q x = \vartheta(b)g(b) - \vartheta(a)g(a) - \int_a^b \vartheta(qx) D_q g(x)\,d_q x. \tag{3}$$

In the literature, there have been found two types of q-analogues of the exponential function introduced as [16–18]

$$E_q(x) = \sum_{n=0}^\infty q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!} \quad (x \in \mathbb{C}) \tag{4}$$

and

$$e_q(x) = \sum_{n=0}^\infty \frac{x^n}{[n]_q!} \quad \left(|x| < |1-q|^{-1}\right), \tag{5}$$

where $[n]_q = q^{n-1} + \cdots + q + 1$ and $[n]_q! = [n]_q \cdots [n]_q$ are the q-analogues of the integer and its factorial, respectively. The q-analogues of the gamma function are also given by [5]

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q(-qx)\,d_q x \quad \text{and} \quad \hat{\Gamma}_q(t) = \int_0^\infty x^{t-1} e_q(-x)\,d_q x. \tag{6}$$

The useful properties of the q-gamma functions Γ_q and $\hat{\Gamma}_q$ are obtained in the literature as follows.

Theorem 1 *Let $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Then the following identities hold*:

(i) $\quad \Gamma_q(t+1) = [t]_q \Gamma_q(t) \quad and \quad \Gamma_q(n+1) = [n]_q!, \qquad$ (ii) $\quad \Gamma_q(t+1) = \dfrac{1-q^t}{1-q} \Gamma_q(t),$

(iii) $\quad \hat{\Gamma}_q(t+1) = q^{-t}[t]_q \hat{\Gamma}_q(t) \quad and \quad \hat{\Gamma}_q(1) = 1, \qquad$ (iv) $\quad \hat{\Gamma}_q(n) = q^{\frac{-n(n-1)}{2}} \Gamma_q(n).$

The q-analogues of the trigonometric functions $\sin x$ and $\cos x$ are given by (see, e.g., [4])

$$\sin_q(at) = \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{n(n+1)}{2}}}{[2n+1]_q!} a^{2n+1} t^{2n+1}, \qquad \cos_q(at) = \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{n(n-1)}{2}}}{[2n]_q!} a^{2n} t^{2n},$$

$$\mathrm{Sin}_q(at) = \sum_{n=0}^\infty \frac{(-1)^n}{[2n+1]_q!} a^{2n+1} t^{2n+1} \quad \text{and} \quad \mathrm{Cos}_q(at) = \sum_{n=0}^\infty \frac{(-1)^n}{[2n]_q!} a^{2n} t^{2n}.$$

This paper is organized as follows. In Sect. 1, we present some preliminaries and notations that are very useful in the sequel. In Sect. 2, we apply the q-analogues of the Laplace-type integral operator to certain polynomials and functions of special-types. In Sect. 3,

we apply the q-analogue $F_{2,q}$ to a certain class of differential operators of arbitrary order. In Sect. 4, we discuss the convolution product and establish a convolution theorem of the $F_{2,q}$ integral operator.

2 $F_{2,q}$ and $S_{2,q}$ analogues of some q-special functions

The Laplace integral operator, among various integral operators, is the most popular and widely used in several branches of engineering sciences and applied mathematics. The Laplace-type integral has been firstly defined by Yürekli and Sadek [19] and extended to a space of generalized functions by Al-Omari [20]. In [4], Ucar et al. have given the q-analogues of the Laplace-type integral operator of some elementary functions by making a free use of the identities of the q-Laplace integral operator. One of the goals of this paper is to provide the q-analogues of the Laplace-type integral operator, with a different approach, and derive results involving elementary functions and some other difference operators. The Laplace-type integral operator [19, (1.4)]

$$L_2\big(\vartheta(x);y\big) = \int_0^\infty x\exp\big(-x^2y^2\big)\vartheta(x)\,dx$$

has a close relation with the familiar Laplace integral operator given as

$$L_2\big(\vartheta(x);y\big) = \frac{1}{2}L\big(\vartheta(\sqrt{x});y^2\big).$$

The q-analogues of the Laplace-type integral operator were introduced by [4]

$$_qL_2\big(\vartheta(x);y\big) = \frac{1}{1-q^2}\int_0^{y^{-1}} xE_{q^2}\big(q^2y^2x^2\big)\vartheta(x)\,d_qx \tag{7}$$

and

$$_ql_2\big(\vartheta(x);y\big) = \frac{1}{1-q^2}\int_0^\infty xe_{q^2}\big(-y^2x^2\big)\vartheta(x)\,d_qx, \tag{8}$$

provided $\mathrm{Re}(y) > 0$. Here, we introduce two q-analogues of the Laplace-type integral operator in the standard way as

$$F_{2,q}\big(\vartheta(x);y\big) = \int_0^\infty x\vartheta(x)E_q\big(-qx^2y^2\big)\,d_qx \tag{9}$$

and

$$S_{2,q}\big(\vartheta(x);y\big) = \int_0^\infty x\vartheta(x)e_q\big(-x^2y^2\big)\,d_qx, \tag{10}$$

provided $\mathrm{Re}(y) > 0$. In what follows, we make a use of Eq. (9) and Eq. (10) and provide a summary of some results related to the $F_{2,q}$ and $S_{2,q}$ analogues in the course of the following theorems.

