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# Strongly extreme points of Orlicz function spaces equipped with $\Phi$-Amemiya norm

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**Abstract**

In this paper, the criterion that points of Orlicz function spaces equipped with $\Phi$-Amemiya norm generated by an Orlicz function are strongly extreme is given. As a corollary, the sufficient and necessary conditions of midpoint local uniform rotundity of Orlicz function spaces equipped with $\Phi$-Amemiya norm are obtained.

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**Keywords:** Orlicz function spaces; $\Phi$-Amemiya norm; Strongly extreme points; Midpoint local uniform rotundity

## 1 Introduction

An extreme point plays a crucial role in functional analysis, convex analysis, and optimization. In fact, any compact convex set is the convex hull of its extreme point set, the result is called Krein–Milman theorem. The notion of a dentable subset of a Banach space was introduced by Rieffel in conjunction with a Radon–Nikodym theorem for Banach space-valued measures. Subsequent work by Maynard and Davis and Phelos has shown those Banach spaces in which Rieffel's Radon–Nikodym theorem is valid and every bounded closed convex set is dentable. This has been a significant breakthrough in studying the nature of Radon–Nikodym as a geometric property. In 1988, Bor-Luh Lin, Pei-Kee Lin, and Troyanski described the characteristic of denting points (see [1, 2]) and obtained that there is a close relationship between denting points and strongly extreme points. It is easy to see that every denting point of Banach space $X$ is a strongly extreme point (see [3]) of $X$, and it is known that every strongly extreme point of $X$ is a $w^*$ extreme point of $X$. Orlicz space is an important class of Banach space, it was introduced by the famous Polish mathematician Wladyslaw Orlicz in 1932. The theory of Orlicz space has been greatly developed because of its important theoretical properties and application value. Up to now, the criterion that an element in the unit sphere of Orlicz spaces equipped with the Orlicz norm, the Luxemburg norm, and the $p$-Amemiya norm is a strongly extreme point has been obtained (see [4–6]). In this paper, we introduce a new norm, namely $\Phi$-Amemiya norm, whose calculation formula is given as follows: $\|x\|_{\Phi,\Phi_1} = \inf_{k>0}\{\frac{1}{k}(1+\Phi(I_{\Phi_1}(kx)))\}$. When we take some special functions, the previous norms are special cases of this new norm. This new norm also has wider applicability than before. We give the criterion that an element in the unit sphere of Orlicz spaces equipped with $\Phi$-Amemiya norm is a strongly extreme point. Incidentally, the sufficient and necessary conditions of midpoint local uniform rotundity of Orlicz function spaces equipped with $\Phi$-Amemiya norm are obtained.

## 2 Preliminaries

Let $[X, \|\cdot\|]$ be a Banach space. $S(X)$ and $B(X)$ denote the unit sphere and the unit ball of $X$, respectively. $X^*$ is said to be the dual space of $X$.

**Definition 2.1** A mapping $\Phi : R \to [0,\infty)$ is called an Orlicz function: if $\Phi$ is even, continuous, convex and $\Phi(u) = 0$ if and only if $u = 0$. If $\Phi$ also satisfies $\lim_{u\to 0} \frac{\Phi(u)}{u} = 0$ and $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$, then $\Phi$ is called an $N$-function.

**Definition 2.2** The function $\Psi$ defined by the formula $\Psi(u) = \sup\{|u|v - \Phi(v) : v \geq 0\}$ is called complementary function of $\Phi$ in the sense of Young.

**Definition 2.3** Let $(G, \Sigma, \mu)$ be a nonatomic finite measure space. Let $L^0$ denote the whole of the measurable real function on $G$. We define the modular $I_\Phi$: $L^0 \to R^+ = [0, +\infty]$ as follows:

$$I_\Phi(x) = \int_G \Phi\bigl(x(t)\bigr)\,dt,$$

it is called the modular (see [7]) of $x$.

**Definition 2.4** The Orlicz function space (see [8]) $L_\Phi$ generated by an Orlicz function is defined by the formula $L_\Phi = \{x \in L^0 : I_\Phi(kx) < +\infty \text{ for some } k > 0\}$.

Those spaces that are equipped with the Orlicz norm (Amemiya norm) (see [9])

$$\|x\|^0_\Phi = \inf_{k>0} \frac{1}{k}\bigl(1 + I_\Phi(kx)\bigr),$$

or equipped with the Luxemburg norm

$$\|x\|_\Phi = \inf\left\{k > 0 : I_\Phi\left(\frac{x}{k}\right) \leq 1\right\},$$

or equipped with the $p$-Amemiya norm $(1 \leq p < +\infty)$ (see [10, 11])

$$\|x\|_{\Phi,p} = \inf_{k>0} \frac{1}{k}\bigl(1 + I^p_\Phi(kx)\bigr)^{\frac{1}{p}},$$

are Banach spaces, abbreviated as

$$L^0_\Phi = \bigl[L_\Phi, \|\cdot\|^0_\Phi\bigr]; \qquad L_\Phi = \bigl[L_\Phi, \|\cdot\|_\Phi\bigr]; \qquad L_{\Phi,p} = \bigl[L_\Phi, \|\cdot\|_{\Phi,p}\bigr].$$

**Definition 2.5** We say that Orlicz function $\Phi$ satisfies the $\Delta_2$ condition if there exist $k > 2$ and $u_0 \geq 0$ such that the inequality

$$\Phi(2u) \leq k\Phi(u)$$

holds for $|u| \geq u_0$.

**Definition 2.6** If, for every $y, z \in R$ and $y \neq z$ with $\frac{y+z}{2} = x$, we have $\Phi(x) < \frac{\Phi(y)+\Phi(z)}{2}$, then $x$ is called a strictly convex point of $\Phi$. The set of all strictly convex points of $\Phi$ will be denoted by $S_\Phi$.

