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Existence and uniqueness of solutions for a class of higher-order fractional boundary value problems with the nonlinear term satisfying some inequalities

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Abstract

This paper focuses on a class of hider-order nonlinear fractional boundary value problems. The boundary conditions contain Riemann–Stieltjes integral and nonlocal multipoint boundary conditions. It is worth mentioning that the nonlinear term and the boundary conditions contain fractional derivatives of different orders. Based on the Schauder fixed point theorem, we obtain the existence of solutions under the hypothesis that the nonlinear term satisfies the Carathéodory conditions. We apply the Banach contraction mapping principle to obtain the uniqueness of solutions. Moreover, by using the theory of spectral radius we prove the uniqueness and nonexistence of positive solutions. Finally, we illustrate our main results by some examples.

MSC: 26A33; 34A08; 34B10

Keywords: Fractional boundary value problems; Riemann–Stieltjes integral; Nonlocal multipoint boundary conditions; Existence and uniqueness of solutions

1 Introduction

In this paper, we consider the class of boundary value problems

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t),D_{0+}^{\alpha_1}u(t),\ldots,D_{0+}^{\alpha_{n-3}}u(t),D_{0+}^{\alpha_{n-2}}u(t))=0, \quad 0<t<1, \\ u(0)=D_{0+}^{\beta_1}u(0)=\cdots=D_{0+}^{\beta_{n-2}}u(0)=0, \\ D_{0+}^{q_0}u(1)=c_1\int_0^1 g_1(s)D_{0+}^{q_1}u(s)\,dA_1(s)+c_2\int_0^{\theta} g_2(s)D_{0+}^{q_2}u(s)\,dA_2(s) \\ \qquad +c_3\sum_{i=1}^m \gamma_i D_{0+}^{q_3}u(\xi_i), \end{cases} \tag{1.1}$$

where D_{0+}^{α} is the Riemann–Liouville fractional derivative of order α, $n-1<\alpha\leq n$ $(n\geq 3)$, $1<\alpha-\alpha_{n-2}\leq 2$; $k-1<\alpha_k,\beta_k\leq k$, $\alpha_{n-2}-\beta_k\leq n-2-k$ $(k=1,2,\ldots,n-2)$; $\alpha_{n-2}\leq q_j\leq q_0\leq n-1$, $c_j\geq 0$ $(j=1,2,3)$, $\alpha-q_0\geq 1$; $\gamma_i\geq 0$, $0<\xi_1<\xi_2<\cdots<\xi_i<\cdots<\xi_m<1$ $(i=1,2,\ldots,m)$; $0\leq\theta\leq 1$, $g_1,g_2:(0,1)\to[0,+\infty)$ are continuous and belong to $L^1(0,1)$; $\int_0^1 g_1(s)u(s)\,dA_1(s)$ and $\int_0^1 g_2(s)u(s)\,dA_2(s)$ are the Riemann–Stieltjes integrals,

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where $A_1, A_2 : [0,1] \to (-\infty, +\infty)$ are functions of bounded variation. In this paper, a function $u \in C[0,1]$ is called a solution of problem (1.1) if it satisfies (1.1) a.e. on $[0,1]$.

Fractional calculus and fractional boundary value problems have been researched extensively to apply them in various areas, including image processing, rheology, electrical networks, virus infection models, and so on. Some interesting results can be found in [1–11] and the references therein. For example, in [1] the authors discovered that the motion frequency of a class of neurons should be characterized by noninteger derivatives. Therefore fractional derivatives are introduced to characterize this behavior, which is not possible by integer-derivative models. In [4] the authors introduced the Riemann–Liouville fractional derivative of order α $(0.5 < \alpha \leq 1)$ into a model of HIV infection of $CD4^+$ T-cells. By using stability analysis the authors obtained a sufficient condition on the parameters for the stability of the infected steady state. It should be noted that this fractional model possessed positive solutions, which is desired in any population dynamics. Indeed, there are many definitions of fractional derivatives. Because the Riemann–Liouville fractional derivative avoids seeking limits, it is widely used in mathematical studies. The definition of Riemann–Liouville fractional derivative shows that it has some important properties such as globality. In fact, the Riemann–Liouville fractional derivative is very suitable for describing viscoelastic material models and processes with memory properties. It has the advantages of simple modeling and accurate description. Recently, the research on the properties of solutions of fractional boundary value problems has received substantial attention. Some interesting results can be found in [12–64] and the references therein. For example, in [15] the authors have obtained the existence of one and two solutions by using the fixed point index theory. In [16], based on the Schaefer fixed point theorem and Banach contraction principle, the existence and uniqueness of solutions for a class of fractional boundary value problem are obtained. Moreover, the higher-order fractional boundary value problems have attracted more attention. We refer to [13–15, 19, 31, 32, 34–37, 44, 45, 58, 59]. For example, in [36] the existence and uniqueness of solutions are obtained by applying the Krasnoselskii theorem and Banach fixed point theorem. Based on the Leggett–Williams and Krasnoselskii fixed point theorems, Zhang and Zhong [31] showed the existence of positive solutions for the following nonlinear fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta} u(1) = \lambda \int_0^{\eta} h(t) D_{0^+}^{\beta} u(t)\, dt, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the Riemann–Liouville derivative, $n-1 < \alpha \leq n$ $(n \geq 3)$, $\beta \geq 1$, $\alpha - \beta - 1 > 0$, $0 < \eta \leq 1$, $0 \leq \lambda \int_0^{\eta} h(t) t^{\alpha-\beta-1}\, dt < 1$. The nonlinearity f may be singular at $t = 0, 1$ and $u = 0$, and $h \in L^1([0,1], [0, +\infty))$ may be singular at $t = 0, 1$.

Furthermore, the condition that the nonlinearity contains the derivative of the unknown function, especially the fractional-order derivative, causes some mathematical difficulties but make the research very interesting. We refer to [19, 21, 25, 27, 29, 37, 44, 50, 59, 61]. For instance, in [21] the authors investigated the following fractional boundary value problem:

$$\begin{cases} -D_t^{\alpha} x(t) + \lambda f(t, x(t), -D_t^{\beta} x(t)) = 0, \quad 0 < t < 1, \\ D_t^{\beta} x(0) = 0, D_t^{\gamma} x(1) = \sum_{j=1}^{p-2} a_j D_t^{\gamma} x(\xi_j), \end{cases}$$

where $D^{\alpha}_{0^+}$ is the standard Riemann–Liouville derivative, $1 < \alpha \leq 2, 0 < \beta \leq \gamma < 1, \alpha - \beta > 1$, $a_j \geq 0$ $(j = 1, 2, \ldots, p-2)$, $\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1} < 1$, and f may be singular at $t = 0, 1$. The existence and uniqueness of positive solutions was proved by the reduction method of fractional order and the monotone iterative technique.

Motivated by the papers mentioned, in this paper, we are lead to study problem (1.1). Evidently, our discussion is novel and meaningful. Firstly, problem (1.1) is more general; especially, the boundary conditions include two types of Riemann–Stieltjes integral boundary conditions and nonlocal multipoint boundary conditions. Secondly, the nonlinear term f contains the fractional derivatives of different orders of the unknown function. Thirdly, the existence of solutions is obtained under the hypothesis that f satisfies the Carathéodory condition, which is weaker than the continuity conditions. Fourthly, we show the uniqueness and nonexistence of positive solutions by using appropriate methods. Moreover, in this paper, our approach in obtaining the corresponding integral operator is the reduction method of fractional order on account of semigroup properties of the Riemann–Liouville derivative. We also illustrate the relationship between higher- and lower-order fractional derivatives.

An outline of this paper is as follows. In Sect. 2, we give some preliminaries and lemmas. We transform problem (1.1) into a relatively low-order problem by using the reduction method and obtain the relevant Green's function. In Sect. 3, we construct two results, one handing the existence of solutions and the other one managing the uniqueness of solutions under two different assumptions. In Sect. 4, we obtain the uniqueness of positive solutions by using spectral radius theory. In Sect. 5, we prove the nonexistence of positive solutions. In Sect. 6, we illustrate the main results by some examples.

2 Preliminaries

Definition 2.1 Let $\alpha > 0$. The Riemann–Liouville fractional integral of order α for a function $u : (0, \infty) \to (-\infty, +\infty)$ is defined by

$$I^{\alpha}_{0^+} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds,$$

where Γ is the Euler gamma function, that is, $\Gamma(t) = \int_0^{+\infty} s^{t-1} e^{-s}\, ds$ $(t > 0)$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 Let $\alpha > 0$. The Riemann–Liouville's fractional derivative of order α for a continuous function $u : (0, \infty) \to (-\infty, +\infty)$ is defined by

$$D^{\alpha}_{0^+} u(t) = \left(\frac{d}{dt}\right)^n I^{n-\alpha}_{0^+} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ is the integer part of α, provided that the right-hand side is pointwise defined on $(0, \infty)$. In particular, if $\alpha = n \in N_+$, then $D^{\alpha}_{0^+} u(t) = u^{(n)}(t)$.

Lemma 2.1 ([64]) *Let* $\alpha > 0$. *Suppose that* $u \in C(0,1) \cap L^1(0,1)$. *Then the equation*

$$D^{\alpha}_{0^+} u(t) = 0$$

has a unique solution

$$u(t) = d_1 t^{\alpha-1} + d_2 t^{\alpha-2} + \cdots + d_N t^{\alpha-N}, \quad d_i \in (-\infty, +\infty), i = 1, 2, \ldots, N,$$

where N is the smallest integer greater than or equal to α.

Lemma 2.2 ([64]) *Suppose that $u \in C(0,1) \cap L^1(0,1)$ and $D_{0^+}^{\alpha} u \in C(0,1) \cap L^1(0,1)$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + d_1 t^{\alpha-1} + d_2 t^{\alpha-2} + \cdots + d_N t^{\alpha-N},$$

where $d_i \in (-\infty, +\infty)$ $(i = 1, 2, \ldots, N)$, N is the smallest integer greater than or equal to α.

Lemma 2.3 ([5]) *If $u \in L^1(0,1)$ and $\alpha \geq \beta > 0$, then*

$$D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t), \qquad D_{0^+}^{\beta} I_{0^+}^{\alpha} u(t) = I_{0^+}^{\alpha-\beta} u(t).$$

Let $v = D_{0^+}^{\alpha_{n-2}} u$. Then we can transform problem (1.1) into the equivalent problem

$$\begin{cases} D_{0+}^{\alpha-\alpha_{n-2}} v(t) + f(t, I_{0+}^{\alpha_{n-2}} v(t), I_{0+}^{\alpha_{n-2}-\alpha_1} v(t), \ldots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v(t), v(t)) = 0, \\ D_{0^+}^{\beta_{n-2}-\alpha_{n-2}} v(0) = 0, \quad 0 < t < 1, \\ D_{0+}^{q_0-\alpha_{n-2}} v(1) = c_1 \int_0^1 g_1(s) D_{0+}^{q_1-\alpha_{n-2}} v(s)\, dA_1(s) \\ \qquad + c_2 \int_0^{\theta} g_2(s) D_{0+}^{q_2-\alpha_{n-2}} v(s)\, dA_2(s) + c_3 \sum_{i=1}^{m} \gamma_i D_{0+}^{q_3-\alpha_{n-2}} v(\xi_i), \end{cases} \tag{2.1}$$

where $1 < \alpha - \alpha_{n-2} \leq 2$.

Lemma 2.4 *Suppose that problem (1.1) has a solution $u \in C[0,1]$. Then problem (2.1) has a solution $v = D_{0^+}^{\alpha_{n-2}} u$. On the contrary, if problem (2.1) has a solution $v \in C[0,1]$, then problem (1.1) has a solution $u = I_{0^+}^{\alpha_{n-2}} v$.*

Proof Suppose that problem (1.1) has a solution $u \in C[0,1]$. Let

$$v(t) = D_{0^+}^{\alpha_{n-2}} u(t), \quad t \in [0,1]. \tag{2.2}$$

In view of Lemma 2.2, we have

$$I_{0^+}^{\alpha_{n-2}} v(t) = I_{0^+}^{\alpha_{n-2}} D_{0^+}^{\alpha_{n-2}} u(t) = u(t) + d_1 t^{\alpha_{n-2}-1} + \cdots + d_{n-2} t^{\alpha_{n-2}-(n-2)}$$

for some $d_i \in (-\infty, +\infty)$ $(i = 1, 2, \ldots, n-2)$, that is,

$$u(t) = I_{0^+}^{\alpha_{n-2}} v(t) - d_1 t^{\alpha_{n-2}-1} - \cdots - d_{n-2} t^{\alpha_{n-2}-(n-2)}.$$

The boundary conditions $u(0) = D_{0+}^{\beta_1} u(0) = \cdots = D_{0+}^{\beta_{n-3}} u(0) = 0$ indicate that $d_{n-2} = \cdots = d_1 = 0$. Hence we have

$$u(t) = I_{0^+}^{\alpha_{n-2}} v(t), \quad t \in [0,1]. \tag{2.3}$$

It follows from Lemma 2.3 that

$$D_{0^+}^{\alpha_i}u(t) = D_{0^+}^{\alpha_i}I_{0^+}^{\alpha_{n-2}}v(t) = I_{0^+}^{\alpha_{n-2}-\alpha_i}v(t), \quad i = 1,2,\ldots,n-3. \tag{2.4}$$

Furthermore, we obtain

$$\begin{aligned} D_{0^+}^{\alpha}u(t) &= D_{0^+}^{\alpha}I_{0^+}^{\alpha_{n-2}}v(t) = \frac{d^n}{dt^n}I_{0^+}^{n-\alpha}I_{0^+}^{\alpha_{n-2}}v(t) = \frac{d^n}{dt^n}I_{0^+}^{n-(\alpha-\alpha_{n-2})}v(t) \\ &= \frac{d^2}{dt^2}\frac{d^{n-2}}{dt^{n-2}}I_{0^+}^{n-2}I_{0^+}^{2-(\alpha-\alpha_{n-2})}v(t) \\ &= \frac{d^2}{dt^2}I_{0^+}^{2-(\alpha-\alpha_{n-2})}v(t) = D_{0^+}^{\alpha-\alpha_{n-2}}v(t). \end{aligned} \tag{2.5}$$

Equivalently, we have

$$D_{0^+}^{\beta_{n-2}}u(t) = D_{0^+}^{\beta_{n-2}-\alpha_{n-2}}v(t), \tag{2.6}$$

$$D_{0^+}^{q_i}u(t) = D_{0^+}^{q_i-\alpha_{n-2}}v(t), \quad i = 0,1,2,3. \tag{2.7}$$

From (2.2)–(2.5) we have

$$\begin{aligned} &D_{0^+}^{\alpha-\alpha_{n-2}}v(t) + f\bigl(t, I_{0^+}^{\alpha_{n-2}}v(t), I_{0^+}^{\alpha_{n-2}-\alpha_1}v(t),\ldots, I_{0^+}^{\alpha_{n-2}-\alpha_{n-3}}v(t), v(t)\bigr) \\ &\quad = D_{0+}^{\alpha}u(t) + f\bigl(t, u(t), D_{0+}^{\beta_1}u(t),\ldots, D_{0^+}^{\beta_{n-3}}u(t), D_{0^+}^{\alpha_{n-2}}u(t)\bigr) = 0. \end{aligned} \tag{2.8}$$

It follows (2.6) and (2.7) that

$$D_{0^+}^{\beta_{n-2}-\alpha_{n-2}}v(0) = D_{0^+}^{\beta_{n-2}}u(0) = 0, \tag{2.9}$$

$$\begin{aligned} D_{0+}^{q_0-\alpha_{n-2}}v(1) &= D_{0^+}^{q_0}u(1) \\ &= c_1\int_0^1 g_1(s)D_{0+}^{q_1-\alpha_{n-2}}v(s)\,dA_1(s) + c_2\int_0^{\theta} g_2(s)D_{0+}^{q_2-\alpha_{n-2}}v(s)\,dA_2(s) \\ &\quad + c_3\sum_{i=1}^{m}\gamma_i D_{0+}^{q_3-\alpha_{n-2}}v(\xi_i). \end{aligned} \tag{2.10}$$

According to (2.8), (2.9), and (2.10), we conclude that problem (2.1) has a solution $v = D_{0^+}^{\alpha_{n-2}}u$.