Theorem 2 *Let δ be a real number. Then the following hold*:

$$(\mathrm{i}) \quad F_{2,q}\big(x^{2\delta};y\big) = \frac{\Gamma_q(\delta+1)}{[2]_q y^{2\delta+2}}, \qquad (\mathrm{ii}) \quad S_{2,q}\big(x^{2\delta};y\big) = \frac{\hat{\Gamma}_q(\delta+1)}{[2]_q y^{2\delta+2}}.$$

Proof By aid of Eq. (9), we proceed by

$$F_{2,q}\left(x^{2\delta};y\right) = \int_0^\infty x^{2\delta} E_q\left(-qx^2y^2\right) d_q x.$$

Therefore, by the change of variables $x^2 y^2 = z$ we obtain

$$F_{2,q}\left(x^{2\delta};y\right) = \frac{1}{[2]_q y^{2\delta+2}} \int_0^\infty z^\delta E_q(-qz)\, d_q z.$$

Hence, the proof of the first part (i) of the theorem follows from Eq. (6). The proof of the second part (ii) follows by similar techniques. The proof of the theorem is completed. □

Consequently, we state without proof the following straightforward corollary.

Corollary 3 *The following identities hold*:

(i) $F_{2,q}\left(x^{2n};y\right) = \dfrac{[n]!}{[2]_q y^{2n+2}}$,   (ii) $S_{2,q}\left(x^{2n};y\right) = q^{-n\frac{(n-1)}{2}} \dfrac{[n]_q!}{[2]_q y^{2n+2}}$.

Theorem 4 *Let δ be a positive real number. Then the following identities hold*:

(i) $F_{2,q}\left(e_q(\delta x^2);y\right) = \dfrac{1}{[2]_q(y^2-\delta)}$ $\left(|\delta| < y^2\right)$,

(ii) $S_{2,q}\left(e_q(\delta x^2);y\right) = \displaystyle\sum_{n=0}^\infty \frac{\delta^n}{[2]_q y^{2n+2} - q^{-n(n+1)}}$,

(iii) $F_{2,q}\left(E_q(\delta x^2);y\right) = \displaystyle\sum_{n=0}^\infty \frac{q^{\frac{n(n-1)}{2}}\delta^n}{[2]_q y^{2n+2}}$,

(iv) $S_{2,q}\left(E_q(\delta x^2);y\right) = \displaystyle\sum_{n=0}^\infty \frac{-q^n}{[2]_q[n]_q y^{2n+2}}\delta^n$.

Proof Let δ be a positive real number. Then, by employing Eq. (9) and Eq. (5) we write

$$F_{2,q}\left(e_q(\delta x^2);y\right) = \sum_{n=0}^\infty \frac{\delta^n}{[n]_q!} \int_0^\infty x^{2n+1} E_q\left(-qx^2y^2\right) d_q x.$$

By using the change of variables $z = x^2 y^2$, the above equation yields

$$F_{2,q}\left(e_q(\delta x^2);y\right) = \sum_{n=0}^\infty \frac{\delta^n}{[n]_q!} \int_0^\infty \frac{z^n}{[2]_q y^{2n+2}} E_q(-qz)\, d_q z.$$

Hence, by invoking the definition of the gamma function given by Eq. (6), we obtain

$$F_{2,q}\left(e_q(\delta x^2);y\right) = \sum_{n=0}^\infty \frac{\delta^n}{[n]_q!} \frac{1}{[2]_q y^{2n+2}} \Gamma_q(n+1). \tag{11}$$

Therefore, by applying Theorem 1, Eq. (11) leads to the geometric series

$$F_{2,q}\big(e_q(\delta x^2);y\big) = \sum_{n=0}^{\infty} \frac{\delta^n}{[2]_q y^{2n+2}} = \frac{1}{[2]_q(y^2 - \delta)} \quad \big(|\delta| < y^2\big).$$

This proves the first part of the theorem. By following a like approach, we get

$$S_{2,q}\big(e_q(\delta x^2);y\big) = \sum_{n=0}^{\infty} \frac{\delta^n}{[2]_q y^{2n+2}[n]_q!} \hat{\Gamma}_q(n+1).$$

This, indeed, proves the second part of the theorem. The proof of the third part (iii) follows from a similar technique. To establish the fourth part (iv) of the theorem, we make use of the definition of the q-analogue E_q to write

$$F_{2,q}\big(E_q(\delta x^2);y\big) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{[n]_q!} \delta^n \int_0^{\infty} x^{2x+1} e_q\big(-x^2 y^2\big)\, d_q x.$$