For any Orlicz functions $\Phi$ and $\Phi_1$, we put $L_{\Phi,\Phi_1} = \{x \in L^0 : \Phi(I_{\Phi_1}(kx)) < +\infty$ for some $k > 0\}$. The calculation formula

$$\|x\|_{\Phi,\Phi_1} = \inf_{k>0} \frac{1}{k}\bigl(1 + \Phi\bigl(I_{\Phi_1}(kx)\bigr)\bigr)$$

is called $\Phi$-Amemiya norm.

*Remark*

If we take $\Phi(u) = \max\{0, u - 1\}$, then $\|x\|_{\Phi,\Phi_1}$ is the Luxemburg norm $\|\cdot\|_\Phi$;
If we take $\Phi(u) = u$, then $\|x\|_{\Phi,\Phi_1}$ is the Orlicz norm $\|\cdot\|^0_\Phi$;
If we take $\max\{0, u - 1\} \leq \Phi(u) \leq u$, then $\|x\|_{\Phi,\Phi_1}$ is the $s$-norm $\|\cdot\|^s_\Phi$;
If we take $\Phi(u) \geq |u|$, then $\|x\|_{\Phi,\Phi_1} \geq \|\cdot\|^0_\Phi$.

An important question is the attainability of the "inf" in $\|x\|_{\Phi,\Phi_1} = \inf_{k>0} \frac{1}{k}(1 + \Phi(I_{\Phi_1}(kx)))$. For any $x \in L_{\Phi,\Phi_1}$, $x \neq 0$, we define

$$K(x) = \left\{k > 0 : \|x\|_{\Phi,\Phi_1} = \frac{1}{k}\bigl(1 + \Phi\bigl(I_{\Phi_1}(kx)\bigr)\bigr)\right\}.$$

We will prove that if $\lim_{u\to\infty} \frac{\Phi_1(u)}{u} = +\infty$, then $K(x) \neq \phi$.

*Proof* Put $F(k) = \frac{1}{k}(1 + \Phi(\int_G \Phi_1(kx(t))\,dt))$ and $\theta(x) = \inf\{k > 0, I_{\Phi_1}(\frac{x}{k}) < \infty\}$.

Then there exists $d > 0$ such that $\mu(\{t \in G : |x(t)| > d\}) > 0$ and $F(k)$ is continuous on $(0, \theta(x))$. So $\lim_{k\to 0^+} F(k) = +\infty$.

Suppose that $\theta(x) = +\infty$. Then

$$\begin{aligned}
\lim_{k\to+\infty} F(k) &= \lim_{k\to+\infty} \frac{\Phi(\int_G \Phi_1(kx(t))\,dt)}{k} \\
&\geq \lim_{k\to+\infty} \Phi\left(\frac{\int_G \Phi_1(kx(t))\,dt}{k}\right) \\
&\geq \lim_{k\to+\infty} \Phi\left(\frac{\int_{\{t\in G:|x(t)|>d\}} \Phi_1(kx(t))\,dt}{k}\right) \\
&\geq d \lim_{k\to+\infty} \Phi\left(\frac{\Phi_1(kd)\cdot\mu(\{t \in G : |x(t)| > d\})}{kd}\right) \\
&= +\infty.
\end{aligned}$$

Since $F(k)$ is a continuous function, then there exists $k_0 \in (0, \theta(x))$ such that $F(k) \geq F(k_0)$.

Suppose that $\theta(x) < +\infty$. If $I_{\Phi_1}(\theta(x)x(t)) = +\infty$, we have

$$\lim_{k\to\theta(x)-0} F\left(k = \frac{1}{\theta(x)}\left(1 + \Phi\left(\int_G \Phi_1\bigl(\theta(x)x(t)\bigr)\,dt\right)\right) = +\infty.\right.$$

If $I_{\Phi_1}(\theta(x)x(t)) < +\infty$, we have

$$\lim_{k\to\theta(x)-0} F(k) = \frac{1}{\theta(x)}\left(1 + \Phi\left(\int_G \Phi_1\big(\theta(x)x(t)\big)\,dt\right)\right) < +\infty.$$

Since $F(k)$ is a continuous function, there exists $k_0 \in (0,\theta(x)]$ such that $F(k) \geq F(k_0)$.
Thus $K(x) \neq \phi$. □

**Definition 2.7** A point $x \in S(X)$ is said to be an extreme point (see [12]) of $B(X)$ if, for any $y,z \in S(X)$ and $x = \frac{y+z}{2}$, we have $y = z$. The set of all extreme points of the unit ball $B(X)$ will be denoted by $ExtB(X)$. $X$ is said to be strictly convex (see [13]) if and only if $ExtB(X) = S(X)$.

**Definition 2.8** A point $x \in S(X)$ is called a strongly extreme point (see [14–16]) of $B(X)$ if, for any $\{x_n\} \subseteq X$, $\{y_n\} \subseteq X$, $\lim_{n\to\infty}\|x_n\| = \lim_{n\to\infty}\|y_n\| = 1$, and $\frac{x_n+y_n}{2} = x$, we have $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

**Definition 2.9** Banach space $X$ is called middle point local uniform convex (see [17, 18]) if and only if each point on $S(X)$ is a strongly extreme point.