On the contrary, if problem (2.1) has a solution $v \in C[0,1]$, then problem (1.1) has a solution $u = I_{0^+}^{\alpha_{n-2}}v$. The proof is similar to that of Lemma 3 in [59], and we omit it. □

Remark 2.4 In view of Lemma 2.4, we infer that researching solutions of problem (1.1) is equivalent to the work on considering solutions of problem (2.1) under the premise that $1 < \alpha - \alpha_{n-2} \leq 2$. Note that the corresponding integral operator of problem (2.1) can be considered in the space $C[0,1]$, which avoids doing the work in a complex space. Therefore our work focusses on problem (2.1) in the following:

Lemma 2.5 *Let $h \in C(0,1) \cap L^1(0,1)$. Then the problem*

$$\begin{cases} D_{0+}^{\alpha-\alpha_{n-2}} v(t) + h(t) = 0, \quad 0 < t < 1, 1 < \alpha - \alpha_{n-2} \leq 2, \\ D_{0+}^{\beta_{n-2}-\alpha_{n-2}} v(0) = 0, \\ D_{0+}^{q_0-\alpha_{n-2}} v(1) = c_1 \int_0^1 g_1(s) D_{0+}^{q_1-\alpha_{n-2}} v(s)\, dA_1(s) \\ \qquad + c_2 \int_0^\theta g_2(s) D_{0+}^{q_2-\alpha_{n-2}} v(s)\, dA_2(s) + c_3 \sum_{i=1}^m \gamma_i D_{0+}^{q_3-\alpha_{n-2}} v(\xi_i) \end{cases} \tag{2.11}$$

is equivalent to

$$v(t) = \int_0^1 H(t,s)h(s)\, ds \tag{2.12}$$

with

$$\begin{aligned} H(t,s) = {} & H_0(t,s) + t^{\alpha-\alpha_{n-2}-1} \left(\int_0^1 H_1(\tau,s) g_1(\tau)\, dA_1(\tau) \right) \\ & + t^{\alpha-\alpha_{n-2}-1} \left(\int_0^\theta H_2(\tau,s) g_2(\tau)\, dA_2(\tau) \right) \\ & + t^{\alpha-\alpha_{n-2}-1} \sum_{i=1}^m \gamma_i H_3(\xi_i,s), \end{aligned} \tag{2.13}$$

where

$$H_0(t,s) = \frac{1}{\Gamma(\alpha-\alpha_{n-2})} \begin{cases} t^{\alpha-\alpha_{n-2}-1}(1-s)^{\alpha-q_0-1} - (t-s)^{\alpha-\alpha_{n-2}-1}, & s \leq t, \\ t^{\alpha-\alpha_{n-2}-1}(1-s)^{\alpha-q_0-1}, & t \leq s, \end{cases}$$

$$H_i(t,s) = \frac{c_i}{\sigma \Gamma(\alpha-\alpha_{n-2})\Gamma(\alpha-q_i)} \begin{cases} t^{\alpha-q_i-1}(1-s)^{\alpha-q_0-1} - (t-s)^{\alpha-q_i-1}, & s \leq t, \\ t^{\alpha-q_i-1}(1-s)^{\alpha-q_0-1}, & t \leq s, \end{cases}$$

$i = 1,2,3$,

$$\begin{aligned} \sigma = {} & \frac{1}{\Gamma(\alpha-q_0)} - \frac{c_1}{\Gamma(\alpha-q_1)} \int_0^1 s^{\alpha-q_1-1} g_1(s)\, dA_1(s) \\ & - \frac{c_2}{\Gamma(\alpha-q_2)} \int_0^\theta s^{\alpha-q_2-1} g_2(s)\, dA_2(s) - \frac{c_3}{\Gamma(\alpha-q_3)} \sum_{i=1}^m \gamma_i \xi_i^{\alpha-q_3-1} \\ & \neq 0. \end{aligned}$$

Proof By Lemma 2.2 problem (2.11) can be rewritten as

$$v(t) = d_1 t^{\alpha-\alpha_{n-2}-1} + d_2 t^{\alpha-\alpha_{n-2}-2} - \int_0^t \frac{(t-s)^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha-\alpha_{n-2})} h(s)\, ds, \tag{2.14}$$

where $d_i \in (-\infty,+\infty)$ $(i = 1,2)$ are arbitrary constants. The condition $D_{0+}^{\beta_{n-2}-2-\alpha_{n-2}} v(0) = 0$ means that $d_2 = 0$. So we have

$$v(t) = d_1 t^{\alpha-\alpha_{n-2}-1} - I_{0+}^{\alpha-\alpha_{n-2}} h(t). \tag{2.15}$$

By Lemma 2.3 we have

$$D_{0+}^{q_0-\alpha_{n-2}}v(1)=d_1\frac{\Gamma(\alpha-\alpha_{n-2})}{\Gamma(\alpha-q_0)}-\frac{1}{\Gamma(\alpha-q_0)}\int_0^1(1-s)^{\alpha-q_0-1}h(s)\,ds, \tag{2.16}$$

$$D_{0+}^{q_1-\alpha_{n-2}}v(t)=d_1\frac{\Gamma(\alpha-\alpha_{n-2})}{\Gamma(\alpha-q_1)}t^{\alpha-q_1-1}-\frac{1}{\Gamma(\alpha-q_1)}\int_0^t(t-s)^{\alpha-q_1-1}h(s)\,ds, \tag{2.17}$$

$$D_{0+}^{q_2-\alpha_{n-2}}v(t)=d_1\frac{\Gamma(\alpha-\alpha_{n-2})}{\Gamma(\alpha-q_2)}t^{\alpha-q_2-1}-\frac{1}{\Gamma(\alpha-q_2)}\int_0^t(t-s)^{\alpha-q_2-1}h(s)\,ds, \tag{2.18}$$

$$D_{0+}^{q_3-\alpha_{n-2}}v(t)=d_1\frac{\Gamma(\alpha-\alpha_{n-2})}{\Gamma(\alpha-q_3)}t^{\alpha-q_3-1}-\frac{1}{\Gamma(\alpha-q_3)}\int_0^t(t-s)^{\alpha-q_3-1}h(s)\,ds. \tag{2.19}$$

Substituting (2.16)–(2.19) into the boundary condition

$$\begin{aligned}D_{0+}^{q_0-\alpha_{n-2}}v(1)&=c_1\int_0^1 g_1(s)D_{0+}^{q_1-\alpha_{n-2}}v(s)\,dA_1(s)\\&\quad+c_2\int_0^\theta g_2(s)D_{0+}^{q_2-\alpha_{n-2}}v(s)\,dA_2(s)+c_3\sum_{i=1}^m\gamma_iD_{0+}^{q_3-\alpha_{n-2}}v(\xi_i),\end{aligned}$$

we have

$$\begin{aligned}d_1&=\frac{1}{\sigma\Gamma(\alpha-\alpha_{n-2})}\left\{\frac{1}{\Gamma(\alpha-q_0)}\int_0^1(1-s)^{\alpha-q_0-1}h(s)\,ds\right.\\&\quad-\frac{c_1}{\Gamma(\alpha-q_1)}\int_0^1\left(\int_0^s(s-\tau)^{\alpha-q_1-1}h(\tau)\,d\tau\right)g_1(s)\,dA_1(s)\\&\quad-\frac{c_2}{\Gamma(\alpha-q_2)}\int_0^\theta\left(\int_0^s(s-\tau)^{\alpha-q_2-1}h(\tau)\,d\tau\right)g_2(s)\,dA_2(s)\\&\quad\left.-\frac{c_3}{\Gamma(\alpha-q_3)}\sum_{i=1}^m\gamma_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-q_3-1}h(s)\,ds\right\},\end{aligned}$$

where

$$\begin{aligned}\sigma&=\frac{1}{\Gamma(\alpha-q_0)}-\frac{c_1}{\Gamma(\alpha-q_1)}\int_0^1 s^{\alpha-q_1-1}g_1(s)\,dA_1(s)\\&\quad-\frac{c_2}{\Gamma(\alpha-q_2)}\int_0^\theta s^{\alpha-q_2-1}g_2(s)\,dA_2(s)-\frac{c_3}{\Gamma(\alpha-q_3)}\sum_{i=1}^m\gamma_i\xi_i^{\alpha-q_3-1}\neq0.\end{aligned} \tag{2.20}$$

Hence we have

$$\begin{aligned}v(t)&=-\frac{1}{\Gamma(\alpha-\alpha_{n-2})}\int_0^t(t-s)^{\alpha-\alpha_{n-2}-1}h(s)\,ds\\&\quad+\frac{t^{\alpha-\alpha_{n-2}-1}}{\sigma\Gamma(\alpha-\alpha_{n-2})}\left\{\frac{1}{\Gamma(\alpha-q_0)}\int_0^1(1-s)^{\alpha-q_0-1}h(s)\,ds\right.\\&\quad-\frac{c_1}{\Gamma(\alpha-q_1)}\int_0^1\left(\int_0^s(s-\tau)^{\alpha-q_1-1}h(\tau)\,d\tau\right)g_1(s)\,dA_1(s)\\&\quad-\frac{c_2}{\Gamma(\alpha-q_2)}\int_0^\theta\left(\int_0^s(s-\tau)^{\alpha-q_2-1}h(\tau)\,d\tau\right)g_2(s)\,dA_2(s)\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{c_3}{\Gamma(\alpha-q_3)} \sum_{i=1}^{m} \gamma_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-q_3-1} h(s)\,ds \Bigg\} \\
&= - \frac{1}{\Gamma(\alpha-\alpha_{n-2})} \int_0^t (t-s)^{\alpha-\alpha_{n-2}-1} h(s)\,ds \\
&\quad + \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha-\alpha_{n-2})} \int_0^1 (1-s)^{\alpha-q_0-1} h(s)\,ds \\
&\quad + \frac{1}{\sigma\Gamma(\alpha-\alpha_{n-2})} \int_0^1 (1-s)^{\alpha-q_0-1} h(s)\,ds \\
&\quad \times \Bigg\{ \frac{c_1 t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha-q_1)} \int_0^1 s^{\alpha-q_1-1} g_1(s)\,dA_1(s) \\
&\quad + \frac{c_2 t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha-q_2)} \int_0^\theta s^{\alpha-q_2-1} g_2(s)\,dA_2(s) + \frac{c_3 t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha-q_3)} \sum_{i=1}^{m} \gamma_i \xi_i^{\alpha-q_3-1} \Bigg\} \\
&\quad - \frac{c_1 t^{\alpha-\alpha_{n-2}-1}}{\sigma\Gamma(\alpha-\alpha_{n-2})\Gamma(\alpha-q_1)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-q_1-1} h(\tau)\,d\tau \right) g_1(s)\,dA_1(s) \\
&\quad - \frac{c_2 t^{\alpha-\alpha_{n-2}-1}}{\sigma\Gamma(\alpha-\alpha_{n-2})\Gamma(\alpha-q_2)} \int_0^\theta \left(\int_0^s (s-\tau)^{\alpha-q_2-1} h(\tau)\,d\tau \right) g_2(s)\,dA_2(s) \\
&\quad - \frac{c_3 t^{\alpha-\alpha_{n-2}-1}}{\sigma\Gamma(\alpha-\alpha_{n-2})\Gamma(\alpha-q_3)} \sum_{i=1}^{m} \gamma_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-q_3-1} h(s)\,ds \\
&= \frac{1}{\Gamma(\alpha-\alpha_{n-2})} \Bigg\{ \int_0^1 t^{\alpha-\alpha_{n-2}-1} (1-s)^{\alpha-q_0-1} h(s)\,ds \\
&\quad - \int_0^t (t-s)^{\alpha-\alpha_{n-2}-1} h(s)\,ds \Bigg\} \\
&\quad + \frac{c_1 t^{\alpha-\alpha_{n-2}-1}}{\sigma\Gamma(\alpha-\alpha_{n-2})\Gamma(\alpha-q_1)} \Bigg\{ \int_0^1 \left(\int_0^1 s^{\alpha-q_1-1} (1-\tau)^{\alpha-q_0-1} h(\tau)\,d\tau \right. \\
&\quad \left. - \int_0^s (s-\tau)^{\alpha-q_1-1} h(\tau)\,d\tau \right) g_1(s)\,dA_1(s) \Bigg\} \\
&\quad + \frac{c_2 t^{\alpha-\alpha_{n-2}-1}}{\sigma\Gamma(\alpha-\alpha_{n-2})\Gamma(\alpha-q_2)} \Bigg\{ \int_0^\theta \left(\int_0^1 s^{\alpha-q_2-1} (1-\tau)^{\alpha-q_0-1} h(\tau)\,d\tau \right. \\
&\quad \left. - \int_0^s (s-\tau)^{\alpha-q_2-1} h(\tau)\,d\tau \right) g_2(s)\,dA_2(s) \Bigg\} \\
&\quad + \frac{c_3 t^{\alpha-\alpha_{n-2}-1}}{\sigma\Gamma(\alpha-\alpha_{n-2})\Gamma(\alpha-q_3)} \sum_{i=1}^{m} \gamma_i \left(\int_0^1 \xi_i^{\alpha-q_3-1} (1-\tau)^{\alpha-q_0-1} h(\tau)\,d\tau \right. \\
&\quad \left. - \int_0^{\xi_i} (\xi_i - \tau)^{\alpha-q_3-1} h(\tau)\,d\tau \right).
\end{aligned}
$$

Thus problem (2.11) has a unique solution

$$
\begin{aligned}
v(t) &= \int_0^1 H_0(t,s) h(s)\,ds + \int_0^1 t^{\alpha-\alpha_{n-2}-1} \left(\int_0^1 H_1(s,\tau) h(\tau)\,d\tau \right) g_1(s)\,dA_1(s) \\
&\quad + \int_0^\theta t^{\alpha-\alpha_{n-2}-1} \left(\int_0^1 H_2(s,\tau) h(\tau)\,d\tau \right) g_2(s)\,dA_2(s)
\end{aligned}
$$

$$
\begin{aligned}
&+ t^{\alpha-\alpha_{n-2}-1}\int_0^1 \sum_{i=1}^m \gamma_i H_3(\xi_i,s)h(s)\,ds \\
&= \int_0^1 H_0(t,s)h(s)\,ds + \int_0^1 t^{\alpha-\alpha_{n-2}-1}\left(\int_0^1 H_1(\tau,s)g_1(\tau)\,dA_1(\tau)\right)h(s)\,ds \\
&\quad + \int_0^1 t^{\alpha-\alpha_{n-2}-1}\left(\int_0^\theta H_2(\tau,s)g_2(\tau)\,dA_2(\tau)\right)h(s)\,ds \\
&\quad + \int_0^1 t^{\alpha-\alpha_{n-2}-1}\sum_{i=1}^m \gamma_i H_3(\xi_i,s)h(s)\,ds \\
&= \int_0^1 H(t,s)h(s)\,ds, \qquad (2.21)
\end{aligned}
$$

where $H(t,s)$ is defined by (2.13). □

Lemma 2.6 *Let $c_i \geq 0$ $(i = 1,2,3)$, $\sigma > 0$ (defined in Lemma 2.5), and*

$$
\int_0^1 t^{\alpha-q_1-1}g_1(t)\,dA_1(t) \geq 0, \qquad \int_0^\theta t^{\alpha-q_2-1}g_2(t)\,dA_2(t) \geq 0.
$$

Then the functions $H_i(t,s)$ $(i = 0,1,2,3)$ defined in Lemma 2.5 satisfy the following properties:

(1) $H_i \in C([0,1]\times[0,1],[0,+\infty))$, *and* $H_i(t,s) > 0$ *for* $t,s \in (0,1)$, $i = 0,1,2,3$.