By applying the change of variables $z = x^2 y^2$, our previous equation becomes

$$F_{2,q}\big(E_q(\delta x^2);y\big) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{[n]_q!} \frac{\delta^n}{[2]_q y^{2n+2}} \hat{\Gamma}_q(n+1).$$

Hence, by using Part (iv) of Theorem 1, the equation can be put into the form

$$F_{2,q}\big(E_q(\delta x^2);y\big) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} \delta^n}{[n]_q![2]_q y^{2n+2}} q^{\frac{-n(n+1)}{2}} \Gamma_q(n).$$

Therefore, simple computations yield

$$F_{2,q}\big(E_q(\delta x^2);y\big) = \sum_{n=0}^{\infty} \frac{-q^n}{[2]_q [n]_q y^{2n+2}} \delta^n.$$

Hence, the proof of this theorem is therefore completed. □

Theorem 5 *Let δ be a positive real number. Then the following identities hold*:

(i) $\quad F_{2,q}\big(\mathrm{Cos}_q(\delta x^2);y\big) = \dfrac{y^2}{[2]_q(y^4 + \delta^2)} \quad \big(\delta^2 < y^4\big),$

(ii) $\quad S_{2,q}\big(\mathrm{Cos}_q(\delta x^2);y\big) = \dfrac{1}{[2]_q y^2} \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n-1)}{2}} \dfrac{\delta^{2n}}{y^{4n}},$

(iii) $\quad F_{2,q}\big(\cos_q(\delta x^2);y\big) = \dfrac{1}{[2]_q y^2} \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \left(\dfrac{-\delta^2}{y^4}\right)^n,$

(iv) $\quad S_{2,q}\big(\cos_q(\delta x^2);y\big) = \sum_{n=0}^{\infty} (-1)^n \dfrac{q^{n(\frac{n-1}{2})}}{[2]_q y^{4n+2}} \quad (2n > -1).$

Proof Let δ be a positive real number. Then, by the definition of $F_{2,q}$ and the definition of Cos_q, we write

$$F_{2,q}\big(\text{Cos}_q\big(\delta x^2\big); y\big) = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_q!} \delta^{2n} \int_0^{\infty} x^{4n+1} E_q\big(-qx^2y^2\big) \, d_q x.$$

By making the change of variables $z = x^2 y^2$ we derive

$$F_{2,q}\big(\text{Cos}_q\big(\delta x^2\big); y\big) = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_q!} \frac{\delta^{2n}}{[2]_q y^{4n+2}} \int_0^{\infty} z^{2n} E_q(-qz) \, d_q z$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_q!} \frac{\delta \, \Gamma_q(2n+1)}{[2]_q y^{4n+2}}. \tag{12}$$

Consequently, by employing Theorem 1, Eq. (12) turns out to be in the form of the geometric series

$$F_{2,q}\big(\text{Cos}_q\big(\delta x^2\big); y\big) = \frac{1}{[2]_q y^2} \sum_{n=0}^{\infty} \left(\frac{-\delta^2}{y^4}\right)^n.$$

Therefore, the convergence condition of the geometric series shows that

$$F_{2,q}\big(\text{Cos}_q\big(\delta x^2\big); y\big) = \frac{y^2}{[2]_q(y^2 + \delta^2)} \quad \big(\delta^2 < y^4\big).$$

The proof of Part (i) is therefore finished. The proof of the Part (ii) follows from Theorem 1, Part (v) and a similar argument to that employed in the first part. Once again, an argument similar to the argument we have employed for Part (i) establishes Part (iii). Finally, Part (iv) is a straightforward result of Theorem 2. This finishes the proof of the theorem.   □

A similar statement to the statement of Theorem 5 can be read as follows.

Theorem 6 *Let δ be a positive real number. Then the following identities hold true*:

(i) $F_{2,q}\big(\text{Sin}_q\big(\delta x^2\big); y\big) = \dfrac{\delta}{[2]_q(y^4 + \delta^2)} \quad \big(\delta^2 < y^4\big),$

(ii) $S_{2,q}\big(\text{Sin}_q\big(\delta x^2\big); y\big) = \dfrac{\delta}{[2]_q y^4} \sum_{n=0}^{\infty} (-1)^n q^{\frac{-n(n-1)}{2}} \dfrac{\delta^{2n}}{y^{4n}},$

(iii) $F_{2,q}\big(\sin_q\big(\delta x^2\big); y\big) = \sum_{n=0}^{\infty} (-1)^n q^{\frac{-n(n-1)}{2}} \dfrac{\delta^{2n+1}}{[2]_q} y^{-4n+4},$

(iv) $S_{2,q}\big(\sin_q\big(\delta x^2\big); y\big) = \sum_{n=0}^{\infty} (-1)^n \dfrac{q^n}{[2n+1]_q} \dfrac{y^{-4n-4}}{[2]_q} \quad (2n > -1).$

The proof of this theorem follows from definitions and a similar argument to that we already checked for Theorem 5. Details are therefore omitted.