**Lemma 2.10** (EropoB theorem) *Let $\{f_n\}_{n=1}^{\infty}$ be a measurable function and $|f_n(x)| < \infty$ a.e. $x \in E$ with $m(E) < +\infty$. If $f_n(x) \to f(x)$ a.e. $x \in E$, then for any $\delta > 0$ there exists $E_0 \subset E$ such that $m(E_0) < \delta$ and $f_n(x) \to f(x)$ uniformly in $x \in E \setminus E_0$.*

**Lemma 2.11** *Assume $\Phi \in \Delta_2$ (see [19]). Then, for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\big|I_\Phi(u+v) - I_\Phi(v)\big| < \varepsilon,$$

*whenever $I_\Phi(u) \leq L$, $I_\Phi(v) \leq \delta$.*

**Lemma 2.12** *Let $\Phi \in \Delta_2$. If $I_\Phi(x_n) \to I_\Phi(x)$, $x_n \xrightarrow{\mu} x$, then $\|x_n - x\|_\Phi \to 0$ (see [19]).*

## 3 Main results

**Theorem 3.1** *Let $\Phi_1$ be an N-function. Then $x_0 \in S(L_{\Phi,\Phi_1})$ is a strongly extreme point of $B(L_{\Phi,\Phi_1})$ if and only if $\Phi_1 \in \Delta_2$ and $k_0x_0(t) \in S_{\Phi_1}$, where $k_0 \in K(x_0)$.*

*Proof* Necessity. Suppose that $\mu(\{t \in G : k_0x_0(t) \notin S_{\Phi_1}\}) > 0$ for some $k_0 \in K(x_0)$. There exists an interval $(a,b)$ such that $\mu(\{t \in G : \frac{a}{k_0} + \varepsilon < x_0(t) < \frac{b}{k_0} - \varepsilon\}) > 0$ ($\varepsilon > 0$) and $\Phi_1$ is affine on $(a,b)$, i.e., $\Phi_1(x) = px + q$. Divide $\{t \in G : \frac{a}{k_0} + \varepsilon < x_0(t) < \frac{b}{k_0} - \varepsilon\}$ into two sets $E$ and $F$ with $E \cap F = \emptyset$ and $\mu(E) = \mu(F)$. Define

$$y(t) = \begin{cases} x_0(t), & t \in G \setminus (E \cup F), \\ x_0(t) - \varepsilon, & t \in E, \\ x_0(t) + \varepsilon, & t \in F, \end{cases}$$

$$z(t) = \begin{cases} x_0(t), & t \in G \setminus (E \cup F), \\ x_0(t) + \varepsilon, & t \in E, \\ x_0(t) - \varepsilon, & t \in F. \end{cases}$$

Then $x_0 = \frac{y+z}{2}$, $y \neq z$ and

$$\begin{aligned} I_{\Phi_1}(k_0 y) &= \int_{E\cup F} \Phi_1\big(k_0 y(t)\big)\,dt + \int_{G\setminus E\cup F} \Phi_1\big(k_0 y(t)\big)\,dt \\ &= \int_E \big(p\big(k_0\big(x_0(t)-\varepsilon\big)\big)+q\big)\,dt + \int_F \big(p\big(k_0\big(x_0(t)+\varepsilon\big)\big)+q\big)\,dt \\ &\quad + \int_{G\setminus E\cup F} \Phi_1\big(k_0 x_0(t)\big)\,dt \\ &= \int_{E\cup F} \big(pk_0 x_0(t)+q\big)\,dt + \int_{G\setminus E\cup F} \Phi_1\big(k_0 x_0(t)\big)\,dt \\ &= \int_{E\cup F} \Phi_1\big(k_0 x_0(t)\big)\,dt + \int_{G\setminus E\cup F} \Phi_1\big(k_0 x_0(t)\big)\,dt \\ &= I_{\Phi_1}(k_0 x_0). \end{aligned}$$

Thus $\|y\|_{\Phi,\Phi_1} \leq \frac{1}{k_0}(1+\Phi(I_{\Phi_1}(k_0 y))) = \frac{1}{k_0}(1+\Phi(I_{\Phi_1}(k_0 x_0))) = \|x_0\|_{\Phi,\Phi_1} = 1$. In the same way, we can get $\|z\|_{\Phi,\Phi_1} \leq 1$. This contradicts the fact that $x_0$ is an extreme point of $S(L_{\Phi,\Phi_1})$.

In order to complete this proof, we need to prove that if $\Phi_1 \notin \Delta_2$, there is not a strongly extreme point on the unit sphere of $L_{\Phi,\Phi_1}$. If $x_0 \in S(L_{\Phi,\Phi_1})$, then there exists $d > 0$ such that $\mu(\{t \in G : |x_0(t)| \leq d\}) > 0$. Suppose $\Phi_1 \notin \Delta_2$, then there exists $u_n > 0$, $u_n \uparrow \infty$ such that $\Phi_1(2u_n) > 2^n \Phi_1(u_n)$ $(n = 1,2,\ldots)$. Without loss of generality, we can assume that $\frac{1}{\Phi_1(u_1)} < \mu(\{t \in G : |x(t)| \leq d\})$. Take $\{G_n\} \subset \{t \in G : |x(t)| < d\}$ with $G_m \cap G_n = \emptyset$ for any $m \neq n$, satisfying

$$\mu(G_n) = \frac{1}{2^n \Phi_1(u_n)} \quad (n = 1,2,\ldots).$$

Define

$$x_n(t) = \begin{cases} x_0(t), & t \in G\setminus G_n, \\ x_0(t) + \frac{u_n}{k_0}, & t \in G_n, \end{cases}$$

$$y_n(t) = \begin{cases} x_0(t), & t \in G\setminus G_n, \\ x_0(t) - \frac{u_n}{k_0}, & t \in G_n. \end{cases}$$

Then $x_0 = \frac{x_n + y_n}{2}$ for each $n \in N$.

Put

$$x_n(t) = x_n'(t) + x_n''(t),$$

where $x_n'(t) = x_0\chi_{G\setminus G_n}(t) + \frac{u_n}{k_0}\chi_{G_n}(t)$, $x_n''(t) = x_0\chi_{G_n}(t)$.