(2) $t^{\alpha-\alpha_{n-2}-1}K_0(s) \leq H_0(t,s) \leq t^{\alpha-\alpha_{n-2}-1}J_0(s)$ *for* $t,s \in [0,1]$, *where*

$$
K_0(s) = \frac{(1-s)^{\alpha-q_0-1}}{\Gamma(\alpha-\alpha_{n-2})}\bigl(1-(1-s)^{q_0-\alpha_{n-2}}\bigr), \qquad J_0(s) = \frac{1}{\Gamma(\alpha-\alpha_{n-2})}(1-s)^{\alpha-q_0-1}.
$$

(3) $t^{\alpha-q_i-1}K_i(s) \leq H_i(t,s) \leq t^{\alpha-q_i-1}J_i(s)$ *for* $t,s \in [0,1]$, *where*

$$
\begin{aligned}
K_i(s) &= \frac{c_i}{\sigma\Gamma(\alpha-\alpha_{n-2})\Gamma(\alpha-q_i)}(1-s)^{\alpha-q_0-1}\bigl(1-(1-s)^{q_0-q_i}\bigr), \quad i = 1,2,3, \\
J_i(s) &= \frac{c_i}{\sigma\Gamma(\alpha-\alpha_{n-2})\Gamma(\alpha-q_i)}(1-s)^{\alpha-q_0-1}, \quad i = 1,2,3.
\end{aligned}
$$

Proof Obviously, (1) holds. We only prove (2) and (3).

(2) For $0 \leq s \leq t \leq 1$,

$$
\begin{aligned}
H_0(t,s) &\geq \frac{1}{\Gamma(\alpha-\alpha_{n-2})}\bigl\{t^{\alpha-\alpha_{n-2}-1}(1-s)^{\alpha-q_0-1} - (t-ts)^{\alpha-\alpha_{n-2}-1}\bigr\} \\
&\geq \frac{1}{\Gamma(\alpha-\alpha_{n-2})}t^{\alpha-\alpha_{n-2}-1}(1-s)^{\alpha-q_0-1}\bigl(1-(1-s)^{q_0-\alpha_{n-2}}\bigr) \\
&= t^{\alpha-\alpha_{n-2}-1}K_0(s), \qquad (2.22)
\end{aligned}
$$

$$
H_0(t,s) \leq \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha-\alpha_{n-2})}(1-s)^{\alpha-q_0-1} = t^{\alpha-\alpha_{n-2}-1}J_0(s). \qquad (2.23)
$$

For $0 \le t \le s \le 1$,

$$H_0(t,s) \ge \frac{1}{\Gamma(\alpha-\alpha_{n-2})} t^{\alpha-\alpha_{n-2}-1}(1-s)^{\alpha-q_0-1}\bigl(1-(1-s)^{q_0-\alpha_{n-2}}\bigr)$$
$$= t^{\alpha-\alpha_{n-2}-1}K_0(s), \tag{2.24}$$

$$H_0(t,s) = \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha-\alpha_{n-2})}(1-s)^{\alpha-q_0-1} = t^{\alpha-\alpha_{n-2}-1}J_0(s). \tag{2.25}$$

It follows from (2.22)–(2.25) that

$$t^{\alpha-\alpha_{n-2}-1}K_0(s) \le H_0(t,s) \le t^{\alpha-\alpha_{n-2}-1}J_0(s).$$

(3) In a similar manner, we have

$$t^{\alpha-q_i-1}K_i(s) \le H_i(t,s) \le t^{\alpha-q_i-1}J_i(s), \quad i = 1,2,3.$$

We omit the details. □

Lemma 2.7 *Let $c_i \ge 0$ ($i = 1,2,3$), $\sigma > 0$ (defined in Lemma 2.5), and*

$$\int_0^1 t^{\alpha-q_1-1}g_1(t)\,dA_1(t) \ge 0, \qquad \int_0^\theta t^{\alpha-q_2-1}g_2(t)\,dA_2(t) \ge 0.$$

Then the Green's function $H(t,s)$ (defined in (2.13)) has the following properties:

(1) *$H(t,s) \in C([0,1]\times[0,1],[0,+\infty))$, and $H(t,s) > 0$ for $t,s \in (0,1)$.*

(2) *$t^{\alpha-\alpha_{n-2}-1}K(s) \le H(t,s) \le t^{\alpha-\alpha_{n-2}-1}J(s)$ for $t,s \in [0,1]$, where*

$$K(s) = K_0(s) + K_1(s)\left(\int_0^1 \tau^{\alpha-q_1-1}g_1(\tau)\,dA_1(\tau)\right)$$
$$+ K_2(s)\left(\int_0^\theta \tau^{\alpha-q_2-1}g_2(\tau)\,dA_2(\tau)\right) + K_3(s)\sum_{i=1}^m \gamma_i\xi_i^{\alpha-q_3-1},$$
$$J(s) = J_0(s) + J_1(s)\left(\int_0^1 \tau^{\alpha-q_1-1}g_1(\tau)\,dA_1(\tau)\right)$$
$$+ J_2(s)\left(\int_0^\theta \tau^{\alpha-q_2-1}g_2(\tau)\,dA_2(\tau)\right) + J_3(s)\sum_{i=1}^m \gamma_i\xi_i^{\alpha-q_3-1}.$$

Proof The conclusion can be directly deduced from Lemma 2.6. So, we omit the details. □

Definition 2.3 ([8]) Let P be a cone in the Banach space E. A positive linear operator $A: E \to E$ is called a u_0-bounded linear operator if there exists $u_0 \in P\setminus\{\theta\}$ such that for any $v \in P\setminus\{\theta\}$, there exist constants $\alpha,\beta > 0$ and $n \in N_+$ such that $\alpha u_0 \le A^n v \le \beta u_0$.

Lemma 2.8 ([9, 10]) *Let P be a cone in the Banach space E, and let $A: E \to E$ be a completely continuous positive linear operator. If there exist $z \in P - P\setminus(-P)$ and $d > 0$ such that $dAz \ge z$, then the spectral radius $r(A) \ne 0$, and A has a positive eigenfunction ψ that belongs to the first eigenvalue $\lambda_1 = (r(A))^{-1}$ such that $\lambda_1 A\psi = \psi$.*

Lemma 2.9 ([9, 10]) *Let P be a generating cone in the Banach space E, that is, $E = P - P$. Let A is a completely continuous u_0-bounded linear operator. If the spectral radius $r(A) \neq 0$, then A has an eigenfunction $\psi \in P \setminus \{\theta\}$ that belongs to the first eigenvalue $\lambda_1 = (r(A))^{-1}$ such that $\lambda_1 A\psi = \psi$, and A has no other positive eigenvalue that has positive eigenfunctions.*

In this paper, we define the Banach space $E = C[0,1]$ with the norm $\|v\| = \sup_{0\leq t\leq 1} |v(t)|$. Let $P = \{v \in E, v(t) \geq 0, t \in [0,1]\}$ be a cone in E. It is easy to check that P is generating in E, that is, $E = P - P$. Now, we define the nonlinear operator $T : E \to E$ by

$$\begin{aligned}(Tv)(t) = \int_0^1 H(t,s)f\big(&s, I_{0+}^{\alpha_{n-2}}v(s), I_{0+}^{\alpha_{n-2}-\alpha_1}v(s),\\ &\ldots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}v(s), v(s)\big)\,ds, \quad t \in [0,1]. \end{aligned} \tag{2.26}$$

Observe that v is a solution of problem (2.1) if and only if v is a fixed point of the operator T in E.

3 Existence and uniqueness of solutions

Now we make the following assumptions:

(C_1) $f : (0,1) \times (-\infty,+\infty)^{n-1} \to (-\infty,+\infty)$ satisfies the Carathéodory conditions, that is:
  (1) $f(\cdot, x_0, x_1, \ldots, x_{n-2}) : (0,1) \to (-\infty,+\infty)$ is measurable for any fixed $(x_0, x_1, \ldots, x_{n-2}) \in (-\infty,+\infty)^{n-1}$;
  (2) $f(t,\cdot,\cdot,\ldots,\cdot) : (-\infty,+\infty)^{n-1} \to (-\infty,+\infty)$ is continuous for a.e. $t \in [0,1]$;
  (3) for any $r > 0$, there exists $\psi_r \in L^1(0,1)$ such that $|f(t,x_0,x_1,\ldots,x_{n-2})| \leq \psi_r(t)$ for all $x_i \in [-r,r]$ $(i = 0,1,\ldots,n-2)$ and a.e. $t \in [0,1]$.

(C_2) For any $r > 0$, there exist nonnegative functions $k_{r,i}(t) \in L^1(0,1)$ $(i = 0,1,\ldots,n-2)$ such that for all $t \in (0,1)$ and $(x_0,x_1,\ldots,x_{n-2}), (y_0,y_1,\ldots,y_{n-2}) \in [-r,r]^{n-1}$,

$$\big|f(t,x_0,\ldots,x_{n-2}) - f(t,y_0,\ldots,y_{n-2})\big| \leq k_{r,0}(t)|x_0 - y_0| + \cdots + k_{r,n-2}(t)|x_{n-2} - y_{n-2}|.$$

(C_3) $f_0(t) = |f(t,0,0,\ldots,0)| \in L^1(0,1)$, and $f_0(t)$ is not identical zero in any closed subinterval of $(0,1)$.

(C_4) There exist nonnegative functions $q_i(t) \in L^1(0,1)$ $(i = 0,1,\ldots,n-2)$ such that for any $t \in (0,1)$ and $(x_0,x_1,\ldots,x_{n-2}), (y_0,y_1,\ldots,y_{n-2}) \in (-\infty,+\infty)^{n-1}$,

$$\big|f(t,x_0,\ldots,x_{n-2}) - f(t,y_0,\ldots,y_{n-2})\big| \leq p_0(t)|x_0 - y_0| + \cdots + p_{n-2}(t)|x_{n-2} - y_{n-2}|.$$

In the following, for convenience, we set

$$m_\Gamma = \min_{t>0} \Gamma(t), \qquad M_H = \max_{t,s\in[0,1]} H(t,s),$$

where Γ is the Euler gamma function, that is, $\Gamma(t) = \int_0^{+\infty} s^{t-1}e^{-s}\,ds$ $(t > 0)$. Obviously, $M_H > 0$. By the definition of Γ we have $0 < m_\Gamma < 1$.

Theorem 3.1 *Assume the condition (C_1) holds. If $\limsup_{r\to+\infty} \frac{\|\psi_r\|_L}{r} < \frac{m_\Gamma}{M_H}$, then there exists at least one solution of problem (1.1) on $[0,1]$, where $\|\psi_r\|_L = \int_0^1 |\psi_r(s)|\,ds$.*

Proof Obviously, $T : E \to E$ is well defined. By the definition of upper limit there exists $r_0 > 0$ such that

$$\frac{\|\psi_r\|_L}{r} \le \frac{m_\Gamma}{M_H}, \quad r \ge r_0.$$

For any $r_1 \ge r_0$, there exists $\psi_{r_1} \in L^1(0,1)$ such that

$$\left|f(t, x_0, \ldots, x_{n-2})\right| \le \psi_{r_1}(t), \quad x_i \in [-r_1, r_1], i = 0, 1, \ldots, n-2, \text{a.e. } t \in [0,1], \tag{3.1}$$

and $r_1 \ge \frac{M_H \|\psi_{r_1}\|_L}{m_\Gamma}$. Let $B_{r_2} = \{v \in E : \|v\| \le r_2, M_H\|\psi_{r_1}\|_L \le r_2 \le m_\Gamma r_1\}$. As a first step, we show that $TB_{r_2} \subset B_{r_2}$.

By calculation, for any $v \in B_{r_2}$, $t \in [0,1]$, we have $v(t) \le r_2 \le r_1$, and

$$\begin{aligned}
\left|I_{0+}^{\alpha_{n-2}} v(t)\right| &= \frac{1}{\Gamma(\alpha_{n-2})}\left|\int_0^t (t-s)^{\alpha_{n-2}-1} v(s)\, ds\right| \\
&\le \frac{r_2}{\Gamma(\alpha_{n-2})}\left|\int_0^t (t-s)^{\alpha_{n-2}-1}\, ds\right| \\
&\le \frac{r_2}{\Gamma(\alpha_{n-2}+1)} \le r_1,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\left|I_{0+}^{\alpha_{n-2}-\alpha_i} v(t)\right| &= \frac{1}{\Gamma(\alpha_{n-2}-\alpha_i)}\left|\int_0^t (t-s)^{\alpha_{n-2}-\alpha_i-1} v(s)\, ds\right| \\
&\le \frac{r_2}{\Gamma(\alpha_{n-2}-\alpha_i)}\left|\int_0^t (t-s)^{\alpha_{n-2}-\alpha_i-1}\, ds\right| \\
&\le \frac{r_2}{\Gamma(\alpha_{n-2}-\alpha_i+1)} \\
&\le \frac{m_\Gamma r_1}{\Gamma(\alpha_{n-2}-\alpha_i+1)} \le r_1, \quad i = 1, 2, \ldots, n-3,
\end{aligned} \tag{3.3}$$

which, together with (3.1), means that

$$\|Tv\| \le M_H \left|\int_0^1 \psi_{r_1}(s)\, ds\right| = M_H\|\psi_{r_1}\|_L \le r_2. \tag{3.4}$$

Thus $TB_{r_2} \subset B_{r_2}$, so that TB_{r_2} is uniformly bounded.

Next, we show that TB_{r_2} is equicontinuous. For any $v \in B_{r_2}$, $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}
&\left|Tv(t_1) - Tv(t_2)\right| \\
&\quad\le \int_0^1 \left|H(t_1,s) - H(t_2,s)\right|\left|f\left(s, I_{0+}^{\alpha_{n-2}} v(s), \ldots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v(s), v(s)\right)\right| ds \\
&\quad\le \int_0^1 \left|H(t_1,s) - H(t_2,s)\right|\left|\psi_{r_1}(s)\right| ds.
\end{aligned} \tag{3.5}$$

Since $H(t,s)$ is continuous on $[0,1] \times [0,1]$, it is uniformly continuous on $[0,1] \times [0,1]$. Since $\psi_{r_1} \in L^1(0,1)$, (3.5) implies that TB_{r_2} is equicontinuous on $[0,1]$.