Now, by following the usual notations of [16] and the facts that

$$\cosh_q x = \frac{e_q(x) + e_q(-x)}{2} \quad \text{and} \quad \sinh_q x = \frac{e_q(x) - e_q(-x)}{2} \tag{13}$$

we state the following straightforward corollary.

Corollary 7 *Let δ be positive real number. Then we have*

(i)　$F_{2,q}\big(\cosh_q\big(\delta x^2\big); y\big) = \dfrac{y^2}{[2]_q(y^2 - \delta^2)},$　　(ii)　$F_{2,q}\big(\sinh_q\big(\delta x^2\big); y\big) = \dfrac{\delta}{[2]_q(y^4 - \delta^2)}.$

The proof of this corollary directly follows from Eq. (13), Theorems 5 and 6. Details are omitted.

3 $F_{2,q}$ of q-differential operators

We devote this section to computations related to the $F_{2,q}$ integral and some differential operators. First of all, we derive the following theorem.

Theorem 8 *Let $y > 0$. Then we have*

$$D_{x,q}E_q\big(-qx^2y^2\big) = -xy^2 \sum_{n=0}^{\infty}(-1)^n q^{\frac{(n+1)n}{2}}\big(1 + q^{n+1}\big)y^{2n}x^{2n}. \tag{14}$$

Proof By Eq. (4), we write

$$D_{x,q}E_q\big(-qx^2y^2\big) = \sum_{n=1}^{\infty}(-1)^n \frac{q^{\frac{(n-1)n}{2}}}{[n]_q!}[2n]_q q^n y^{2n}x^{2n-1}.$$

But simple computations then yield

$$[2n]_q = \frac{1 - q^{2n}}{1 - q} = \frac{1 - (q^n)^2}{1 - q} = [n]_q\big(1 + q^n\big).$$

Hence, shifting the lower bound of the above summation implies

$$D_{x,q}E_q\big(-qx^2y^2\big) = \sum_{n=0}^{\infty}(-1)^{n+1} \frac{q^{\frac{(n-1)n}{2}}}{[n]_q!}\big(1 + q^{n+1}\big)y^{2n+2}x^{2n+1}$$

$$= -xy^2 \sum_{n=0}^{\infty}(-1)^n q^{\frac{(n+1)n}{2}}\big(1 + q^{n+1}\big)y^{2n}x^{2n}.$$

The proof of the theorem is therefore finished.　　　　　　　　　　　　　　□

Theorem 9 *Let $\Delta_q(x) = \frac{1}{x}D_{x,q}$. Then we have*

$$F_{2,q}(\Delta_q\vartheta; y) = -\vartheta(0) + \frac{1 + q}{q^2}y^2 F_{2,q}\bigg(\vartheta; \frac{y}{q}\bigg).$$

Proof By the q-integration by parts given by Eq. (3), we write

$$
\begin{aligned}
F_{2,q}(\Delta_q \vartheta; y) &= \int_0^\infty D_{x,q} \vartheta(x) e_q(-qx^2 y^2) \, d_q x \\
&= -f(0) - \int_0^\infty \vartheta(qx) D_q E_q(-qx^2 y^2) \, d_q x.
\end{aligned}
\tag{15}
$$

Hence, by applying Theorem 8, Eq. (15) can be expressed as

$$
F_{2,q}(\Delta_q \vartheta; y) = -\vartheta(0) + y^2 \int_0^\infty x\vartheta(qx) \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{(n+1)n}{2}}}{[n]_q!} (1 + q^{n+1}) y^{2x} x^{2x} \, d_q x.
\tag{16}
$$

Therefore, by changing the variables as $qx = z$ $(d_q x = \frac{1}{q} d_q z)$, we transfer Eq. (16) into the form

$$
\begin{aligned}
F_{2,q}(\Delta_q \vartheta; y) &= -\vartheta(0) + y^2 \int_0^\infty q^{-1} z\vartheta(z) \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{(n+1)n}{2}}}{[n]_q!} (1 + q^{n+1}) y^{2n} z^{2n} q^{-1} \, d_q z \\
&= -\vartheta(0) + y^2 + q^{-2} y^2 \int_0^\infty z\vartheta(z) \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{(n+1)n}{2}}}{[n]_q!} y^{2n} q^{-2n} \, d_q z \\
&\quad + q^{-2} y^2 \int_0^\infty z\vartheta(z) \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{(n+1)n}{2}}}{[n]_q!} q^{n+1} y^{2n} q^{-2n} z^{2n} \, d_q z.
\end{aligned}
$$

Thus, by multiplying the previous equation by $q^{-n} q^n$, we obtain

$$
\begin{aligned}
F_{2,q}(\Delta_q \vartheta; y) &= -\vartheta(0) + q^{-2} y^2 \int_0^\infty z\vartheta(z) \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{(n+1)n}{2}}}{[n]_q!} q^{-3n} y^{2n} q^n z^{2n} \, d_q z \\
&\quad + q^{-2} y^2 \int_0^\infty z\vartheta(z) \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{(n+1)n}{2}}}{[n]_q!} q^{n+1} q^{-3n} y^{2n} q^n z^{2n} \, d_q z.
\end{aligned}
$$