Since $\|x_n''\|_{\Phi,\Phi_1} = \|x_0\chi_{G_n}\|_{\Phi,\Phi_1} \leq d\|\chi_{G_n}\|_{\Phi,\Phi_1} \to 0$ $(n \to \infty)$, we have the inequality $\|x_n'\|_{\Phi,\Phi_1} \geq \|x_0\chi_{G\setminus G_n}\|_{\Phi,\Phi_1} \geq \|x_0\|_{\Phi,\Phi_1} - \|x_0\chi_{G_n}\|_{\Phi,\Phi_1}$ holds. Therefore $\underline{\lim}_{n\to\infty} \|x_n'\|_{\Phi,\Phi_1} \geq \|x_0\|_{\Phi,\Phi_1} = 1$.

By the definition of $\Phi$-Amemiya norm, we deduce that

$$\|x_n'\|_{\Phi,\Phi_1} = \inf_{k>0} \frac{1}{k}\big(1 + \Phi\big(I_{\Phi_1}\big(kx_n'\big)\big)\big)$$

$$
\begin{aligned}
&\le \frac{1}{k_0}\bigl(1+\Phi\bigl(I_{\Phi_1}\bigl(k_0x_n'\bigr)\bigr)\bigr)\\
&\le \frac{1}{k_0}+\frac{1}{k_0}\Phi\left(\int_G \Phi_1\left(k_0\left(x_0\chi_{G\setminus G_n}(t)+\frac{u_n}{k_0}\chi_{G_n}(t)\right)\right)dt\right)\\
&\le \frac{1}{k_0}+\frac{1}{k_0}\Phi\left(\int_{G\setminus G_n}\Phi_1\bigl(k_0x_0\chi_{G\setminus G_n}(t)\bigr)\,dt+\int_{G_n}\Phi_1\bigl(u_n\chi_{G_n}(t)\bigr)\,dt\right)\\
&\le \frac{1}{k_0}+\frac{1}{k_0}\Phi\bigl(I_{\Phi_1}(k_0x_0)+\Phi_1(u_n)\mu(G_n)\bigr)\\
&= \frac{1}{k_0}\left(1+\Phi\left(I_{\Phi_1}(k_0x_0)+\frac{1}{2^n}\right)\right).
\end{aligned}
$$

Then

$$\overline{\lim_{n\to\infty}}\bigl\|x_n'\bigr\|_{\Phi,\Phi_1}\le\|x_0\|_{\Phi,\Phi_1}=1.$$

Hence

$$\lim_{n\to\infty}\|x_n\|_{\Phi,\Phi_1}=1.$$

In the same way, we have

$$\lim_{n\to\infty}\|y_n\|_{\Phi,\Phi_1}=1.$$

But

$$
\begin{aligned}
I_{\Phi_1}\bigl(k_0(x_n-y_n)\bigr)&=\int_{G_n}\Phi_1\left(k_0\frac{2u_n(t)}{k_0}\right)dt\\
&=\Phi_1(2u_n)\mu(G_n)\ge 1\quad (n=1,2,\ldots).
\end{aligned}
$$

Therefore

$$\|x_n-y_n\|_{\Phi,\Phi_1}=\frac{1}{k_0}\|2u_n\chi_{G_n}\|_{\Phi,\Phi_1}\ge\frac{1}{k_0}\|2u_n\chi_{G_n}\|_{\Phi_1}\ge\frac{1}{k_0},$$

a contradiction.

Sufficiency. Let $\Phi_1\in\triangle_2$ and $x_0\in S(L_{\Phi,\Phi_1})$ with $k_0x_0(t)\in S_{\Phi_1}$ for $k_0\in K(x_0)$. For any $x_n,y_n\in L_{\Phi,\Phi_1}$ such that

$$
\begin{aligned}
&\lim_{n\to\infty}\|x_n\|_{\Phi,\Phi_1}=1,\\
&\lim_{n\to\infty}\|y_n\|_{\Phi,\Phi_1}=1,\\
&x_n+y_n=2x_0
\end{aligned}
$$

for each $n\in N$.

Take sequences of positive numbers $\{k_n\}$ and $\{h_n\}$ such that

$$
\begin{aligned}
\|x_n\|_{\Phi,\Phi_1}&\ge\frac{1}{k_n}\bigl(1+\Phi\bigl(I_{\Phi_1}(k_nx_n)\bigr)\bigr)-\frac{1}{n},\\
\|y_n\|_{\Phi,\Phi_1}&\ge\frac{1}{h_n}\bigl(1+\Phi\bigl(I_{\Phi_1}(h_ny_n)\bigr)\bigr)-\frac{1}{n}.
\end{aligned}
$$

Define

$$\widetilde{x_n}(t) = \frac{x_n + x_0}{2},$$
$$\widetilde{y_n}(t) = \frac{y_n + x_0}{2}.$$

Then

$$\widetilde{x_n} + \widetilde{y_n} = 2x_0$$

and

$$\lim_{n\to\infty} \|\widetilde{x_n}\|_{\Phi,\Phi_1} \le 1,$$
$$\lim_{n\to\infty} \|\widetilde{y_n}\|_{\Phi,\Phi_1} \le 1.$$

Now, we will prove that $\lim_{n\to\infty} \|\widetilde{x_n}\|_{\Phi,\Phi_1} = \lim_{n\to\infty} \|\widetilde{y_n}\|_{\Phi,\Phi_1} = 1$. Otherwise, we can assume that $\lim_{n\to\infty} \|\widetilde{x_n}\|_{\Phi,\Phi_1} < 1$ and there exist $\delta > 0$, $n_0 \in N$ such that

$$\|\widetilde{x_n}\|_{\Phi,\Phi_1} \le 1 - \delta,$$
$$\|\widetilde{y_n}\|_{\Phi,\Phi_1} \le 1 + \frac{\delta}{2}$$

for all $n \ge n_0$. Then

$$1 = \|x_0\|_{\Phi,\Phi_1} = \left\| \frac{\widetilde{x_n} + \widetilde{y_n}}{2} \right\|_{\Phi,\Phi_1} \le \frac{1}{2}\left(1 - \delta + 1 + \frac{\delta}{2}\right) < 1,$$

a contradiction.