In the following, we show that $T : B_{r_2} \to B_{r_2}$ is continuous. Let $\{v_m\} \subset B_{r_2}$ be a sequence such that $\lim_{m\to\infty} v_m = v$ in E. For fixed $s \in [0,1] \setminus \Xi$, where $\operatorname{mes}(\Xi) = 0$,

$f(s,x_0,x_1,\ldots,x_{n-2})$ is continuous on $(-\infty,+\infty)^{n-1}$ with respect to $(x_0,x_1,\ldots,x_{n-2})$. It follows that $f(s,x_0,x_1,\ldots,x_{n-2})$ is uniformly continuous with respect to $(x_0,x_1,\ldots,x_{n-2}) \in [-r_1,r_1]^{n-1}$. Hence, for fixed $s \in [0,1] \setminus \Xi$, for all $\varepsilon > 0$, there exists $\delta = \delta(s,\varepsilon) > 0$ such that, for any $x_i, y_i \in [-r_1,r_1]$ such that $|x_i - y_i| < \delta$ $(i = 0,1,\ldots,n-2)$, we have

$$\bigl|f(s,x_0,x_1,\ldots,x_{n-2}) - f(s,y_0,y_1,\ldots,y_{n-2})\bigr| < \varepsilon. \tag{3.6}$$

Since $\lim_{m\to\infty} v_m = v$, for the above $\delta > 0$, there exists $N_0 \in N_+$ such that, for any $m \geq N_0$,

$$\|v_m - v\| < m_\Gamma \delta < \delta.$$

Especially, we have

$$\bigl|v_m(s) - v(s)\bigr| < m_\Gamma \delta < \delta. \tag{3.7}$$

Furthermore, we calculate that

$$\begin{aligned}
&\bigl|I_{0+}^{\alpha_{n-2}} v_m(s) - I_{0+}^{\alpha_{n-2}} v(s)\bigr| \\
&\quad = \frac{1}{\Gamma(\alpha_{n-2})}\left|\int_0^s (s-\tau)^{\alpha_{n-2}-1} v_m(\tau)\,d\tau - \int_0^s (s-\tau)^{\alpha_{n-2}-1} v(\tau)\,d\tau\right| \\
&\quad \leq \frac{\|v_m - v\|}{\Gamma(\alpha_{n-2})}\int_0^s (s-\tau)^{\alpha_{n-2}-1}\,d\tau \\
&\quad < \frac{m_\Gamma \delta}{\Gamma(\alpha_{n-2})}\int_0^s (s-\tau)^{\alpha_{n-2}-1}\,d\tau \\
&\quad \leq \frac{m_\Gamma \delta}{\Gamma(\alpha_{n-2}+1)} \leq \delta,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\bigl|I_{0+}^{\alpha_{n-2}-\alpha_i} v_m(s) - I_{0+}^{\alpha_{n-2}-\alpha_i} v(s)\bigr| \\
&\quad = \frac{1}{\Gamma(\alpha_{n-2}-\alpha_i)}\left|\int_0^s (s-\tau)^{\alpha_{n-2}-\alpha_i-1} v_m(\tau)\,d\tau - \int_0^s (s-\tau)^{\alpha_{n-2}-\alpha_i-1} v(\tau)\,d\tau\right| \\
&\quad \leq \frac{\|v_m - v\|}{\Gamma(\alpha_{n-2}-\alpha_i)}\int_0^s (s-\tau)^{\alpha_{n-2}-\alpha_i-1}\,d\tau \\
&\quad < \frac{m_\Gamma \delta}{\Gamma(\alpha_{n-2}-\alpha_i)}\int_0^s (s-\tau)^{\alpha_{n-2}-\alpha_i-1}\,d\tau \\
&\quad \leq \frac{m_\Gamma \delta}{\Gamma(\alpha_{n-2}-\alpha_i+1)} \leq \delta, \quad i = 1,2,\ldots,n-3.
\end{aligned} \tag{3.9}$$

Hence by (3.6)–(3.9) we infer that for fixed $s \in [0,1] \setminus \Gamma$, for $m \geq N_0$,

$$\begin{aligned}
&\bigl|f\bigl(s, I_{0+}^{\alpha_{n-2}} v_m(s), I_{0+}^{\alpha_{n-2}-\alpha_1} v_m(s), \ldots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v_m(s), v_m(s)\bigr) \\
&\quad - f\bigl(s, I_{0+}^{\alpha_{n-2}} v(s), I_{0+}^{\alpha_{n-2}-\alpha_1} v(s), \ldots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v(s), v(s)\bigr)\bigr| < \varepsilon.
\end{aligned}$$

Evidently, we derive that, for a.e. $s \in [0,1]$,

$$\begin{aligned}
&\bigl|f\bigl(s, I_{0+}^{\alpha_{n-2}} v_m(s), I_{0+}^{\alpha_{n-2}-\alpha_1} v_m(s), \ldots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v_m(s), v_m(s)\bigr) \\
&\quad - f\bigl(s, I_{0+}^{\alpha_{n-2}} v(s), I_{0+}^{\alpha_{n-2}-\alpha_1} v(s), \ldots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v(s), v(s)\bigr)\bigr| \to 0, \quad m \to \infty.
\end{aligned} \tag{3.10}$$

On the other hand, condition (C_1) implies that, for a.e. $s \in [0,1]$,

$$\begin{aligned} &\Big|f\big(s, I_{0+}^{\alpha_{n-2}} v_m(s), I_{0+}^{\alpha_{n-2}-\alpha_1} v_m(s), \dots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v_m(s), v_m(s)\big) \\ &\quad - f\big(s, I_{0+}^{\alpha_{n-2}} v(s), I_{0+}^{\alpha_{n-2}-\alpha_1} v(s), \dots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v(s), v(s)\big)\Big| \\ &\leq 2\psi_{r_1}(s), \end{aligned} \tag{3.11}$$

where $\psi_{r_1} \in L^1(0,1)$. The Lebesgue dominated convergence theorem and (3.10) guarantee that, for any $t \in [0,1]$,

$$\begin{aligned} &\big|Tv_m(t) - Tv(t)\big| \\ &= \bigg| \int_0^1 H(t,s)\big(f\big(s, I_{0+}^{\alpha_{n-2}} v_m(s), \dots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v_m(s), v_m(s)\big) \\ &\quad - f\big(s, I_{0+}^{\alpha_{n-2}} v(s), \dots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v(s), v(s)\big)\big)\, ds \bigg| \\ &\leq M_H \int_0^1 \Big|f\big(s, I_{0+}^{\alpha_{n-2}} v_m(s), I_{0+}^{\alpha_{n-2}-\alpha_1} v_m(s), \dots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v_m(s), v_m(s)\big) \\ &\quad - f\big(s, I_{0+}^{\alpha_{n-2}} v(s), \dots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v(s), v(s)\big)\Big|\, ds \to 0, \quad m \to \infty, \end{aligned} \tag{3.12}$$

that is,

$$\|Tv_m - Tv\| = \max_{t\in[0,1]} \big|Tv_m(t) - Tv(t)\big| \to 0, \quad m \to \infty. \tag{3.13}$$

Thus $T : B_{r_2} \to B_{r_2}$ is continuous. From the above proof we deduce that the operator $T : B_{r_2} \to B_{r_2}$ is completely continuous. By using the Schauder fixed point theorem T has a fixed point v in B_{r_2}, that is, v is a solution of problem (2.1) on $[0,1]$. Furthermore, in view of Lemma 2.4 and (3.2), we derive that $u = I_{0+}^{\alpha_{n-2}} v$ is a solution of problem (1.1) that satisfies $\|u\| \leq \frac{r_2}{\Gamma(\alpha_{n-2}+1)}$. □

Theorem 3.2 *Assume that a function $f \in C((0,1) \times (-\infty,+\infty)^{n-1}, (-\infty,+\infty))$ satisfies (C_2) and (C_3). Then problem (1.1) has a unique solution, provided that*

$$\liminf_{r\to+\infty} \left(m_\Gamma - M_H \sum_{i=0}^{n-2} \|k_{r,i}\|_L \right) r > M_H \|f_0\|_L > 0,$$

where $\|k_{r,i}\|_L = \int_0^1 k_{r,i}(t)\,dt$ $(i = 0,1,\dots,n-2)$ and $\|f_0\|_L = \int_0^1 |f(t,0,0,\dots,0)|\,dt$.

Proof By the definition of lower limit, for $r_3 > 0$ large enough, there exist nonnegative functions $k_{r_3,i}(t) \in L^1(0,1)$ $(i = 0,1,\dots,n-2)$ satisfying (C_2) and

$$0 < M_H \|f_0\|_L \leq \left(m_\Gamma - M_H \sum_{i=0}^{n-2} \|k_{r_3,i}\|_L \right) r_3. \tag{3.14}$$

Let $B_{r_4} = \{u \in E : \|x\| \leq r_4, M_H(r_3 \sum_{i=0}^{n-2} \|k_{r_3,i}\|_L + \|f_0\|_L) \leq r_4 \leq m_\Gamma r_3\}$. We first prove that $TB_{r_4} \subset B_{r_4}$. By the same proof as that of Theorem 3.1, for any $v \in B_{r_4}$, $t \in (0,1)$, we have

$v(t) \le r_4 \le r_3$,

$$\left|I_{0+}^{\alpha_{n-2}} v(t)\right| \le \frac{r_4}{\Gamma(\alpha_{n-2}+1)} \le \frac{m_\Gamma r_3}{\Gamma(\alpha_{n-2}+1)} \le r_3 \tag{3.15}$$

and

$$\left|I_{0+}^{\alpha_{n-2}-\alpha_i} v(t)\right| \le \frac{r_4}{\Gamma(\alpha_{n-2}-\alpha_i+1)} \le r_3, \quad i = 1,2,\ldots,n-3. \tag{3.16}$$

Hence we get that, for any $v \in B_{r_4}$ and $t \in (0,1)$,

$$\begin{aligned}
&\left|f\bigl(t, I_{0+}^{\alpha_{n-2}} v(t), I_{0+}^{\alpha_{n-2}-\alpha_1} v(t), \ldots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v(t), v(t)\bigr)\right| \\
&\quad \le \left|f\bigl(t, I_{0+}^{\alpha_{n-2}} v(t), I_{0+}^{\alpha_{n-2}-\alpha_1} v(t), \ldots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v(t), v(t)\bigr) - f(t,0,0,\ldots,0)\right| \\
&\qquad + \left|f(t,0,0,\ldots,0)\right| \\
&\quad \le \bigl(k_{r_3,0}(t) + k_{r_3,1}(t) + \cdots + k_{r_3,n-2}(t)\bigr) r_3 + \left|f(t,0,0,\ldots,0)\right|,
\end{aligned} \tag{3.17}$$

which implies

$$\begin{aligned}
\|Tv\| &\le M_H \int_0^1 \left|f\bigl(s, I_{0+}^{\alpha_{n-2}} v(s), I_{0+}^{\alpha_{n-2}-\alpha_1} v(s), \ldots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v(s), v(s)\bigr)\right| \, ds \\
&\le M_H \int_0^1 \Bigl(\bigl(k_{r_3,0}(s) + k_{r_3,1}(s) + \cdots + k_{r_3,n-2}(s)\bigr) r_3 + \left|f(s,0,0,\ldots,0)\right|\Bigr) \, ds \\
&= M_H \left(r_3 \sum_{i=0}^{n-2} \|k_{r_3,i}\|_L + \|f_0\|_L \right) \\
&\le r_4,
\end{aligned} \tag{3.18}$$

that is, $TB_{r_4} \subset B_{r_4}$. In the following, we prove that T is a contraction mapping. In fact, for any $u, v \in B_{r_4}$ and $t \in (0,1)$, we have

$$\left|I_{0+}^{\alpha_{n-2}} u(t) - I_{0+}^{\alpha_{n-2}} v(t)\right| \le \frac{\|u-v\|}{\Gamma(\alpha_{n-2}+1)} \le \frac{1}{m_\Gamma}\|u-v\|, \tag{3.19}$$

$$\begin{aligned}
\left|I_{0+}^{\alpha_{n-2}-\alpha_i} u(t) - I_{0+}^{\alpha_{n-2}-\alpha_i} v(t)\right| &\le \frac{\|u-v\|}{\Gamma(\alpha_{n-2}-\alpha_i+1)} \\
&\le \frac{1}{m_\Gamma}\|u-v\| \quad (i = 1,\ldots,n-3).
\end{aligned} \tag{3.20}$$

Moreover, from (3.15) and (3.16) we deduce that, for any $u, v \in B_{r_4}$ and $t \in (0,1)$,

$$\begin{aligned}
&\bigl|f\bigl(t, I_{0+}^{\alpha_{n-2}} u(t), I_{0+}^{\alpha_{n-2}-\alpha_1} u(t), \ldots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} u(t), u(t)\bigr) \\
&\qquad - f\bigl(t, I_{0+}^{\alpha_{n-2}} v(t), I_{0+}^{\alpha_{n-2}-\alpha_1} v(t), \ldots, I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v(t), v(t)\bigr)\bigr| \\
&\quad \le k_{r_3,0}(t)\left|I_{0+}^{\alpha_{n-2}} u(t) - I_{0+}^{\alpha_{n-2}} v(t)\right| + k_{r_3,1}(t)\left|I_{0+}^{\alpha_{n-2}-\alpha_1} u(t) - I_{0+}^{\alpha_{n-2}-\alpha_1} v(t)\right| \\
&\qquad + \cdots \\
&\qquad + k_{r_3,n-3}(t)\left|I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} u(t) - I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v(t)\right| + k_{r_3,n-2}(t)\left|u(t) - v(t)\right|
\end{aligned}$$

$$= k_{r_3,0}(t)\left|I_{0+}^{\alpha_{n-2}}\big(u(t)-v(t)\big)\right| + k_{r_3,1}(t)\left|I_{0+}^{\alpha_{n-2}}\big(u(t)-v(t)\big)\right| + \cdots$$
$$+ k_{r_3,n-3}(t)\left|I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}\big(u(t)-v(t)\big)\right| + k_{r_3,n-2}(t)\left|u(t)-v(t)\right|. \tag{3.21}$$

Therefore from (3.19)–(3.21) we deduce that

$$\begin{aligned}
&\|Tu-Tv\| \\
&\quad\le \sup_{t\in[0,1]}\int_0^1 \left|H(t,s)\right|\left|f\big(s, I_{0+}^{\alpha_{n-2}}u(s), I_{0+}^{\alpha_{n-2}-\alpha_1}u(s),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}u(s),u(s)\big)\right. \\
&\qquad \left.-f\big(s, I_{0+}^{\alpha_{n-2}}v(s), I_{0+}^{\alpha_{n-2}-\alpha_1}v(s),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}v(s),v(s)\big)\right| ds \\
&\quad\le M_H\int_0^1 \left|f\big(s, I_{0+}^{\alpha_{n-2}}u(s), I_{0+}^{\alpha_{n-2}-\alpha_1}u(s),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}u(s),u(s)\big)\right. \\
&\qquad \left.-f\big(s, I_{0+}^{\alpha_{n-2}}v(s), I_{0+}^{\alpha_{n-2}-\alpha_1}v(s),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}v(s),v(s)\big)\right| ds \\
&\quad\le M_H\int_0^1 \Big(k_{r_3,0}(s)\left|I_{0+}^{\alpha_{n-2}}\big(u(s)-v(s)\big)\right| + k_{r_3,1}(s)\left|I_{0+}^{\alpha_{n-2}}\big(u(s)-v(s)\big)\right| + \cdots \\
&\qquad + k_{r_3,n-3}(s)\left|I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}\big(u(s)-v(s)\big)\right| + k_{r_3,n-2}(s)\left|u(s)-v(s)\right|\Big)\, ds \\
&\quad\le \frac{M_H}{m_\Gamma}\|u-v\|\int_0^1 \big(k_{r_3,0}(s)+k_{r_3,1}(s)+\cdots+k_{r_3,n-2}(s)\big)\, ds \\
&\quad= \frac{M_H}{m_\Gamma}\sum_{i=0}^{n-2}\|k_{r_3,i}\|_L\|u-v\|.
\end{aligned} \tag{3.22}$$