Thus, this equation can be nicely expressed as

$$
\begin{aligned}
F_{2,q}(\Delta_q \vartheta; y) &= -\vartheta(0) + q^{-2} y^2 \int_0^\infty z\vartheta(z) \sum_{n=0}^\infty \frac{(-1)^n}{[n]_q!} q^{\frac{(n-1)n}{2}} \left(\frac{y^2}{q^2}\right)^n q^n z^{2n} \, d_q z \\
&\quad + q^{-2} y^2 \int_0^\infty z\vartheta(z) \sum_{n=0}^\infty \frac{(-1)^n}{[n]_q!} q^{\frac{(n-1)n}{2}} \cdot \left(\frac{y^2}{q^2}\right)^n q^n z^{2n} q \, d_q z.
\end{aligned}
$$

Finally, the preceding equation can be written as

$$
\begin{aligned}
F_{2,q}(\Delta_q \vartheta; y) &= -\vartheta(0) + q^{-2} y^2 \int_0^\infty z\vartheta(z) E_q\left(-q\left(\frac{y}{q}\right)^{2n} z^{2n}\right) d_q z \\
&\quad + q^{-1} y^2 \int_0^\infty z\vartheta(z) E_q\left(-q\left(\frac{y}{q}\right)^{2n} z^{2n}\right) d_q z \\
&= -\vartheta(0) + q^{-2} y^2 L_{2,q}\left(\vartheta; \frac{y}{q}\right) + q^{-1} y^2 F_{2,q}\left(\vartheta; \frac{y}{q}\right).
\end{aligned}
$$

This finishes the proof of the theorem.   □

Theorem 10 *Let y be a positive real number. Then we have*

$$F_{2,q}\big(\Delta_q^2;y\big) = -(\Delta_q\vartheta)(0) - \frac{1+q}{q^2}y^2\vartheta(0) + \frac{(1+q)^2}{(q^2)^2}\frac{y^2}{q^2}y^2 F_{2,q}\bigg(\vartheta,\frac{y}{q^2}\bigg). \tag{17}$$

Proof Let $y > 0$. Then, by using Theorem 9, we obtain

$$F_{2,q}\big(\Delta_q^2;y\big) = -(\Delta_q\vartheta)(0) + \frac{1+q}{q^2}y^2\bigg(-\vartheta(0) + \frac{1+q}{q^2}\bigg(\frac{y}{q}\bigg)^2 F_{2,q}\bigg(\vartheta,\frac{y}{q^2}\bigg)\bigg)$$

$$= -(\Delta_q\vartheta)(0) - \frac{1+q}{q^2}y^2\vartheta(0) + \bigg(\frac{1+q}{q^2}\bigg)^2\frac{y^2}{q^2}y^2 F_{2,q}\bigg(\vartheta;\frac{y}{q^2}\bigg).$$

Hence, the proof of the theorem is finished.　　　　　　　　　　　　　　　　　　□

Theorem 11 *Let y be a positive real number. Then we have*

$$F_{2,q}\big(\Delta_q^3\vartheta;y\big) = -\big(\Delta_q^2\vartheta\big)(0) - \frac{1+q}{q^2}y^2\Delta_q\vartheta(0) - \frac{(1+q)^2}{(q^2)^2}\frac{y^2}{q^2}y^2\vartheta(0)$$

$$+ \frac{(1+q)^3}{(q^2)^3}\frac{y^2}{q^4}\frac{y^2}{q^2}y^2 F_{2,q}\bigg(\vartheta,\frac{y}{q^3}\bigg).$$

Proof By taking into account Theorem 9 and using simple computations, we have

$$F_{2,q}\big(\Delta_q^3\vartheta;y\big) = F_{2,q}\big(\Delta_q\Delta_q^2\vartheta;y\big) = -\Delta_q^2\vartheta(0) + \frac{1+q}{q^2}y^2 F_{2,q}\bigg(\Delta^2\vartheta;\frac{y}{q}\bigg). \tag{18}$$

By utilizing Theorem 10, Eq. (18) together with computations reveals

$$F_{2,q}\big(\Delta_q^3\vartheta;y\big) = -\Delta_q^2\vartheta(0) + \frac{1+q}{q^2}y\bigg(-\frac{1+q}{q^2}\bigg(\frac{y}{q}\bigg)^2\vartheta(0) - \Delta_q\vartheta(0)\bigg)$$

$$+ \frac{1+q}{q^2}y\bigg(\frac{(1+q)^2}{(q^2)^2}\frac{y^2}{q^2-q^2}\frac{y^2}{q^2}F_{2,q}\bigg(\vartheta,\frac{y}{q^3}\bigg)\bigg)$$

$$= -\big(\Delta_q^2\vartheta\big)(0) - \frac{1+q}{q^2}y^2\Delta_q\vartheta(0) - \frac{(1+q)^2}{(q^2)^2}\frac{y^2}{q^2}y^2\vartheta(0)$$

$$+ \frac{(1+q)^3}{(q^2)^3}\frac{y^2}{q^4}\frac{y^2}{q^2}y^2 F_{2,q}\bigg(\vartheta,\frac{y}{q^3}\bigg).$$