Thus

$$\lim_{n\to\infty} \|\widetilde{x_n}\|_{\Phi,\Phi_1} = \lim_{n\to\infty} \|\widetilde{y_n}\|_{\Phi,\Phi_1} = 1.$$

Since $\|\widetilde{x_n} - \widetilde{y_n}\|_{\Phi,\Phi_1} \to 0$ if and only if $\|x_n - y_n\|_{\Phi,\Phi_1} \to 0$ $(n \to \infty)$, we will use the sequences $\{\widetilde{x_n}\}$ and $\{\widetilde{y_n}\}$ instead of $\{x_n\}$ and $\{y_n\}$, respectively. Put $k'_n = \frac{2k_nk_0}{k_n+k_0}$, $h'_n = \frac{2h_nk_0}{h_n+k_0}$. Then $d = \sup\{k'_n, h'_n\} < +\infty$.

Taking advantage of the forced convergence theorem and

$$\begin{aligned}
\|\widetilde{x_n}\|_{\Phi,\Phi_1} &\le \frac{1}{k'_n}\big(1 + \Phi\big(I_{\Phi_1}\big(k'_n\widetilde{x_n}\big)\big)\big) \\
&\le \frac{k_n + k_0}{2k_nk_0}\left(1 + \Phi\left(I_{\Phi_1}\left(\frac{k_nk_0}{k_n + k_0}(x_n + x_0)\right)\right)\right) \\
&\le \frac{1}{2}\left(\frac{1}{k_0} + \frac{1}{k_0}\Phi\big(I_{\Phi_1}(k_0x_0)\big) + \frac{1}{k_n} + \frac{1}{k_n}\Phi\big(I_{\Phi_1}(k_nx_n)\big)\right) \\
&\le \frac{1}{2}\left(\|x_0\|_{\Phi,\Phi_1} + \|x_n\|_{\Phi,\Phi_1} + \frac{1}{n}\right) \\
&\to 1 \quad (n \to \infty),
\end{aligned}$$

we have

$$\lim_{n\to\infty}\frac{1}{k'_n}\big(1+\Phi\big(I_{\Phi_1}\big(k'_n\widetilde{x}_n\big)\big)\big)=1.$$

In the same way, we also have

$$\lim_{n\to\infty}\frac{1}{h'_n}\big(1+\Phi\big(I_{\Phi_1}\big(h'_n\widetilde{y}_n\big)\big)\big)=1.$$

Assume $k'_n\to k$, $h'_n\to h$ $(n\to\infty)$. We will prove that $k,h\geq 1$. Since $\lim_{n\to\infty}\frac{1}{k'_n}(1+\Phi(I_{\Phi_1}(k'_n\widetilde{x}_n)))=1$, then $\lim_{n\to\infty}\Phi(I_{\Phi_1}(k'_n\widetilde{x}_n))=k-1$. If $k<1$, then $\Phi(I_{\Phi_1}(k'_n\widetilde{x}_n))<0$ as $n\to\infty$, a contradiction. Therefore, $k\geq 1$. Similarly, $h\geq 1$.

Hence

$$\frac{k}{k+h},\frac{h}{k+h}\in\left[\frac{1}{1+d},\frac{d}{1+d}\right].$$

In order to finish the proof of the theorem, we divide the left proof of the theorem into three steps.

Step 1: We will show that $k_0=\frac{2kh}{k+h}\in K(x_0)$. In fact

$$\begin{aligned}
\|x_0\|_{\Phi,\Phi_1} &\leq \frac{k'_n+h'_n}{2k'_nh'_n}\left(1+\Phi\left(I_{\Phi_1}\left(\frac{2k'_nh'_n}{k'_n+h'_n}x_0\right)\right)\right)\\
&\leq \frac{k'_n+h'_n}{2k'_nh'_n}\left(1+\Phi\left(I_{\Phi_1}\left(\frac{k'_nh'_n}{k'_n+h'_n}(\widetilde{x}_n+\widetilde{y}_n)\right)\right)\right)\\
&\leq \frac{k'_n+h'_n}{2k'_nh'_n}\left(1+\Phi\left(I_{\Phi_1}\left(\frac{h'_n}{k'_n+h'_n}k'_n\widetilde{x}_n+\frac{k'_n}{k'_n+h'_n}h'_n\widetilde{y}_n\right)\right)\right))\\
&\leq \frac{1}{2}\left(\frac{1}{k'_n}\big(1+\Phi\big(I_{\Phi_1}\big(k'_n\widetilde{x}_n\big)\big)\big)+\frac{1}{h'_n}\big(1+\Phi\big(I_{\Phi_1}\big(h'_n\widetilde{y}_n\big)\big)\big)\right)\\
&\to 1\quad(n\to\infty).
\end{aligned}$$

Since $\|x_0\|_{\Phi,\Phi_1}=1$, we get $\frac{2k'_nh'_n}{k'_n+h'_n}\to\frac{2kh}{k+h}=k_0\in K(x_0)$.

Step 2: We will prove that $k'_n\widetilde{x}_n-k_0x_0\xrightarrow{\mu}0$ $(n\to\infty)$.