From (3.14) we get that $\frac{M_H}{m_\Gamma}\sum_{i=0}^{n-2}\|k_{r_3,i}\|_L < 1$, which allows us to infer that T is a contraction mapping. In consequence, by the Banach contraction mapping principle we deduce that problem (2.1) has a unique solution v on $[0,1]$. In addition, according to Lemma 2.4, we find that $u = I_{0+}^{\alpha_{n-2}}v$ is the unique solution of problem (1.1) on $[0,1]$, which satisfies $\|u\| \le \frac{r_4}{\Gamma(\alpha_{n-2}+1)}$. □

Theorem 3.3 *Let $f : (0,1)\times(-\infty,+\infty)^{n-1} \to (-\infty,+\infty)$ be a continuous function such that condition (C_4) holds. Then problem (1.1) has a unique solution, provided that $\frac{M_H}{m_\Gamma}\sum_{i=0}^{n-2}\|p_i\|_L < 1$, where $\|p_i\|_L = \int_0^1 p_i(t)\,dt$ $(i = 0,1,\ldots,n-2)$.*

Proof For any $u, v \in E$ and $t \in (0,1)$, we have

$$\left|I_{0+}^{\alpha_{n-2}}u(t) - I_{0+}^{\alpha_{n-2}}v(t)\right| \le \frac{\|u-v\|}{\Gamma(\alpha_{n-2}+1)} \le \frac{1}{m_\Gamma}\|u-v\|, \tag{3.23}$$

$$\begin{aligned}
\left|I_{0+}^{\alpha_{n-2}-\alpha_i}u(t) - I_{0+}^{\alpha_{n-2}-\alpha_i}v(t)\right| &\le \frac{\|u-v\|}{\Gamma(\alpha_{n-2}-\alpha_i+1)} \\
&\le \frac{1}{m_\Gamma}\|u-v\|, \quad i = 1,\ldots,n-3.
\end{aligned} \tag{3.24}$$

Moreover, by (C_4) we have

$$\begin{aligned}
&\left|f\big(t, I_{0+}^{\alpha_{n-2}}u(t), I_{0+}^{\alpha_{n-2}-\alpha_1}u(t),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}u(t),u(t)\big)\right. \\
&\quad \left.-f\big(t, I_{0+}^{\alpha_{n-2}}v(t), I_{0+}^{\alpha_{n-2}-\alpha_1}v(t),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}v(t),v(t)\big)\right|
\end{aligned}$$

$$\begin{aligned}
&\leq p_0(t)\bigl|I_{0+}^{\alpha_{n-2}}u(t)-I_{0+}^{\alpha_{n-2}}v(t)\bigr|+p_1(t)\bigl|I_{0+}^{\alpha_{n-2}-\alpha_1}u(t)-I_{0+}^{\alpha_{n-2}-\alpha_1}v(t)\bigr|+\cdots\\
&\quad+p_{n-3}(t)\bigl|I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}u(t)-I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}v(t)\bigr|+p_{n-2}(t)\bigl|u(t)-v(t)\bigr|\\
&=p_0(t)\bigl|I_{0+}^{\alpha_{n-2}}\bigl(u(t)-v(t)\bigr)\bigr|+p_1(t)\bigl|I_{0+}^{\alpha_{n-2}}\bigl(u(t)-v(t)\bigr)\bigr|+\cdots\\
&\quad+p_{n-3}(t)\bigl|I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}\bigl(u(t)-v(t)\bigr)\bigr|+p_{n-2}(t)\bigl|u(t)-v(t)\bigr|.
\end{aligned}\tag{3.25}$$

Hence it follows that

$$\begin{aligned}
&\|Tu-Tv\|\\
&\quad\leq\sup_{t\in[0,1]}\int_0^1\bigl|H(t,s)\bigr|\bigl|f\bigl(s,I_{0+}^{\alpha_{n-2}}u(s),I_{0+}^{\alpha_{n-2}-\alpha_1}u(s),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}u(s),u(s)\bigr)\\
&\quad\quad-f\bigl(s,I_{0+}^{\alpha_{n-2}}v(s),I_{0+}^{\alpha_{n-2}-\alpha_1}v(s),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}v(s),v(s)\bigr)\bigr|\,ds\\
&\quad\leq M_H\int_0^1\bigl|f\bigl(s,I_{0+}^{\alpha_{n-2}}u(s),I_{0+}^{\alpha_{n-2}-\alpha_1}u(s),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}u(s),u(s)\bigr)\\
&\quad\quad-f\bigl(s,I_{0+}^{\alpha_{n-2}}v(s),I_{0+}^{\alpha_{n-2}-\alpha_1}v(s),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}v(s),v(s)\bigr)\bigr|\,ds\\
&\quad\leq M_H\int_0^1\bigl(p_0(s)\bigl|I_{0+}^{\alpha_{n-2}}\bigl(u(s)-v(s)\bigr)\bigr|+p_1(s)\bigl|I_{0+}^{\alpha_{n-2}}\bigl(u(s)-v(s)\bigr)\bigr|+\cdots\\
&\quad\quad+p_{n-3}(s)\bigl|I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}\bigl(u(s)-v(s)\bigr)\bigr|+p_{n-2}(s)\bigl|u(s)-v(s)\bigr|\bigr)\,ds\\
&\quad\leq\frac{M_H}{m_\Gamma}\|u-v\|\int_0^1\bigl(p_0(s)+p_1(s)+\cdots+p_{n-2}(s)\bigr)\,ds\\
&\quad=\frac{M_H}{m_\Gamma}\sum_{i=0}^{n-2}\|p_i\|_L\|u-v\|.
\end{aligned}\tag{3.26}$$

Since $\frac{M_H}{m_\Gamma}\sum_{i=0}^{n-2}\|p_i\|_L<1$, T is a contraction mapping. By the Banach contraction mapping principle we derive that problem (2.1) has a unique solution v on $[0,1]$. Furthermore, according to Lemma 2.4, we have that $u=I_{0+}^{\alpha_{n-2}}v$ is the unique solution of problem (1.1) on $[0,1]$. □

Remark 3.3 In Theorems 3.2 and 3.3, we get the uniqueness of solutions by using the Banach contraction mapping principle. It should be noted that in Theorem 3.2 the range of the solution is a ball, which is more accurate than Theorem 3.3. However, Theorem 3.3 has less restrictions on the corresponding parameters.

4 The uniqueness of positive solutions

Now, we make the following assumptions:

(C_5) there exist nonnegative functions $l_i(t)\in C(0,1)\cap L^1(0,1)$ $(i=0,1,\ldots,n-2)$ and a constant $k_1\geq 0$ such that for any $t\in[0,1]$ and $(x_0,\ldots,x_{n-2}),(y_0,\ldots,y_{n-2})\in[0,+\infty)^{n-1}$,

$$\begin{aligned}
&\bigl|f(t,x_0,\ldots,x_{n-2})-f(t,y_0,\ldots,y_{n-2})\bigr|\\
&\quad\leq k_1\bigl(l_0(t)|x_0-y_0|+\cdots+l_{n-2}(t)|x_{n-2}-y_{n-2}|\bigr).
\end{aligned}$$

Define the operator $L_1 : E \to E$ by

$$(L_1 v)(t) = \int_0^1 H(t,s)\big(l_0(s)I_{0+}^{\alpha_{n-2}} v(s) + l_1(s)I_{0+}^{\alpha_{n-2}-\alpha_1} v(s) + \cdots \\ + l_{n-3}(s)I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} v(s) + l_{n-2}(s)v(s)\big)\,ds, \quad t \in [0,1]. \tag{4.1}$$

Lemma 4.1 *The spectral radius $r(L_1) \neq 0$, and L_1 has a positive eigenfunction ψ_1 that belongs to the first eigenvalue $\lambda_{L_1} = (r(L_1))^{-1}$ such that $\lambda_{L_1} L_1 \psi_1 = \psi_1$.*

Proof We have

$$\begin{aligned}
(L_1 u)(t) &= \int_0^1 H(t,s)\big(l_0(s)I_{0+}^{\alpha_{n-2}} u(s) + l_1(s)I_{0+}^{\alpha_{n-2}-\alpha_1} u(s) \\
&\quad + \cdots + l_{n-3}(s)I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} u(s) + l_{n-2}(s)u(s)\big)\,ds \\
&= \int_0^1 \left(\frac{1}{\Gamma(\alpha_{n-2})} \int_\nu^1 H(t,s)l_0(s)(s-\nu)^{\alpha_{n-2}-1}\,ds\right)u(\nu)\,d\nu \\
&\quad + \int_0^1 \left(\frac{1}{\Gamma(\alpha_{n-2}-\alpha_1)} \int_\nu^1 H(t,s)l_1(s)(s-\nu)^{\alpha_{n-2}-\alpha_1-1}\,ds\right)u(\nu)\,d\nu \\
&\quad + \cdots \\
&\quad + \int_0^1 \left(\frac{1}{\Gamma(\alpha_{n-2}-\alpha_{n-3})} \int_\nu^1 H(t,s)l_{n-3}(s)(s-\nu)^{\alpha_{n-2}-\alpha_{n-3}-1}\,ds\right)u(\nu)\,d\nu \\
&\quad + \int_0^1 H(t,s)l_{n-2}(s)u(s)\,ds \\
&= \int_0^1 \left(\frac{1}{\Gamma(\alpha_{n-2})} \int_s^1 H(t,\nu)l_0(\nu)(\nu-s)^{\alpha_{n-2}-1}\,d\nu\right)u(s)\,ds \\
&\quad + \int_0^1 \left(\frac{1}{\Gamma(\alpha_{n-2}-\alpha_1)} \int_s^1 H(t,\nu)l_1(\nu)(\nu-s)^{\alpha_{n-2}-\alpha_1-1}\,d\nu\right)u(s)\,ds \\
&\quad + \cdots \\
&\quad + \int_0^1 \left(\frac{1}{\Gamma(\alpha_{n-2}-\alpha_{n-3})} \int_s^1 H(t,\nu)l_{n-3}(\nu)(\nu-s)^{\alpha_{n-2}-\alpha_{n-3}-1}\,d\nu\right)u(s)\,ds \\
&\quad + \int_0^1 H(t,s)l_{n-2}(s)u(s)\,ds \\
&= \int_0^1 K(t,s)u(s)\,ds,
\end{aligned}$$

where

$$\begin{aligned}
K(t,s) &= \frac{1}{\Gamma(\alpha_{n-2})} \int_s^1 H(t,\nu)l_0(\nu)(\nu-s)^{\alpha_{n-2}-1}\,d\nu \\
&\quad + \frac{1}{\Gamma(\alpha_{n-2}-\alpha_1)} \int_s^1 H(t,\nu)l_1(\nu)(\nu-s)^{\alpha_{n-2}-\alpha_1-1}\,d\nu \\
&\quad + \cdots
\end{aligned}$$

$$+ \frac{1}{\Gamma(\alpha_{n-2}-\alpha_{n-3})} \int_s^1 H(t,\nu) l_{n-3}(\nu)(\nu-s)^{\alpha_{n-2}-\alpha_{n-3}-1}\,d\nu + H(t,s) l_{n-2}(s). \tag{4.2}$$

By standard arguments the operator $L_1 : P \to P$ is completely continuous. By the properties of $H(t,s)$ there exists a closed interval $[t_0,t_1] \subset (0,1)$ such that $H(t,s) > 0$ for $t,s \in [t_0,t_1]$. Take a function $z \in C[0,1]$ satisfying $z(t) > 0$ for $t \in (t_0,t_1)$ and $z(t) = 0$ for $t \notin (t_0,t_1)$. Then, for any $t \in [t_0,t_1]$,

$$\begin{aligned}
(L_1 z)(t) &= \int_0^1 H(t,s)\bigl(l_0(s) I_{0+}^{\alpha_{n-2}} z(s) + l_1(s) I_{0+}^{\alpha_{n-2}-\alpha_1} z(s) \\
&\quad + \cdots + l_{n-3}(s) I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} z(s) + l_{n-2}(s) z(s)\bigr)\,ds \\
&\ge \int_a^b H(t,s)\bigl(l_0(s) I_{0+}^{\alpha_{n-2}} z(s) + l_1(s) I_{0+}^{\alpha_{n-2}-\alpha_1} z(s) \\
&\quad + \cdots + l_{n-3}(s) I_{0+}^{\alpha_{n-2}-\alpha_{n-3}} z(s) + l_{n-2}(s) z(s)\bigr)\,ds \\
&> 0.
\end{aligned}$$

On the basis of the density of $(-\infty,+\infty)$, there exists $d > 0$ such that $d(L_1 z)(t) \ge z(t)$, $t \in [0,1]$. On the other hand, P is generating in $C[0,1]$, that is, $C[0,1] = P - P$. Thus by Lemma 2.8 we infer that $r(L_1) \neq 0$ and L_1 has a positive eigenfunction ψ_1, which belongs to the first eigenvalue $\lambda_{L_1} = (r(L_1))^{-1}$, such that $\lambda_{L_1} L_1 \psi_1 = \psi_1$. □

Lemma 4.2 *The operator L_1 is a u_0-bounded linear operator with $u_0(t) = t^{\alpha-\alpha_{n-2}-1}$.*