This finishes the proof of the theorem.　　　　　　　　　　　　　　　　　　□

Theorem 12 *Let y be a positive real number. Then we have*

$$F_{2,q}\big(\Delta_q^4\vartheta;y\big) = -\Delta_q^3\vartheta(0) - \frac{1+q}{q^2}y^2\Delta_q^2\vartheta(0) - \bigg(\frac{1+q}{q^2}\bigg)^2\bigg(\frac{y}{q}\bigg)^2 y^2\Delta_q\vartheta(0)$$

$$- \bigg(\frac{1+q}{q^2}\bigg)^3\frac{y^2}{q^4}\frac{y^2}{q^2}.y^2\vartheta(0)$$

$$+ \bigg(\frac{1+q}{q^2}\bigg)^4\frac{y^2}{q^6}\frac{y^2}{q^4}\frac{y^2}{q^2}y^2 F_{2,q}\bigg(\vartheta;\frac{y}{q^4}\bigg).$$

Proof Let $y > 0$ be given arbitrary. Then, by employing Theorem 10, we derive that

$$
F_{2,q}\left(\Delta_q^4 \vartheta; y\right) = -\Delta_q^3 \vartheta(0) + \frac{1+q}{q^2} y^2 L_{2,q}\left(\Delta_q^3 \vartheta; \frac{y}{q}\right)
$$

$$
= -\Delta_q^3 \vartheta(0) + \frac{1+q}{q^2} y^2 \left(-\Delta_q^2 \vartheta(0) - \frac{1+q}{q^2}\left(\frac{y}{q}\right)^2 \Delta_q \vartheta(0)\right)
$$

$$
- \frac{1+q}{q^2} y^2 \left(\left(\frac{1+q}{q^2}\right)^2 \frac{y^2}{q^4} \frac{y^2}{q^2} \vartheta(0) + \left(\frac{1+q}{q^2}\right)^3 \frac{y^2}{q^6} \frac{y^2}{q^4} \frac{y^2}{q^2} F_{2,q}\left(\vartheta, \frac{y}{q^4}\right)\right).
$$

Therefore, further simplifications on the above equation yield

$$
F_{2,q}\left(\Delta_q^4 \vartheta; y\right) = -\Delta_q^3 \vartheta(0) - \frac{1+q}{q^2} y^2 \Delta_q^2 \vartheta(0) - \left(\frac{1+q}{q^2}\right)^2 \left(\frac{y}{q}\right)^2 y^2 \Delta_q \vartheta(0)
$$

$$
\times \left(\frac{1+q}{q^2}\right)^3 \frac{y^2}{q^4} \frac{y^2}{q^2} y^2 \vartheta(0) + \left(\frac{1+q}{q^2}\right)^4 y^2 \frac{y^2}{q^6} \frac{y^2}{q^4} \frac{y^2}{q^2} F_{2,q}\left(\vartheta, \frac{y}{q^4}\right).
$$

Hence, the proof of the theorem is finished. □

By following techniques similar to the techniques already used for Theorems 10–12, we reach the following result.

Corollary 13 *Let $y > 0$. Then we have*

$$
F_{2,q}\left(\Delta_q^n \vartheta; y\right) = -\sum_{j=1}^{n-1}\left(\frac{1+q}{q^2}\right)^{n-j-1} \prod_{k=1}^{n-j-1} \frac{y^2}{q^{2k-2}} \Delta_q^j \vartheta(0) - \left(\frac{1+q}{q^2}\right)^{n-1}
$$

$$
\times \prod_{k=1}^{n-1} \frac{y^2}{q^{2k-2}} \vartheta(0) + \left(\frac{1+q}{q^2}\right)^n \prod_{k=1}^{n} \frac{y^2}{q^{2k-2}} F_{2,q}\left(\vartheta, \frac{y}{q^n}\right).
$$

Let us now check the following differentiation formula.

Theorem 14 *Let y be a positive real number. Then we have*

$$
D_{x,q} e_q\left(-x^2 y^2\right) = -y^2 x \sum_{n=0}^{\infty} \frac{(-1)^n}{[n]_q!}\left(1 + q^{n+1}\right) y^{2n} x^{2n}.
$$

Proof Let y be a positive real number. Then, from the definitions, we have

$$
D_{x,q} e_q\left(-x^2 y^2\right) = \sum_{n=1}^{\infty} (-1)^n \frac{[2n]_q}{[n]_q!} y^{2n} x^{2n-1}.
$$

But, as earlier, we know that

$$
[2n]_q = \frac{1-q^{2n}}{1-q} = \frac{1-(q^n)^2}{1-q} = [n]_q\left(1 + q^n\right).
$$