Firstly, we will show that

$$k\widetilde{x}_n-h\widetilde{y}_n\xrightarrow{\mu}0\quad(n\to\infty).$$

Otherwise, there exist $\sigma_0,\varepsilon_0>0$ such that

$$\mu\big(\big\{t\in G:\big|k\widetilde{x}_n(t)-h\widetilde{y}_n(t)\big|\geq\sigma_0\big\}\big)\geq\varepsilon_0.$$

Let

$$D=\Phi_1^{-1}\left(\frac{3}{\varepsilon_0}\right),$$

$$D_1=2kD.$$

Put $G_n = \{t \in G : |k\widetilde{x_n}(t)| \leq D_1, |h\widetilde{y_n}(t)| \leq D_1, |k\widetilde{x_n}(t) - h\widetilde{y_n}(t)| \geq \sigma_0\}$. We will show that

$$\mu(G_n) > \frac{\varepsilon_0}{3}.$$

Indeed, since $\lim_{n\to\infty} \|\widetilde{x_n}\|_{\Phi,\Phi_1} = 1$, we may assume $\|\widetilde{x_n}\|_{\Phi} \leq \|\widetilde{x_n}\|_{\Phi,\Phi_1} \leq 2$. Then

$$\begin{aligned}
1 &\geq I_{\Phi_1}\left(\frac{\widetilde{x_n}}{2}\right) \\
&\geq \int_{\{t\in G:|\frac{\widetilde{x_n}(t)}{2}|>D\}} \Phi_1\left(\frac{\widetilde{x_n}(t)}{2}\right) dt \\
&> \Phi_1(D)\mu\left(\left\{t \in G : \left|\frac{\widetilde{x_n}(t)}{2}\right| > D\right\}\right) \\
&= \frac{3}{\varepsilon_0}\mu\left(\left\{t \in G : \left|\frac{\widetilde{x_n}(t)}{2}\right| > D\right\}\right).
\end{aligned}$$

Hence

$$\mu\left(\left\{t \in G : \left|\frac{\widetilde{x_n}(t)}{2}\right| > D\right\}\right) < \frac{\varepsilon_0}{3}.$$

Consequently,

$$\mu\left(\left\{t \in G : \left|k\widetilde{x_n}(t)\right| > D_1\right\}\right) < \frac{\varepsilon_0}{3}.$$

Therefore,

$$\begin{aligned}
\mu(G_n) &= \mu\left(\left\{t \in G : \left|k\widetilde{x_n}(t) - h\widetilde{y_n}(t)\right| \geq \sigma_0\right\}\right) - \mu\left(\left\{t \in G : \left|k\widetilde{x_n}(t)\right| > D_1\right\}\right) \\
&\quad - \mu\left(\left\{t \in G : \left|h\widetilde{y_n}(t)\right| > D_1\right\}\right) \\
&> \varepsilon_0 - \frac{\varepsilon_0}{3} - \frac{\varepsilon_0}{3} \\
&= \frac{\varepsilon_0}{3}.
\end{aligned}$$

Let

$$F = \left\{(x,y) : |x| \leq D_1, |y| \leq D_1, |x-y| \geq \sigma_0, \frac{h}{k+h}x + \frac{k}{k+h}y \in S_\Phi\right\}.$$

By virtue of the fact that $S_{\Phi_1}$ is a closed set, we know that $F$ is a bounded closed set and

$$f(x,y) = \frac{\Phi_1(\frac{h}{k+h}x + \frac{k}{k+h}y)}{\frac{h}{k+h}\Phi_1(x) + \frac{k}{k+h}\Phi_1(y)} < 1$$

for every $(x,y) \in F$.

By $f(x,y)$ is continuous on $F$, there exists $(x_0,y_0) \in F$ such that $f(x,y) \leq f(x_0,y_0)$. We next will prove that $f(x_0,y_0) < 1$. If $f(x_0,y_0) = 1$, then $\frac{\Phi_1(\frac{h}{k+h}x_0 + \frac{k}{k+h}y_0)}{\frac{h}{k+h}\Phi_1(x_0) + \frac{k}{k+h}\Phi_1(y_0)} = 1$, this contradicts $\frac{h}{k+h}x_0 + \frac{k}{k+h}y_0 \in S_{\Phi_1}$. Put $f(x_0,y_0) = 1 - \delta$. For every $(x,y) \in F$, we have

$$\Phi_1\left(\frac{h}{k+h}x + \frac{k}{k+h}y\right) \leq (1-\delta)\left(\frac{h}{k+h}\Phi_1(x) + \frac{k}{k+h}\Phi_1(y)\right).$$

By the definition of $k_0$ and $x_0$, we derive that

$$\frac{h}{k+h}k\widetilde{x_n}(t)+\frac{k}{k+h}h\widetilde{y_n}(t)=\frac{2kh}{k+h}x_0(t)=k_0x_0(t)\in S_{\Phi_1}.$$

Since $k_0x_0(t)\in S_{\Phi_1}$, then $(k\widetilde{x_n}(t),h\widetilde{y_n}(t))\in F$, i.e., for $t\in G_n$ and

$$\Phi_1\left(\frac{h}{k+h}k\widetilde{x_n}(t)+\frac{k}{k+h}h\widetilde{y_n}(t)\right)\le(1-\delta)\left(\frac{h}{k+h}\Phi_1\big(k\widetilde{x_n}(t)\big)+\frac{k}{k+h}\Phi_1\big(h\widetilde{y_n}(t)\big)\right).$$