Proof By Lemma 2.7 we have

$$\begin{aligned}
K(t,s) &= \frac{1}{\Gamma(\alpha_{n-2})} \int_s^1 H(t,\nu) l_0(\nu)(\nu-s)^{\alpha_{n-2}-1}\,d\nu \\
&\quad + \frac{1}{\Gamma(\alpha_{n-2}-\alpha_1)} \int_s^1 H(t,\nu) l_1(\nu)(\nu-s)^{\alpha_{n-2}-\alpha_1-1}\,d\nu \\
&\quad + \cdots \\
&\quad + \frac{1}{\Gamma(\alpha_{n-2}-\alpha_{n-3})} \int_s^1 H(t,\nu) l_{n-3}(\nu)(\nu-s)^{\alpha_{n-2}-\alpha_{n-3}-1}\,d\nu \\
&\quad + H(t,s) l_{n-2}(s) \\
&\ge t^{\alpha-\alpha_{n-2}-1}\biggl(\frac{1}{\Gamma(\alpha_{n-2})} \int_s^1 K(\nu) l_0(\nu)(\nu-s)^{\alpha_{n-2}-1}\,d\nu \\
&\quad + \frac{1}{\Gamma(\alpha_{n-2}-\alpha_1)} \int_s^1 K(\nu) l_1(\nu)(\nu-s)^{\alpha_{n-2}-\alpha_1-1}\,d\nu \\
&\quad + \cdots \\
&\quad + \frac{1}{\Gamma(\alpha_{n-2}-\alpha_{n-3})} \int_s^1 K(\nu) l_{n-3}(\nu)(\nu-s)^{\alpha_{n-2}-\alpha_{n-3}-1}\,d\nu \\
&\quad + K(s) l_{n-2}(s)\biggr)
\end{aligned}$$

and

$$
\begin{aligned}
K(t,s) \le t^{\alpha-\alpha_{n-2}-1}\biggl(&\frac{1}{\Gamma(\alpha_{n-2})}\int_s^1 J(\nu)l_0(\nu)(\nu-s)^{\alpha_{n-2}-1}\,d\nu \\
&+ \frac{1}{\Gamma(\alpha_{n-2}-\alpha_1)}\int_s^1 J(\nu)l_1(\nu)(\nu-s)^{\alpha_{n-2}-\alpha_1-1}\,d\nu \\
&+ \cdots \\
&+ \frac{1}{\Gamma(\alpha_{n-2}-\alpha_{n-3})}\int_s^1 J(\nu)l_{n-3}(\nu)(\nu-s)^{\alpha_{n-2}-\alpha_{n-3}-1}\,d\nu \\
&+ J(s)l_{n-2}(s)\biggr).
\end{aligned}
$$

Thus, for any $u \in P \setminus \{\theta\}$, $t \in [0,1]$,

$$
\begin{aligned}
(L_1 u)(t) &= \int_0^1 K(t,s)u(s)\,ds \\
&\ge t^{\alpha-\alpha_{n-2}-1}\biggl(\int_0^1 \biggl(\frac{1}{\Gamma(\alpha_{n-2})}\int_s^1 K(\nu)l_0(\nu)(\nu-s)^{\alpha_{n-2}-1}\,d\nu \\
&\quad + \frac{1}{\Gamma(\alpha_{n-2}-\alpha_1)}\int_s^1 K(\nu)l_1(\nu)(\nu-s)^{\alpha_{n-2}-\alpha_1-1}\,d\nu \\
&\quad + \cdots \\
&\quad + \frac{1}{\Gamma(\alpha_{n-2}-\alpha_{n-3})}\int_s^1 K(\nu)l_{n-3}(\nu)(\nu-s)^{\alpha_{n-2}-\alpha_{n-3}-1}\,d\nu \\
&\quad + K(s)l_{n-2}(s)\biggr)u(s)\,ds\biggr), \\
(L_1 u)(t) &= \int_0^1 K(t,s)u(s)\,ds \\
&\le t^{\alpha-\alpha_{n-2}-1}\biggl(\int_0^1 \biggl(\frac{1}{\Gamma(\alpha_{n-2})}\int_s^1 J(\nu)l_0(\nu)(\nu-s)^{\alpha_{n-2}-1}\,d\nu \\
&\quad + \frac{1}{\Gamma(\alpha_{n-2}-\alpha_1)}\int_s^1 J(\nu)l_1(\nu)(\nu-s)^{\alpha_{n-2}-\alpha_1-1}\,d\nu \\
&\quad + \cdots \\
&\quad + \frac{1}{\Gamma(\alpha_{n-2}-\alpha_{n-3})}\int_s^1 J(\nu)l_{n-3}(\nu)(\nu-s)^{\alpha_{n-2}-\alpha_{n-3}-1}\,d\nu \\
&\quad + J(s)l_{n-2}(s)\biggr)u(s)\,ds\biggr).
\end{aligned}
$$

This implies that L_1 is a u_0-bounded operator with $u_0(t) = t^{\alpha-\alpha_{n-2}-1}$. The proof is complete. □

Remark 4.2 Since L_1 is a u_0-bounded linear operator, we have that, for the positive eigenfunction ψ_1 defined in Lemma 4.1, there exist constants $\alpha(\psi_1) > 0$ and $\beta(\psi_1) > 0$ such that $\alpha(\psi_1)u_0 \le L_1\psi_1 = \lambda_{L_1}^{-1}\psi_1 \le \beta(\psi_1)u_0$. Accordingly, we have $(\beta(\psi_1)\lambda_{L_1})^{-1}\psi_1 \le u_0 \le (\alpha(\psi_1)\lambda_{L_1})^{-1}\psi_1$.

Theorem 4.1 *Assume that a function $f \in C([0,1]\times[0,+\infty)^{n-1},[0,+\infty))$ satisfies condition (C_5). If $k_1 \in [0,\lambda_{L_1})$, then problem (1.1) has a unique positive solution.*

Proof For any $v_0 \in P\setminus\{\theta\}$, let $v_m = Tv_{m-1}$ $(m = 1,2,\ldots)$. Since L_1 is a u_0-bounded linear operator, by Remark 4.2 we have that there exists $\beta(|v_1 - v_0|) > 0$ such that

$$L_1\bigl(|v_1 - v_0|\bigr) \le \beta\bigl(|v_1 - v_0|\bigr)u_0 \le \beta\bigl(|v_1 - v_0|\bigr)\bigl(\alpha(\psi_1)\lambda_{L_1}\bigr)^{-1}\psi_1.$$

Then we get

$$\begin{aligned}
&\bigl|v_2(t) - v_1(t)\bigr| \\
&\quad= \biggl|\int_0^1 H(t,s)\bigl(f\bigl(s, I_{0+}^{\alpha_{n-2}}v_1(s), I_{0+}^{\alpha_{n-2}-\alpha_1}v_1(s),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}v_1(s),v_1(s)\bigr) \\
&\qquad - f\bigl(s, I_{0+}^{\alpha_{n-2}}v_0(s), I_{0+}^{\alpha_{n-2}-\alpha_1}v_0(s),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}v_0(s),v_0(s)\bigr)\bigr)\,ds\biggr| \\
&\quad\le k_1\int_0^1 H(t,s)\bigl(l_0(s)\bigl|I_{0+}^{\alpha_{n-2}}\bigl(v_1(s)-v_0(s)\bigr)\bigr| + l_1(s)\bigl|I_{0+}^{\alpha_{n-2}-\alpha_1}\bigl(v_1(s)-v_0(s)\bigr)\bigr| \\
&\qquad + \cdots + l_{n-3}(s)\bigl|I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}\bigl(v_1(s)-v_0(s)\bigr)\bigr| + l_{n-2}(s)\bigl|v_1(s)-v_0(s)\bigr|\bigr)\,ds \\
&\quad\le k_1\int_0^1 H(t,s)\bigl(l_0(s)I_{0+}^{\alpha_{n-2}}\bigl|v_1(s)-v_0(s)\bigr| + l_1(s)I_{0+}^{\alpha_{n-2}-\alpha_1}\bigl|v_1(s)-v_0(s)\bigr| \\
&\qquad + \cdots + l_{n-3}(s)I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}\bigl|v_1(s)-v_0(s)\bigr| + l_{n-2}(s)\bigl|v_1(s)-v_0(s)\bigr|\bigr)\,ds \\
&\quad= k_1L_1\bigl(|v_1-v_0|\bigr)(t), \\
&\bigl|v_3(t) - v_2(t)\bigr| \\
&\quad= \biggl|\int_0^1 H(t,s)\bigl(f\bigl(s, I_{0+}^{\alpha_{n-2}}v_2(s), I_{0+}^{\alpha_{n-2}-\alpha_1}v_2(s),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}v_2(s),v_2(s)\bigr) \\
&\qquad - f\bigl(s, I_{0+}^{\alpha_{n-2}}v_1(s), I_{0+}^{\alpha_{n-2}-\alpha_1}v_1(s),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}v_1(s),v_1(s)\bigr)\bigr)\,ds\biggr| \\
&\quad\le k_1\int_0^1 H(t,s)\bigl(l_0(s)I_{0+}^{\alpha_{n-2}}\bigl|v_2(s)-v_1(s)\bigr| + l_1(s)I_{0+}^{\alpha_{n-2}-\alpha_1}\bigl|v_2(s)-v_1(s)\bigr| \\
&\qquad + \cdots + l_{n-3}(s)I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}\bigl|v_2(s)-v_1(s)\bigr| + l_{n-2}(s)\bigl|v_2(s)-v_1(s)\bigr|\bigr)\,ds \\
&\quad= k_1L_1\bigl(|v_2-v_1|\bigr)(t) \le k_1^2L_1^2\bigl(|v_1-v_0|\bigr)(t), \\
&\ldots \\
&\bigl|v_{m+1}(t) - v_m(t)\bigr| \\
&\quad= \biggl|\int_0^1 H(t,s)\bigl(f\bigl(s, I_{0+}^{\alpha_{n-2}}v_m(s), I_{0+}^{\alpha_{n-2}-\alpha_1}v_m(s),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}v_m(s),v_m(s)\bigr) \\
&\qquad - f\bigl(s, I_{0+}^{\alpha_{n-2}}v_{m-1}(s), I_{0+}^{\alpha_{n-2}-\alpha_1}v_{m-1}(s),\ldots,I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}v_{m-1}(s),v_{m-1}(s)\bigr)\bigr)\,ds\biggr| \\
&\quad\le k_1\int_0^1 H(t,s)\bigl(l_0(s)I_{0+}^{\alpha_{n-2}}\bigl|v_m(s)-v_{m-1}(s)\bigr| + l_1(s)I_{0+}^{\alpha_{n-2}-\alpha_1}\bigl|v_m(s)-v_{m-1}(s)\bigr| \\
&\qquad + \cdots + l_{n-3}(s)I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}\bigl|v_m(s)-v_{m-1}(s)\bigr| + l_{n-2}(s)\bigl|v_m(s)-v_{m-1}(s)\bigr|\bigr)\,ds
\end{aligned}$$

$$
\begin{aligned}
&= k_1 L_1\big(|v_m - v_{m-1}|\big)(t) \\
&\le k_1^m L_1^m\big(|v_1 - v_0|\big)(t) \\
&\le k_1^m \beta\big(|v_1 - v_0|\big)\big(\alpha(\psi_1)\lambda_{L_1}\big)^{-1} L_1^{m-1}\big(\psi_1(t)\big) \\
&\le \frac{\beta(|v_1 - v_0|)}{\alpha(\psi_1)} \frac{k_1^m}{\lambda_{L_1}^2} L_1^{m-2}\big(\psi_1(t)\big) \\
&\le \cdots \\
&\le \left(\frac{k_1}{\lambda_{L_1}}\right)^m \frac{\beta(|v_1 - v_0|)}{\alpha(\psi_1)} \psi_1(t).
\end{aligned}
$$

It follows from $k_1 \in [0, \lambda_{L_1})$ that

$$\|v_{m+1} - v_m\| \to 0, \quad m \to \infty.$$

Hence there exists $v^* \in P$ such that

$$v_m \to v^*, \quad m \to \infty.$$

Since $v_m = Tv_{m-1}$, we get that v^* is a fixed point of the operator T in P.

In the following, we prove that v^* is the unique fixed point of T in P. If not, there exists an element $v^{**} \in P$ such that $v^{**} = Tv^{**}$. Similarly, there exists $\beta(|v^{**} - v^*|) > 0$ such that

$$L_1\big(\big|v^{**} - v^*\big|\big) \le \beta\big(\big|v^{**} - v^*\big|\big)u_0 \le \beta\big(\big|v^{**} - v^*\big|\big)\big(\alpha(\psi_1)\lambda_{L_1}\big)^{-1}\psi_1.$$

Thus we get that, for any $t \in [0,1]$,

$$
\begin{aligned}
\big|v^{**}(t) - v^*(t)\big| &= \big|Tv^{**}(t) - Tv^*(t)\big| \\
&= \big|T^m v^{**}(t) - T^m v^*(t)\big| \\
&= \big|T\big(T^{m-1}v^{**}(t)\big) - T\big(T^{m-1}v^*(t)\big)\big| \\
&\le k_1 L_1\big|T^{m-1}v^{**}(t) - T^{m-1}v^*(t)\big| \\
&\le \cdots \\
&\le k_1^m L_1^m\big|v^{**}(t) - v^*(t)\big| \\
&\le \frac{\beta(|v^{**} - v^*|)}{\alpha(\psi_1)} \frac{k_1^m}{\lambda_{L_1}^m}\psi_1(t).
\end{aligned}
$$

Since

$$\frac{k_1^m}{\lambda_{L_1}^m} \to 0, \quad m \to \infty,$$

we get that $v^{**} = v^*$. Hence v^* is the unique fixed point of the operator T in P. By Lemma 2.4 we deduce that $u^* = I_{0+}^{\alpha_n - 2} v^*$ is the unique positive solution of problem (2.1) on $[0,1]$. □

5 Nonexistence of positive solutions

Now we make the following assumptions:

(C_6) there exist nonnegative functions $\rho_i(t) \in C(0,1) \cap L^1(0,1)$ $(i = 0,1,\dots,n-2)$ and a constant $k_2 \geq 0$ such that for any $t \in [0,1]$ and $(x_0,x_1,\dots,x_{n-2}) \in [0,+\infty)^{n-1}$,

$$\bigl|f(t,x_0,x_1,\dots,x_{n-2})\bigr| \leq k_2\bigl(\rho_0(t)x_0 + \cdots + \rho_{n-2}(t)x_{n-2}\bigr).$$

(C_7) there exist nonnegative functions $q_i(t) \in C(0,1) \cap L^1(0,1)$ $(i = 0,1,\dots,n-2)$ and a constant $k_3 \geq 0$ such that for any $t \in [0,1]$ and $(x_0,x_1,\dots,x_{n-2}) \in [0,+\infty)^{n-1}$,

$$\bigl|f(t,x_0,x_1,\dots,x_{n-2})\bigr| \geq k_3\bigl(q_0(t)x_0 + \cdots + q_{n-2}(t)x_{n-2}\bigr).$$

Define the operator $L_2 : E \to E$ by

$$\begin{aligned}(L_2v)(t) = \int_0^1 &H(t,s)\bigl(\rho_0(s)I_{0+}^{\alpha_{n-2}}v(s) + \rho_1(s)I_{0+}^{\alpha_{n-2}-\alpha_1}v(s) + \cdots \\ &+ \rho_{n-3}(s)I_{0+}^{\alpha_{n-2}-\alpha_{n-3}}v(s) + \rho_{n-2}(s)v(s)\bigr)\,ds. \end{aligned} \tag{5.1}$$

Lemma 5.1 *The spectral radius $r(L_2) \neq 0$, and L_2 has a positive eigenfunction ψ_2, which belongs to the first eigenvalue $\lambda_{L_2} = (r(L_2))^{-1}$, such that $\lambda_{L_2}L_2\psi_2 = \psi_2$.*

Proof The proof is similar to that of Lemma 4.1. So, we omit the details. □

Lemma 5.2 *The operator L_2 is a u_0-bounded linear operator with $u_0(t) = t^{\alpha-\alpha_{n-2}-1}$.*

Proof The proof is similar to that of Lemma 4.2. So, we omit the details. □

Remark 5.2 For the positive eigenfunction ψ_2 defined in Lemma 5.1, there exist $\alpha(\psi_2) > 0$ and $\beta(\psi_2) > 0$ such that $\alpha(\psi_2)u_0 \leq L_2\psi_2 = \lambda_{L_2}^{-1}\psi_2 \leq \beta(\psi_2)u_0$. Accordingly, we have $(\beta(\psi_2)\lambda_{L_2})^{-1}\psi_2 \leq u_0 \leq (\alpha(\psi_2)\lambda_{L_2})^{-1}\psi_2$.