Hence, invoking these results in the previous summation gives

$$
\begin{aligned}
D_{x,q}e_q\big(-x^2y^2\big) &= \sum_{n=1}^{\infty}(-1)^n\frac{[n]_q}{[n]_q!}\big(1+q^n\big)y^{2n}x^{2n-1} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^n}{[n-1]_q!}\big(1+q^n\big)y^{2n}x^{2n-1} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{[n]_q!}\big(1+q^{n+1}\big)y^{2n+2}x^{2n+1} \\
&= -y^2x\sum_{n=0}^{\infty}\frac{(-1)^n}{[n]_q!}\big(1+q^{n+1}\big)y^{2n}x^{2n}.
\end{aligned}
$$

This finishes the proof of the theorem.   □

Theorem 15 *Let $\Delta_q(x) = \frac{1}{x}D_{x,q}$ be given. Then we have*

$$
S_{2,q}(\Delta_q\vartheta;y) = -\vartheta(0) + q^{-1}y^2 S_{2,q}\left(\vartheta;\frac{y}{q}\right) + y^2 S_{2,q}\left(\vartheta;\frac{y}{\sqrt{q}}\right).
$$

Proof By making use of Eq. (9) and employing Eq. (3), we establish that

$$
\begin{aligned}
S_{2,q}(\Delta_q\vartheta;y) &= \int_0^{\infty} D_{x,q}\vartheta(x)e_q\big(-x^2y^2\big)\,d_qx \\
&= \vartheta(x)e_q\big(-x^2y^2\big)\big|_0^{\infty} - \int_0^{\infty}\vartheta(qx)D_{x,q}e_q\big(-x^2y^2\big)\,d_qx.
\end{aligned}
$$

Therefore, substituting the integral bounds in the first part suggests we have

$$
S_{2,q}(\Delta_q\vartheta;y) = -\vartheta(0) - \int_0^{\infty}\vartheta(qx)D_{x,q}e_q\big(-x^2y^2\big)\,d_qx. \tag{19}
$$

Hence, by virtue of Theorem 10, Eq. (19) reveals

$$
S_{2,q}(\Delta_q\vartheta;y) = -\vartheta(0) + y^2\int_0^{\infty}\vartheta(qx)x\sum_{n=0}^{\infty}\frac{(-1)^n}{[n]_q!}\big(1+q^{n+1}\big)y^{2n}x^{2n}\,d_qx.
$$

The change of variables $qx = z$ yields $d_qx = \frac{1}{q}d_qz$. Therefore, the above equation together with certain technical computations implies

$$
\begin{aligned}
S_{2,q}(\Delta_q\vartheta;y) &= -\vartheta(0) + y^2\int_0^{\infty}\vartheta(z)\frac{z}{q}\sum_{n=0}^{\infty}\frac{(-1)^n}{[n]_q!}\big(1+q^{n+1}\big)y^{2n}\frac{z^{2n}}{q^{2n}}\frac{1}{q}\,d_qz \\
&= -\vartheta(0) + y^2\int_0^{\infty}\vartheta(z)\sum_{n=0}^{\infty}\frac{(-1)^n}{[n]_q!}\frac{(1+q^{n+1})}{q^{2(n+1)}}y^{2n}z^{2n}\,d_qz \\
&= -\vartheta(0) + y^2\int_0^{\infty}z\vartheta(z)\sum_{n=0}^{\infty}\frac{(-1)^n}{[n]_q!}\big(q^{-2n-1-n}+q\big)y^{2n}z^{2n}\,d_qz
\end{aligned}
$$

$$= -\vartheta(0) + q^{-1} y^2 \int_0^\infty z \vartheta(z) \sum_{n=0}^\infty \frac{(-1)^n}{[n]_q!} \left(\frac{y^2}{q^2} \right)^n z^{2n} \, d_q z$$

$$+ y^2 \int_0^\infty z \vartheta(z) \sum_{n=0}^\infty \frac{(-1)^n}{[n]_q!} \left(\frac{y^2}{\sqrt{q}^2} \right)^n z^{2n} \, d_q z$$

$$= -\vartheta(0) + q^{-1} y^2 \int_0^\infty z \vartheta(z) e_q \left(\frac{y^2}{q^2} z^2 \right) d_q z$$

$$+ y^2 \int_0^\infty z \vartheta(z) e_q \left(\frac{y^2}{q^2} z^2 \right) z^{2n} \, d_q z$$

$$= -\vartheta(0) + q^{-1} y^2 S_{2,q} \left(\vartheta; \frac{y}{q} \right) + y^2 S_{2,q} \left(\vartheta; \frac{y}{\sqrt{q}} \right).$$

Hence, the proof of this theorem is completed. □

4 $F_{2,q}$ of q-convolution products

In this section we focus on giving a convolution theorem for the $F_{2,q}$ integral operator. For, let us assume $\vartheta(t) = t^{2\delta}$ and $\theta(t) = t^{2\gamma-1}$, $\delta, \gamma > 0$. Then the q-convolution product is defined for ϑ and θ as (see [15, Eq. (44)])

$$(\vartheta * \theta)_q(t) = \int_0^t \vartheta(x) \theta(t - qx) \, d_q x, \tag{20}$$

provided the integral exists. The following is a very useful property.