Hence

$$\begin{aligned}
&\|\widetilde{x_n}+\widetilde{y_n}\|_{\Phi,\Phi_1}\\
&\quad\le\frac{k+h}{kh}\left(1+\Phi\left(I_{\Phi_1}\left(\frac{kh}{k+h}(\widetilde{x_n}+\widetilde{y_n})\right)\right)\right)\\
&\quad\le\frac{k+h}{kh}+\frac{k+h}{kh}\Phi\left(\int_G\Phi_1\left(\frac{kh}{k+h}\big(\widetilde{x_n}(t)+\widetilde{y_n}(t)\big)\right)dt\right)\\
&\quad\le\frac{k+h}{kh}+\frac{k+h}{kh}\Phi\left((1-\delta)\int_{G_n}\left[\frac{h}{k+h}\Phi_1\big(k\widetilde{x_n}(t)\big)+\frac{k}{k+h}\Phi_1\big(h\widetilde{y_n}(t)\big)\right]dt\right)\\
&\qquad+\frac{k+h}{kh}\Phi\left(\int_{G\setminus G_n}\left[\frac{h}{k+h}\Phi_1\big(k\widetilde{x_n}(t)\big)+\frac{k}{k+h}\Phi_1\big(h\widetilde{y_n}(t)\big)\right]dt\right)\\
&\quad\le\frac{k+h}{kh}+\frac{k+h}{kh}\Phi\left(\int_G\left[\frac{h}{k+h}\Phi_1\big(k\widetilde{x_n}(t)\big)+\frac{k}{k+h}\Phi_1\big(h\widetilde{y_n}(t)\big)\right]dt\right)\\
&\qquad-\frac{k+h}{kh}\Phi\left(\delta\int_{G_n}\left[\frac{h}{k+h}\Phi_1\big(k\widetilde{x_n}(t)\big)+\frac{k}{k+h}\Phi_1\big(h\widetilde{y_n}(t)\big)\right]dt\right)\\
&\quad\le\frac{1}{k}(1+\Phi\big(I_{\Phi_1}\big(k\widetilde{x_n}(t)\big)\big)+\frac{1}{h}(1+\Phi\big(I_{\Phi_1}\big(k\widetilde{y_n}(t)\big)\big)\\
&\qquad-\frac{k+h}{kh}\Phi\left(\delta\int_{G_n}\left[\frac{h}{k+h}\Phi_1\big(k\widetilde{x_n}(t)\big)+\frac{k}{k+h}\Phi_1\big(h\widetilde{y_n}(t)\big)\right]dt\right).
\end{aligned}$$

Notice that

$$I_{\Phi_1}\big((k-k_n')\widetilde{x_n}\big)\le\big|k-k_n'\big|I_{\Phi_1}(\widetilde{x_n})\to 0\quad(n\to\infty).$$

By Lemma 2.11, we get

$$I_{\Phi_1}(k\widetilde{x_n})-I_{\Phi_1}\big(k_n'\widetilde{x_n}\big)=I_{\Phi_1}\big(k_n'\widetilde{x_n}+\big(k-k_n'\big)\widetilde{x_n}\big)-I_{\Phi_1}\big(k_n'\widetilde{x_n}\big)\to 0\quad(n\to\infty).$$

Thus

$$\begin{aligned}
0&\le\frac{1}{k}\big(1+\Phi\big(I_{\Phi_1}(k\widetilde{x_n})\big)\big)-\|\widetilde{x_n}\|_{\Phi,\Phi_1}\\
&=\frac{1}{k}\big(1+\Phi\big(I_{\Phi_1}(k\widetilde{x_n})\big)\big)-\frac{1}{k_n'}\big(1+\Phi\big(I_{\Phi_1}\big(k_n'\widetilde{x_n}\big)\big)\big)+\frac{1}{n}\to 0\quad(n\to\infty).
\end{aligned}$$

Similarly, $\frac{1}{h}(1+\Phi(I_{\Phi_1}(h\widetilde{y_n})))-\|\widetilde{y_n}\|_{\Phi,\Phi_1}\to 0$ as $n\to\infty$. Since $u>0$, $\Phi(u)>0$, and

$$\|\widetilde{x_n}+\widetilde{y_n}\|_{\Phi,\Phi_1}\le 2-\frac{k+h}{kh}\left(\Phi\left(\frac{2\delta}{1+d}\Phi_1\left(\frac{\delta_0}{2}\right)\frac{\varepsilon_0}{3}\right)\right)\quad(n\to\infty),$$

we have $\lim_{n\to\infty}\|\widetilde{x_n}+\widetilde{y_n}\|_{\Phi,\Phi_1}<2$. A contradiction. Hence $k\widetilde{x_n}-h\widetilde{y_n}\xrightarrow{\mu}0$.

Thanks to the $\Phi$-Amemiya norm being equivalent with the Luxemburg norm, their weak topology and weak star topology are all equivalent. So $L_{\Phi,\Phi_1}$ is $w^*$ compact. Take $\{x_n''\} \subset \{\widetilde{x_n}\}$, $\{y_n''\} \subset \{\widetilde{y_n}\}$ such that $x_n'' \xrightarrow{w^*} x'$ and $y_n'' \xrightarrow{w^*} y'$. We get $x' + y' = 2x_0$.

Since $\Phi_1 \in \triangle_2$, we have

$$\|x\|_{\Phi,\Phi_1} = \sup\left\{\int_G x(t)y(t)\,dt : y \in B\bigl(L^*_{\Phi,\Phi_1}\bigr)\right\},$$

where $L^*_{\Phi,\Phi_1}$ is the dual space of $L_{\Phi,\Phi_1}$.