Theorem 5.1 *Assume that a function $f \in C([0,1] \times [0,+\infty)^{n-1}, [0,+\infty))$ satisfies condition (C_6). If $k_2 \in [0,\lambda_{L_2})$, then problem (1.1) has no positive nontrivial solution.*

Proof Suppose the statement is false. Then there exists $\bar{v} \in P$ such that $T\bar{v} = \bar{v}$. Since L_2 is a u_0-bounded linear operator, there exists $\beta(\bar{v}) > 0$ satisfying

$$L_2\bar{v} \leq \beta(\bar{v})u_0 \leq \beta(\bar{v})\bigl(\alpha(\psi_2)\lambda_{L_2}\bigr)^{-1}\psi_2.$$

Hence we have

$$\begin{aligned}\bar{v}(t) &= (T\bar{v})(t) \\ &= \bigl(T^m\bar{v}\bigr)(t) \\ &\leq k_2\bigl(L_2\bigl(T^{m-1}\bar{v}\bigr)\bigr)(t) \\ &\leq k_2^m\bigl(L_2^m\bar{v}\bigr)(t)\end{aligned}$$

$$
\begin{aligned}
&\le k_2^m \beta(\bar{v})\bigl(\alpha(\psi_2)\lambda_{L_2}\bigr)^{-1} L_2^{m-1}\bigl(\psi_2(t)\bigr)\\
&\le \frac{\beta(\bar{v})}{\alpha(\psi_2)} \frac{k_2^m}{\lambda_{L_2}^2} L_2^{m-2}\bigl(\psi_2(t)\bigr)\\
&\le \cdots\\
&\le \left(\frac{k_2}{\lambda_{L_2}}\right)^m \frac{\beta(\bar{v})}{\alpha(\psi_2)} \psi_2(t).
\end{aligned}
$$

It follows from $k_2 \in [0, \lambda_{L_2})$ that

$$\left(\frac{k_2}{\lambda_{L_2}}\right)^m \to 0, \quad m \to \infty.$$

Thus we can conclude that $\bar{v} = \theta$. Therefore we deduce that problem (1.1) has no positive nontrivial solution. □

Theorem 5.2 *Assume that a function $f \in C([0,1] \times [0,+\infty)^{n-1}, [0,+\infty))$ satisfies condition (C_6). If $k_3 \in (\lambda_{L_3}, +\infty)$, then problem (1.1) has no positive nontrivial solution.*

Proof The proof is similar to that of Theorem 5.1. So, we omit the details. □

6 Examples

Now, we give five explicit examples illustrating the main results.

Example 6.1 Consider the problem

$$
\begin{cases}
D_{0+}^{\frac{13}{4}} u(t) + \frac{u^{\frac{1}{2}}(t)(D_{0+}^{\frac{1}{2}} u(t))^{\frac{1}{3}}}{80t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}} + (\ln\frac{1}{t})\frac{u(t)\sin(D_{0+}^{\frac{1}{2}} u(t))}{200(|D_{0+}^{\frac{1}{2}} u(t)|+1)}\\
\quad + \frac{3D_{0+}^{\frac{7}{4}} u(t)\ln t}{32(t-1)(u^2(t)+1)} = 0, \quad 0 < t < 1,\\
u(0) = D_{0+}^{\frac{3}{4}} u(0) = D_{0+}^{\frac{15}{8}} u(0) = 0,\\
D_{0+}^{\frac{9}{4}} u(1) = \frac{1}{5}\Gamma(\frac{2}{5})\int_0^1 \frac{1}{s^{\frac{2}{5}}(4s^2+1)} D_{0+}^{\frac{37}{20}} u(s)\,dA_1(s)\\
\qquad + \frac{3}{20}\Gamma(\frac{3}{10})\int_0^{\frac{2}{3}} \frac{1}{s^{\frac{3}{10}}(4s^2+3)} D_{0+}^{\frac{39}{20}} u(s)\,dA_2(s)\\
\qquad + \frac{1}{8}\Gamma(\frac{1}{5})(\frac{1}{2}D_{0+}^{\frac{41}{20}} u(\frac{1}{32}) + \frac{3}{4}D_{0+}^{\frac{41}{20}} u(\frac{1}{243}) + \frac{1}{5}D_{0+}^{\frac{41}{20}} u(\frac{1}{1024})),
\end{cases}
\tag{6.1}
$$

where

$$
A_1(t) = \begin{cases} \frac{1}{3}, & t \in [0, \frac{1}{2}),\\ \frac{4}{3}, & t \in [\frac{1}{2}, 1],\end{cases} \qquad A_2(t) = \begin{cases} \frac{5}{2}, & t \in [0, \frac{1}{2}),\\ \frac{7}{2}, & t \in [\frac{1}{2}, 1].\end{cases}
$$

Let

$$f(t,x,y,z) = \frac{x^{\frac{1}{2}} y^{\frac{1}{3}}}{80t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}} + \left(\ln\frac{1}{t}\right)\frac{x\sin y}{200(|y|+1)} + \frac{3z\ln t}{32(t-1)(x^2+1)},$$

$\alpha = \frac{13}{4}$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{7}{4}$, $\beta_1 = \frac{3}{4}$, $\beta_2 = \frac{15}{8}$, $q_0 = \frac{9}{4}$, $q_1 = \frac{37}{20}$, $q_2 = \frac{39}{20}$, $q_3 = \frac{41}{20}$, $c_1 = \frac{1}{5}\Gamma(\frac{2}{5})$, $c_2 = \frac{3}{20}\Gamma(\frac{3}{10})$, $c_3 = \frac{1}{8}\Gamma(\frac{1}{5})$, $\theta = \frac{2}{3}$, $\gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{3}{4}$, $\gamma_3 = \frac{1}{5}$, $\xi_1 = \frac{1}{32}$, $\xi_2 = \frac{1}{243}$, $\xi_3 = \frac{1}{1024}$, $g_1(t) =$

$\frac{1}{t^{\frac{2}{5}}(4t^2+1)}$, $g_2(t) = \frac{1}{t^{\frac{3}{10}}(4t^2+3)}$. Then problem (6.1) can be transformed into problem (1.1). With the given data, we get that $\int_0^1 t^{\alpha-q_1-1} g_1(t)\,dA_1(t) = 0.5$, $\int_0^\theta t^{\alpha-q_2-1} g_2(t)\,dA_2(t) = 0.25$, $\sum_{i=1}^3 \gamma_i \xi_i^{\alpha-q_3-1} = 0.55$,

$$\begin{aligned}
\sigma &= \frac{1}{\Gamma(\alpha-q_0)} - \frac{c_1}{\Gamma(\alpha-q_1)} \int_0^1 s^{\alpha-q_1-1} g_1(s)\,dA_1(s) \\
&\quad - \frac{c_2}{\Gamma(\alpha-q_2)} \int_0^\theta s^{\alpha-q_2-1} g_2(s)\,dA_2(s) - \frac{c_3}{\Gamma(\alpha-q_3)} \sum_{i=1}^m \gamma_i \xi_i^{\alpha-q_3-1} \\
&= 0.28125, \\
M_H &\le \frac{1}{\Gamma(\alpha-\alpha_2)} + \frac{c_1}{\sigma\Gamma(\alpha-\alpha_2)\Gamma(\alpha-q_1)} \int_0^1 s^{\alpha-q_1-1} g_1(s)\,dA_1(s) \\
&\quad + \frac{c_2}{\sigma\Gamma(\alpha-\alpha_2)\Gamma(\alpha-q_2)} \int_0^\theta s^{\alpha-q_2-1} g_2(s)\,dA_2(s) \\
&\quad + \frac{c_3}{\sigma\Gamma(\alpha-\alpha_2)\Gamma(\alpha-q_3)} \sum_{i=1}^m \gamma_i \xi_i^{\alpha-q_3-1} \\
&= \frac{2}{\pi^{\frac{1}{2}}} \left(1 + \frac{1}{0.28125}(0.25 + 0.125 + 0.34375)\right) \approx 4.012.
\end{aligned}$$

Then, for any $(x, y, z) \in [-r, r]^3$ and a.e. $t \in [0, 1]$,

$$\bigl|f(t,x,y,z)\bigr| \le \frac{r^{\frac{5}{6}}}{80t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}} + \frac{r}{200} \ln\left(\frac{1}{t}\right) + \frac{3r\ln t}{32(t-1)}.$$

We set $\psi_r(t) = \frac{r^{\frac{5}{6}}}{80t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}} + \frac{r}{200} \ln \frac{1}{t} + \frac{3r\ln t}{32(t-1)}$. Obviously, $\psi_r \in L^1(0, 1)$, and

$$\begin{aligned}
\|\psi_r\|_L &= \int_0^1 \left(\frac{r^{\frac{5}{6}}}{80t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}} + \frac{r}{200} \ln \frac{1}{t} + \frac{3r\ln t}{32(t-1)}\right) dt \\
&= \frac{\pi r^{\frac{5}{6}}}{80} + \frac{r}{200} + \frac{\pi^2 r}{64}.
\end{aligned}$$

Accordingly, we have

$$\lim_{r\to+\infty} \frac{\|\psi_r\|_L}{r} = \frac{1}{200} + \frac{\pi^2}{64} \approx 0.159 < \frac{m_\Gamma}{M_H} \approx 0.219.$$

Thus the assumptions of Theorem 3.1 are satisfied. So, by Theorem 3.1, problem (6.1) has at least one solution.

Example 6.2 Consider the problem

$$\begin{cases} D_{0+}^{\frac{25}{8}}u(t) + \frac{t}{10(1-t^4)^{\frac{1}{2}}} + \frac{u(t)}{100t^{\frac{5}{6}}(1-t)^{\frac{1}{6}}} + \frac{\sqrt{2}D_{0+}^{\frac{1}{3}}u(t)}{200((D_{0+}^{\frac{1}{3}}u(t))^2+\sqrt{2})}\ln\frac{1}{t} \\ \quad + \frac{\ln t}{40(t-1)}\sin(D_{0+}^{\frac{1}{3}}u(t))\cos(D_{0+}^{\frac{13}{8}}u(t)) = 0, \quad 0 < t < 1, \\ u(0) = D_{0+}^{\frac{5}{8}}u(0) = D_{0+}^{\frac{7}{4}}u(0) = 0, \\ D_{0+}^{\frac{17}{8}}u(1) = \frac{1}{5}\Gamma(\frac{19}{16})\int_0^1 \frac{2}{s^{\frac{3}{16}}(4s^2+1)}D_{0+}^{\frac{31}{16}}u(s)\,dA_1(s) \\ \qquad + \frac{1}{10}\Gamma(\frac{21}{16})\int_0^{\frac{2}{3}}\frac{1}{s^{\frac{5}{16}}(8s^2+3)}D_{0+}^{\frac{29}{16}}u(s)\,dA_2(s) \\ \qquad + \frac{1}{2}\Gamma(\frac{5}{4})(\frac{1}{4}D_{0+}^{\frac{15}{8}}u(\frac{1}{16}) + \frac{3}{4}D_{0+}^{\frac{15}{8}}u(\frac{1}{81}) + \frac{4}{5}D_{0+}^{\frac{15}{8}}u(\frac{1}{256})), \end{cases} \tag{6.2}$$

where

$$A_1(t) = \begin{cases} \frac{2}{7}, & t \in [0, \frac{1}{2}), \\ \frac{9}{7}, & t \in [\frac{1}{2}, 1], \end{cases} \qquad A_2(t) = \begin{cases} \frac{3}{7}, & t \in [0, \frac{1}{2}), \\ \frac{10}{7}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Let

$$\begin{aligned} f(t,x,y,z) = {} & \frac{t}{10(1-t^4)^{\frac{1}{2}}} + \frac{x}{100t^{\frac{5}{6}}(1-t)^{\frac{1}{6}}} + \frac{\sqrt{2}y}{200(y^2+\sqrt{2})}\ln\frac{1}{t} \\ & + \frac{\ln t}{40(t-1)}\sin y\cos z, \end{aligned}$$

$\alpha = \frac{25}{8}$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{13}{8}$, $\beta_1 = \frac{5}{8}$, $\beta_2 = \frac{7}{4}$, $q_0 = \frac{17}{8}$, $q_1 = \frac{31}{16}$, $q_2 = \frac{29}{16}$, $q_3 = \frac{15}{8}$, $c_1 = \frac{1}{5}\Gamma(\frac{19}{16})$, $c_2 = \frac{1}{10}\Gamma(\frac{21}{16})$, $c_3 = \frac{1}{2}\Gamma(\frac{5}{4})$, $\theta = \frac{2}{3}$, $\gamma_1 = \frac{1}{4}$, $\gamma_2 = \frac{3}{4}$, $\gamma_3 = \frac{4}{5}$, $\xi_1 = \frac{1}{16}$, $\xi_2 = \frac{1}{81}$, $\xi_3 = \frac{1}{256}$, $g_1(t) = \frac{2}{t^{\frac{3}{16}}(4t^2+1)}$, $g_2(t) = \frac{1}{t^{\frac{5}{16}}(8t^2+3)}$. Then problem (6.2) can be transformed into problem (1.1). By simple computation we get $\sum_{i=1}^3 \gamma_i\xi_i^{\alpha-q_3-1} = 0.575$, $\int_0^1 t^{\alpha-q_1-1}g_1(t)\,dA_1(t) = 1$, $\int_0^\theta t^{\alpha-q_2-1}g_2(t)\,dA_2(t) = 0.2$,

$$\begin{aligned} \sigma = {} & \frac{1}{\Gamma(\alpha-q_0)} - \frac{c_1}{\Gamma(\alpha-q_1)}\int_0^1 s^{\alpha-q_1-1}g_1(s)\,dA_1(s) \\ & - \frac{c_2}{\Gamma(\alpha-q_2)}\int_0^\theta s^{\alpha-q_2-1}g_2(s)\,dA_2(s) - \frac{c_3}{\Gamma(\alpha-q_3)}\sum_{i=1}^m \gamma_i\xi_i^{\alpha-q_3-1} \\ = {} & 0.4925, \end{aligned}$$

$$\begin{aligned} M_H \le {} & \frac{1}{\Gamma(\alpha-\alpha_2)} + \frac{c_1}{\sigma\Gamma(\alpha-\alpha_2)\Gamma(\alpha-q_1)}\int_0^1 s^{\alpha-q_1-1}g_1(s)\,dA_1(s) \\ & + \frac{c_2}{\sigma\Gamma(\alpha-\alpha_2)\Gamma(\alpha-q_2)}\int_0^\theta s^{\alpha-q_2-1}g_2(s)\,dA_2(s) \\ & + \frac{c_3}{\sigma\Gamma(\alpha-\alpha_2)\Gamma(\alpha-q_3)}\sum_{i=1}^m \gamma_i\xi_i^{\alpha-q_3-1} \\ = {} & \frac{2}{0.4925\sqrt{\pi}}. \end{aligned}$$

Let

$$k_{r,0}(t)=\frac{1}{100t^{\frac{5}{6}}(1-t)^{\frac{1}{6}}},\qquad k_{r,1}(t)=\frac{1}{200}\ln\frac{1}{t}+\frac{\ln t}{40(t-1)},\qquad k_{r,2}(t)=\frac{\ln t}{40(t-1)}.$$

Then

$$\begin{aligned}
\|k_{r,0}\|_L&=\int_0^1\frac{1}{100t^{\frac{5}{6}}(1-t)^{\frac{1}{6}}}\,dt=\frac{\pi}{50},\\
\|k_{r,1}\|_L&=\int_0^1\frac{1}{200}\ln\frac{1}{t}\,dt+\int_0^1\frac{\ln t}{40(t-1)}\,dt=\frac{1}{200}+0.0411=0.04612,\\
\|k_{r,2}\|_L&=\int_0^1\frac{\ln t}{40(t-1)}\,dt=\frac{\pi^2}{240},\\
\|f_0\|_L&=\int_0^1\bigl|f(t,0,0,0)\bigr|\,dt=\int_0^1\frac{t}{10(1-t^4)^{\frac{1}{2}}}\,dt=\frac{\pi}{40},
\end{aligned}$$

and for any $t\in(0,1)$ and $(x_0,y_0,y_0),(x_1,y_1,z_1)\in[-r,r]\times[-r,r]\times[-r,r]$,

$$\begin{aligned}
&\bigl|f(t,x_0,y_0,z_0)-f(t,x_1,y_1,z_1)\bigr|\\
&\quad\le k_{r_3,0}(t)|x_0-x_1|+k_{r,1}(t)|y_0-y_1|+k_{r,2}(t)|z_0-z_1|,
\end{aligned}$$

which implies that assumptions (C_2) and (C_3) hold. Moreover, we have

$$0<M_H\|f_0\|_L<\liminf_{r\to+\infty}\left(m_\Gamma-M_H\sum_{i=0}^{n-2}\|k_{r,i}\|_L\right)r=+\infty.$$

Thus, by Theorem 3.2, problem (6.2) has a unique solution.