Theorem 16 *Let $\vartheta(t) = t^{2\delta}$ and $\theta(t) = t^{2\gamma-1}$, $\delta, \gamma > 0$. Then we have*

$$(\vartheta * \theta)_q(t) = t^{2\delta+2\gamma} \frac{\Gamma_q(2\delta + 1)\Gamma_q(2\gamma)}{\Gamma_q(2\delta + 2\gamma + 1)}.$$

Proof By using the integral equation given by Eq. (20), we get

$$(\vartheta * \theta)_q(t) = \int_0^t x^{2\delta}(t - qx)^{2\gamma-1} \, d_q x.$$

The change of variables $x = tz$ transforms the equation into the form

$$(\vartheta * \theta)_q(t) = \int_0^1 t^{2+1} z^{2\delta}(t - qtz)^{2\gamma-1} \, d_q z$$

$$= \int_0^1 t^{2+1} z^{2\delta} t^{2\gamma-1}(1 - qz)^{2\gamma-1} \, d_q z$$

$$= t^{2\delta+2\gamma} \int_0^1 z^{2\delta}(1 - qz)^{2\gamma-1} \, d_q z$$

$$= t^{2\delta+2\gamma} B_q(2\delta + 1, 2\gamma),$$

where B_q is the q-analogue of the Beta function,

$$B_q(\delta, \gamma) = \int_0^1 t^{\delta}(1 - qt)^{\gamma-1} \, d_q t. \tag{21}$$

Hence, by the well-known formula $B_q(\delta, \gamma) = \frac{\Gamma_q(\delta)\Gamma_q(\gamma)}{\Gamma_q(\delta+\gamma)}$, we get

$$(\vartheta * \theta)_q(t) = t^{2\delta+2\gamma} \frac{\Gamma_q(2\delta + 1)\Gamma_q(2\gamma)}{\Gamma_q(2\delta + 2\gamma + 1)}.$$

This finishes the proof of the above result. □

Now, by the aid of Theorem 16, we establish the following convolution theorem.

Theorem 17 *Let* $\delta, \gamma > 0$, $\delta + \gamma > -\frac{1}{2}$, $\vartheta(x) = x^{2\delta}$ *and* $\theta(x) = x^{2\gamma-1}$. *Then we have*

$$F_{2,q}\big((\vartheta * \theta)_q(x); y\big) = \frac{1}{[2]_q} \frac{\Gamma_q(2\delta + 1)\Gamma_q(2\gamma)\Gamma_q(\delta + \gamma + 1)}{\Gamma_q(2\delta + 2\gamma + 1)} y^{-2\delta-2\gamma-2}.$$

Proof Let $\delta, \gamma > 0$, $\vartheta(x) = x^{2\delta}$ and $\theta(x) = x^{2\gamma-1}$ be given. Then we obtain

$$\begin{aligned}
F_{2,q}\big((\vartheta * \theta)_q; y\big) &= \int_0^\infty (\vartheta * \theta)_q(x) x E_q\big(-qx^2y^2\big)\, d_q x \\
&= \frac{\Gamma_q(2\delta + 1)\Gamma_q(2\gamma)}{\Gamma_q(2\delta + 2\gamma + 1)} \int_0^\infty x^{2\delta+2\gamma} x E_q\big(-qx^2y^2\big)\, d_q x.
\end{aligned}$$

Hence, from above, we obtain

$$F_{2,q}\big((\vartheta * \theta)_q; y\big) = \frac{\Gamma_q(2\delta + 1)\Gamma_q(2\gamma)}{\Gamma_q(2\delta + 2\gamma + 1)} F_{2,q}\big(x^{2(\delta+\gamma)}; y\big). \qquad (22)$$

Therefore, by aid of Theorem 2, we put Eq. (22) into the form

$$F_{2,q}\big((\vartheta * \theta)_q; y\big) = \frac{\Gamma_q(2\delta + 1)\Gamma_q(2\gamma)}{\Gamma_q(2\delta + 2\gamma + 1)} \frac{\Gamma_q(\delta + \gamma + 1)}{[2]_q y^{2\delta+2\gamma+2}},$$

$\delta, \gamma > 0$, $\delta + \gamma > -\frac{1}{2}$.

This finishes the proof of the theorem. □

5 Conclusion

The q-Laplace-type integral operator was applied to certain polynomials and various functions of q-trigonometric types. Such results were extended to q-hyperbolic functions and some other q-differential operators. By using q-differentiation formulas, several derivatives of the q-Laplace-type integral operator were obtained. On top of that, some related formulas as well as a convolution theorem were also discussed in detail.

Authors' contributions

The author was the only one to write this paper. The author read and approved the final manuscript.

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