So

$$\|x\|_{\Phi,\Phi_1} = \sup\left\{\int_G x(t)y(t)\,dt : y \in B\bigl(E^*_{\Phi,\Phi_1}\bigr)\right\}.$$

Since $\|2x_0\|_{\Phi,\Phi_1} = 2$, then

$$\|2x_0\|_{\Phi,\Phi_1} \le \|x'\|_{\Phi,\Phi_1} + \|y'\|_{\Phi,\Phi_1} \le \varliminf_{n\to\infty} \|\widetilde{x_n}\|_{\Phi,\Phi_1} + \varliminf_{n\to\infty} \|\widetilde{y_n}\|_{\Phi,\Phi_1} = 2.$$

This shows

$$\|x'\|_{\Phi,\Phi_1} = \|y'\|_{\Phi,\Phi_1} = 1.$$

Hence, there exist $k, h > 1$ such that

$$\begin{aligned} 1 &= \|x'\|_{\Phi,\Phi_1} = \frac{1}{k}\bigl(1 + \Phi\bigl(I_{\Phi_1}(kx')\bigr)\bigr), \\ 1 &= \|y'\|_{\Phi,\Phi_1} = \frac{1}{h}\bigl(1 + \Phi\bigl(I_{\Phi_1}(hy')\bigr)\bigr). \end{aligned}$$

Since $\|x'\|_{\Phi,\Phi_1} + \|y'\|_{\Phi,\Phi_1} = 2$, then

$$\begin{aligned} \|x'\|_{\Phi,\Phi_1} + \|y'\|_{\Phi,\Phi_1} &= \frac{1}{k}\bigl(1 + \Phi\bigl(I_{\Phi_1}(kx')\bigr)\bigr) + \frac{1}{h}\bigl(1 + \Phi\bigl(I_{\Phi_1}(hy')\bigr)\bigr) \\ &= \frac{k+h}{kh}\left[1 + \frac{h}{k+h}\Phi\bigl(I_{\Phi_1}(kx')\bigr) + \frac{k}{k+h}\Phi\bigl(I_{\Phi_1}(hy')\bigr)\right] \\ &\ge \frac{k+h}{kh}\left[1 + \Phi\left(\frac{h}{k+h}\bigl(I_{\Phi_1}(kx')\bigr) + \frac{k}{k+h}\bigl(I_{\Phi_1}(hy')\bigr)\right)\right] \\ &\ge \frac{k+h}{kh}\left[1 + \Phi\left(I_{\Phi_1}\left(\frac{kh}{k+h}x' + \frac{kh}{k+h}y'\right)\right)\right] \\ &= 2\cdot\frac{1}{k_0}\bigl(1 + \Phi\bigl(I_{\Phi_1}(k_0x_0)\bigr)\bigr) \\ &= 2, \end{aligned}$$

and

$$\|x_0\|_{\Phi,\Phi_1} = 1 = \frac{1}{k_0}\bigl(1 + \Phi\bigl(I_{\Phi_1}(k_0x_0)\bigr)\bigr).$$

Hence $x' = y' = x_0$. Combining this with $kx_n'' - hy_n'' \xrightarrow{\mu} 0$, we can prove that $kx_n'' - hy_n'' \xrightarrow{w^*} 0$. Since $v(t) \in E_\Psi$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|v\chi_{G_0}\|_\Phi < \varepsilon$, whence $\mu(G_0) <$

$\delta$. Using the EropoB theorem, there exists $G_0 \subset G$, $\mu(G_0) < \delta$ such that $kx''_n - hy''_n \xrightarrow{\mu} 0$ for $t \in G \setminus G_0$.

Put $\|x''_n\|^o_\Phi = 1$ and $\|y''_n\|^o_\Phi = 1$. We have

$$\begin{aligned}\int_G (kx''_n - hy''_n)\nu(t)\,dt &= \int_{G\setminus G_0} (kx''_n - hy''_n)\nu(t)\,dt + \int_{G_0} (kx''_n - hy''_n)\nu(t)\,dt \\ &\le M \cdot \mu(G \setminus G_0) \cdot \varepsilon + k\|x''_n\|^o_\Phi \|\nu(t)\|_\Phi + h\|y''_n\|^o_\Phi \|\nu(t)\|_\Phi \\ &\le M \cdot \mu(G \setminus G_0) \cdot \varepsilon + k \cdot \varepsilon + h \cdot \varepsilon.\end{aligned}$$

By the arbitrariness of $\varepsilon$, we have $\int_G (kx''_n - hy''_n)\nu(t)\,dt < \varepsilon$. Thus $kx''_n - hy''_n \xrightarrow{w^*} 0$. Since $x''_n - y''_n \xrightarrow{w^*} 0$, then $k = h$. Thus $\widetilde{x}_n - \widetilde{y}_n \xrightarrow{\mu} 0$ as $n \to \infty$. Therefore

$$k'_n\widetilde{x}_n - k_0x_0 \xrightarrow{\mu} 0 \quad (n \to \infty).$$

Step 3: We will prove that $I_{\Phi_1}(k_nx_n) \to I_{\Phi_1}(k_0x_0)$. In fact

$$\begin{aligned}&\Phi\bigl(I_{\Phi_1}(k_0x_0)\bigr) = k_0 - 1, \\ &\Phi\bigl(I_{\Phi_1}(k'_n\widetilde{x}_n)\bigr) \to k \quad (n \to \infty).\end{aligned}$$

We deduce that $\Phi(I_{\Phi_1}(k'_n\widetilde{x}_n)) \to \Phi(I_{\Phi_1}(k_0x_0))$ $(n \to \infty)$. Using $u > 0$, $\Phi(u) > 0$ and $\Phi(u)$ is strictly increasing, we get

$$I_{\Phi_1}\bigl(k'_n\widetilde{x}_n\bigr) \to I_{\Phi_1}(k_0x_0) \quad (n \to \infty).$$

By Lemma 2.12, we have

$$\bigl\|k'_n\widetilde{x}_n - k_0x_0\bigr\|_{\Phi,\Phi_1} \to 0. \qquad \square$$

**Corollary 3.2** *Let $\Phi$ be an Orlicz function. Then $L_{\Phi,\Phi_1}$ is midpoint local uniform rotundity if and only if $\Phi_1 \in \Delta_2$ and $L_{\Phi,\Phi_1}$ is strictly convex.*

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