Example 6.3 Consider the problem

$$\begin{cases}
D_{0+}^{\frac{7}{2}}u(t)+\frac{u(t)}{100t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}}+\cos(t^2+D_{0+}^{\frac{1}{3}}u(t))\frac{\pi^2}{10(16+\pi^2t^2)}\\
\quad+\frac{\ln t}{10(t-1)}\arctan(\frac{D_{0+}^{\frac{3}{2}}u(t)}{2})=0,\quad 0<t<1,\\
u(0)=D_{0+}^{\frac{1}{4}}u(0)=D_{0+}^{\frac{5}{4}}u(0)=0,\\
D_{0+}^{\frac{5}{2}}u(1)=\frac{1}{8}\Gamma(\frac{1}{8})\int_0^1\frac{1}{20s^{\frac{1}{8}}(1-s)}D_{0+}^{\frac{19}{8}}u(s)\,dA_1(s)\\
\qquad+\frac{3}{8}\Gamma(\frac{3}{8})\int_0^{\frac{3}{4}}\frac{1}{8s^{\frac{3}{8}}(1+s^2)}D_{0+}^{\frac{17}{8}}u(s)\,dA_2(s)\\
\qquad+\frac{1}{8}\Gamma(\frac{1}{4})(\frac{1}{4}D_{0+}^{\frac{9}{4}}u(\frac{1}{16})+\frac{3}{4}D_{0+}^{\frac{9}{4}}u(\frac{1}{81})+\frac{4}{5}D_{0+}^{\frac{9}{4}}u(\frac{1}{256})),
\end{cases}\tag{6.3}$$

where

$$A_1(t)=\begin{cases}\frac{1}{9},& t\in[0,\frac{1}{2}),\\ \frac{11}{9},& t\in[\frac{1}{2},1],\end{cases}\qquad A_2(t)=\begin{cases}\frac{4}{7},& t\in[0,\frac{1}{2}),\\ \frac{11}{7},& t\in[\frac{1}{2},1].\end{cases}$$

Let

$$f(t,x,y,z)=\frac{x}{100t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}}+\cos\bigl(t^2+y\bigr)\frac{\pi^2}{10(16+\pi^2t^2)}+\frac{\ln t}{10(t-1)}\arctan\frac{z}{2},$$

$\alpha = \frac{7}{2}, \alpha_1 = \frac{1}{3}, \alpha_2 = \frac{3}{2}, \beta_1 = \frac{1}{4}, \beta_2 = \frac{5}{4}, q_0 = \frac{5}{2}, q_1 = \frac{19}{8}, q_2 = \frac{17}{8}, q_3 = \frac{9}{4}, c_1 = \frac{1}{8}\Gamma(\frac{1}{8}), c_2 = \frac{3}{8}\Gamma(\frac{3}{8})$, $c_3 = \frac{1}{8}\Gamma(\frac{1}{4})$, $\theta = \frac{3}{4}$, $\gamma_1 = \frac{1}{4}$, $\gamma_2 = \frac{3}{4}$, $\gamma_3 = \frac{4}{5}$, $\xi_1 = \frac{1}{16}$, $\xi_2 = \frac{1}{81}$, $\xi_3 = \frac{1}{256}$, $g_1(t) = \frac{1}{20t^{\frac{1}{8}}(1-t)}$, $g_2(t) = \frac{1}{8t^{\frac{3}{8}}(1+t^2)}$. Then problem (6.3) can be transformed into problem (1.1). By simple computation we get $\int_0^1 t^{\alpha-q_1-1}g_1(t)\,dA_1(t) = 0.1$, $\int_0^\theta t^{\alpha-q_2-1}g_2(t)\,dA_2(t) = 0.1$, $\sum_{i=1}^3 \gamma_i\xi_i^{\alpha-q_3-1} = 0.575$. So, we have

$$\begin{aligned}
\sigma &= \frac{1}{\Gamma(\alpha-q_0)} - \frac{c_1}{\Gamma(\alpha-q_1)}\int_0^1 s^{\alpha-q_1-1}g_1(s)\,dA_1(s) \\
&\quad - \frac{c_2}{\Gamma(\alpha-q_2)}\int_0^\theta s^{\alpha-q_2-1}g_2(s)\,dA_2(s) - \frac{c_3}{\Gamma(\alpha-q_3)}\sum_{i=1}^m \gamma_i\xi_i^{\alpha-q_3-1} \\
&= 0.6065, \\
M_H &\le \frac{1}{\Gamma(\alpha-\alpha_2)} + \frac{c_1}{\sigma\Gamma(\alpha-\alpha_2)\Gamma(\alpha-q_1)}\int_0^1 s^{\alpha-q_1-1}g_1(s)\,dA_1(s) \\
&\quad + \frac{c_2}{\sigma\Gamma(\alpha-\alpha_2)\Gamma(\alpha-q_2)}\int_0^\theta s^{\alpha-q_2-1}g_2(s)\,dA_2(s) \\
&\quad + \frac{c_3}{\sigma\Gamma(\alpha-\alpha_2)\Gamma(\alpha-q_3)}\sum_{i=1}^m \gamma_i\xi_i^{\alpha-q_3-1} \\
&< 1.7.
\end{aligned}$$

Let

$$p_0(t) = \frac{1}{100t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}}, \qquad p_1(t) = \frac{\pi^2}{10(16+\pi^2t^2)}, \qquad p_2(t) = \frac{\ln t}{20(t-1)}.$$

Then

$$\begin{aligned}
\|p_0\|_L &= \int_0^1 \frac{1}{100t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}}\,dt = \frac{\pi}{100}, \\
\|p_1\|_L &= \int_0^1 \frac{\pi^2}{10(16+\pi^2t^2)}\,dt = \frac{\pi}{40}, \\
\|p_2\|_L &= \int_0^1 \frac{\ln t}{20(t-1)}\,dt = \frac{\pi^2}{120},
\end{aligned}$$

and for any $t \in (0,1)$ and $(x_0, y_0, y_0), (x_1, y_1, z_1) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty)$,

$$\bigl|f(t,x_0,y_0,z_0) - f(t,x_1,y_1,z_1)\bigr| \le p_0(t)|x_0-x_1| + p_1(t)|y_0-y_1| + p_2(t)|z_0-z_1|,$$

which implies that assumption (C_4) holds. Moreover, we have

$$\frac{M_H}{m_\Gamma}\sum_{i=0}^{n-2}\|p_i\|_L < 1.$$

Thus, in view of Theorem 3.3, problem (6.3) has a unique solution.

Example 6.4 Consider the problem

$$\begin{cases} D_{0+}^{\frac{7}{2}}u(t) + \frac{1}{4}\sqrt{t}u^3(t) - \frac{1}{4}\sqrt{t}|u^3(t)| + \frac{1}{3}(\arctan D_{0+}^{\frac{1}{3}}u(t)) \\ \quad - \frac{1}{3}\arctan(|D_{0+}^{\frac{1}{3}}u(t)|) + 20D_{0+}^{\frac{3}{2}}u(t) = 0, \quad 0 < t < 1, \\ u(0) = D_{0+}^{\frac{2}{3}}u(0) = D_{0+}^{\frac{3}{2}}u(0) = 0, \\ D_{0+}^{\frac{3}{2}}u(1) = 0, \end{cases} \tag{6.4}$$

where $f(t,x,y,z) = \frac{1}{4}\sqrt{t}x^3 - \frac{1}{4}\sqrt{t}|x^3| + \frac{1}{3}\arctan y - \frac{1}{3}\arctan|y| + 20z$, $\alpha = \frac{7}{2}$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{3}{2}$, $\beta_1 = \frac{2}{3}$, $\beta_2 = \frac{3}{2}$, $q_0 = \frac{3}{2}$, $q_1 = \frac{3}{2}$, $q_2 = \frac{3}{2}$, $q_3 = \frac{3}{2}$, $c_1 = \frac{1}{5}$, $c_2 = 3$, $c_3 = 0$, $\theta = 0$, $\gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{3}{4}$, $\gamma_3 = \frac{1}{5}$, $\xi_1 = \frac{1}{32}$, $\xi_2 = \frac{1}{243}$, $\xi_3 = \frac{1}{1024}$, $g_1(t) = 0$, $g_2(t) = \frac{t}{4t^2+3}$, and

$$A_1(t) = \begin{cases} \frac{1}{2}, & t \in [0, \frac{1}{2}), \\ \frac{4}{3}, & t \in [\frac{1}{2}, 1], \end{cases} \qquad A_2(t) = \begin{cases} \frac{5}{3}, & t \in [0, \frac{1}{2}), \\ \frac{7}{2}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Let $v = I_{0+}^{\frac{3}{2}}u$. By Lemma 2.4 we reduce problem (6.4) to the problem

$$\begin{cases} v''(t) + f(t, I_{0+}^{\frac{3}{2}}v(t), I_{0+}^{\frac{7}{6}}v(t), v(t)) = 0, \quad 0 < t < 1, \\ v(0) = v(1) = 0. \end{cases} \tag{6.5}$$

By calculating we can obtain the relevant Green's function of problem (6.5)

$$H(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Obviously, for any $x_i, y_i, z_i \in [0, +\infty)$ $(i = 0, 1)$ and $t \in [0, 1]$,

$$\left|f(t, x_0, y_0, z_0) - f(t, x_1, y_1, x_1)\right| \le 20|z_0 - z_1|.$$

Let $l_0(t) = l_1(t) = 0$, $l_2(t) = 20$, and $k_1 = 1$. Then the corresponding Green's function of L_1 is $K(t,s) = 20H(t,s)$. Obviously, L_1 is a u_0-bounded operator, where $u_0(t) = t(1-t)$. The fact that

$$\int_0^1 K(t,s)\sin(\pi s)\,ds = \frac{20}{\pi^2}\sin(\pi t), \quad t \in [0,1],$$

means that $\sin(\pi t)$ is a positive eigenfunction of the operator L_1, which belongs to its first eigenvalue $\lambda_{L_1} = (r(L))^{-1} = \frac{20}{\pi^2}$. Obviously, $k_1 = 1 < \frac{20}{\pi^2}$. Then, by Theorem 4.1, we get that problem (6.4) has a unique positive solution.

Example 6.5 Consider the problem

$$\begin{cases} D_{0+}^{\frac{15}{4}}u(t) + \frac{2\sqrt{t}}{1+u(t)} - \frac{2\sqrt{t}}{1+|u(t)|} + \frac{1}{2}(D_{0+}^{\frac{1}{4}}u(t))^{\frac{1}{5}} \\ \quad - \frac{1}{2}|D_{0+}^{\frac{1}{4}}u(t)|^{\frac{1}{5}} + \frac{1}{10}D_{0+}^{\frac{7}{4}}u(t) = 0, \quad 0 < t < 1, \\ u(0) = D_{0+}^{\frac{2}{3}}u(0) = D_{0+}^{\frac{7}{4}}u(0) = 0, \\ D_{0+}^{\frac{7}{4}}u(1) = 0, \end{cases} \tag{6.6}$$

where $f(t,x,y,z) = \frac{2\sqrt{t}}{1+x} - \frac{2\sqrt{t}}{1+|x|} + \frac{1}{2}y^{\frac{1}{5}} - \frac{1}{2}|y|^{\frac{1}{5}} + \frac{1}{10}z$, $\alpha = \frac{15}{4}$, $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{7}{4}$, $\beta_1 = \frac{2}{3}$, $\beta_2 = \frac{7}{4}$, $q_0 = \frac{7}{4}$, $q_1 = \frac{7}{4}$, $q_2 = \frac{7}{4}$, $q_3 = \frac{7}{4}$, $c_1 = 1$, $c_2 = 1$, $c_3 = 0$, $\theta = 0$, $\gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{3}{4}$, $\gamma_3 = \frac{1}{5}$, $\xi_1 = \frac{1}{32}$, $\xi_2 = \frac{1}{243}$, $\xi_3 = \frac{1}{1024}$, $g_1(t) = 0$, $g_2(t) = \frac{t}{4t^2+3}$, and

$$A_1(t) = \begin{cases} \frac{1}{9}, & t \in [0, \frac{1}{2}), \\ \frac{2}{7}, & t \in [\frac{1}{2}, 1], \end{cases} \qquad A_2(t) = \begin{cases} \frac{4}{3}, & t \in [0, \frac{1}{2}), \\ \frac{7}{2}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Let $v = I_{0+}^{\frac{7}{4}} u$. By Lemma 2.4 we reduce problem (6.6) to the problem

$$\begin{cases} v''(t) + f(t, I_{0+}^{\frac{7}{4}} v(t), I_{0+}^{\frac{3}{2}} v(t), v(t)) = 0, & 0 < t < 1, \\ v(0) = v(1) = 0. \end{cases} \tag{6.7}$$

By calculating we can obtain the relevant Green's function

$$H(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Obviously, for any $x_i, y_i, z_i \in [0, +\infty)$ $(i = 0, 1)$ and $t \in [0, 1]$,

$$\left| f(t, x_0, y_0, z_0) - f(t, x_1, y_1, x_1) \right| \le \frac{1}{10} |z_0 - z_1|.$$

Let $l_0(t) = l_1(t) = 0$, $l_2(t) = 1$, and $k_2 = \frac{1}{10}$. Then the Green's function of L_1 is $K(t,s) = H(t,s)$. Obviously, L_1 is a u_0-bounded operator, where $u_0(t) = t(1-t)$. By the proof similar to that in Example 6.4 we get that $\lambda_{L_2} = (r(L_2))^{-1} = \frac{1}{\pi^2}$. The fact $k_2 = \frac{1}{10} < \frac{1}{\pi^2}$, together with Theorem 5.1, means that problem (6.6) has no positive solution.

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