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The existence and boundedness of linear and multilinear Marcinkiewicz integrals on central Campanato spaces

Jiao Ma¹, Mingquan Wei^{2*} and Dunyan Yan¹

*Correspondence:
weimingquan11@mails.ucas.ac.cn

²School of Mathematics and Statistics, Xinyang Normal University, Xinyang, 464000, China
Full list of author information is available at the end of the article

Abstract

In this paper, we obtain the existence and boundedness of Marcinkiewicz integrals with homogeneous kernels on central Campanato spaces. Moreover, the existence and boundedness of multilinear Marcinkiewicz integrals on central Campanato spaces are also deduced. We extend various previous results to central Campanato spaces which can be seen as the local version of Campanato spaces.

Keywords: Marcinkiewicz integral; Homogeneous kernels; Central Campanato spaces

1 Introduction

Let $\Omega \in L^1(\mathbb{S}^{n-1})$ be homogeneous of degree zero and satisfy

$$\int_{\mathbb{S}^{n-1}} \Omega(x') d\sigma(x') = 0, \quad (1.1)$$

where $x' = x/|x|$ for any $x \neq 0$. The Marcinkiewicz integral operator of higher dimension is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

where $F_{\Omega,t}(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy$, and \mathbb{S}^{n-1} is the unit sphere of \mathbb{R}^n ($n \geq 2$) equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$.

Stein [1] proved that, if Ω is continuous and satisfies a Lip_α ($0 < \alpha \leq 1$) condition on \mathbb{S}^{n-1} , then μ_Ω is of type (p,p) ($1 < p \leq 2$) and of weak type $(1,1)$. Benedek et al. [2] proved that, if $\Omega \in C^1(\mathbb{S}^{n-1})$, then μ_Ω is of type (p,p) ($1 < p \leq \infty$). Ding [3] proved that if Ω is homogeneous of degree zero satisfying a class of L^q -Dini ($1 < q \leq \infty$) conditions, and then μ_Ω is bounded on Campanato spaces.

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Definition 1.1 Suppose $1 \leq p < \infty$ and $-\frac{n}{p} \leq \alpha < 1$. Then the Campanato space $\varepsilon^{\alpha,p}(\mathbb{R}^n)$ is defined as

$$\varepsilon^{\alpha,p}(\mathbb{R}^n) := \left\{ f \in L_{\text{loc}}^p(\mathbb{R}^n) : \|f\|_{\varepsilon^{\alpha,p}} < \infty \right\},$$

where

$$\|f\|_{\varepsilon^{\alpha,p}} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x,r)|^{\frac{\alpha}{n}}} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(x) - f_{B(x,r)}|^p dx \right)^{\frac{1}{p}},$$

here $B(x,r) = \{y \in \mathbb{R}^n : |x - y| < r\}$, $|B(x,r)|$ is its Lebesgue measure and $f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy$.

Remark 1.1 When $\alpha \in (0, 1)$ and $p \in [1, \infty)$, $\varepsilon^{\alpha,p} = \text{Lip}_\alpha$, where Lip_α is the Lipschitz space; when $\alpha = 0$, the Campanato space is the BMO space; when $\alpha \in (-\frac{n}{p}, 0)$, $\varepsilon^{\alpha,p} = L^{p,\alpha p+n}$, where $L^{p,\alpha p+n}$ is the Morrey space.

Theorem A ([3]) Let $1 < p < \infty$, $1 \leq q' \leq p$, Ω satisfy (1.1) and the L^q -Dini condition. Suppose that $f \in \varepsilon^{\alpha,p}(\mathbb{R}^n)$ for $-\infty < \alpha < 0$ and there is a measurable set $E \subset \mathbb{R}^n$ with $|E| > 0$ such that $\mu_\Omega(f)(x) < \infty$ for any $x \in E$. Then $\mu_\Omega(f)(x) < \infty$ a.e. on \mathbb{R}^n and $\|\mu_\Omega(f)\|_{\varepsilon^{\alpha,p}} \leq C \|f\|_{\varepsilon^{\alpha,p}}$, where the constant C is independent off.

The multilinear theory has attracted much attention since the pioneering work of Coifman and Meyer [4, 5]. The topic was reconsidered by several authors, including Christ and Journé [6], Kenig and Stein [7], Grafakos and Torres [8, 9], and Lerner et al. [10]. Chen, Xue and Yabuta [11] proved the boundedness of a multilinear Marcinkiewicz integral on Lebesgue spaces. Xue and Yabuta [12] proved the boundedness of a multilinear Marcinkiewicz integral on Campanato spaces.

Definition 1.2 ([11]) Let Ω be a function defined on $(\mathbb{R}^n)^m$ with the following properties:

- (1) Ω is homogeneous of degree 0, i.e., for any $\lambda > 0$ and $\vec{y} = (y_1, \dots, y_m) \in (\mathbb{R}^n)^m$,

$$\Omega(\lambda \vec{y}) = \Omega(\vec{y}); \quad (1.2)$$

- (2) Ω is Lipschitz continuous on $(\mathbb{S}^{n-1})^m$, i.e. there are $0 < \alpha < 1$ and $C > 0$ such that, for any $\vec{\xi} = (\xi_1, \dots, \xi_m)$, $\vec{\eta} = (\eta_1, \dots, \eta_m) \in (\mathbb{R}^n)^m$,

$$|\Omega(\vec{\xi}) - \Omega(\vec{\eta})| \leq C |\vec{\xi}' - \vec{\eta}'|^\alpha, \quad (1.3)$$

where $(y_1, \dots, y_m)' = \frac{(y_1, \dots, y_m)}{|y_1| + \dots + |y_m|}$, and we should note that $(y_1, \dots, y_m)'$ is not an element of $(\mathbb{S}^{n-1})^m$;

- (3) the integration of Ω on each unit sphere vanishes,

$$\int_{\mathbb{S}^{n-1}} \Omega(y_1, \dots, y_m) dy_j = 0, \quad j = 1, \dots, m. \quad (1.4)$$

For any $\vec{f} = (f_1, \dots, f_m) \in (S(\mathbb{R}^n))^m$, we can define the operator F_t for any $t > 0$ as

$$\begin{aligned} F_t(\vec{f})(x) &= \frac{\chi_{(B(0,t))^m} \Omega(\cdot)}{t^m |\cdot|^{m(n-1)}} * (f_1 \otimes \cdots \otimes f_m)(x) \\ &= \frac{1}{t^m} \int_{(B(0,t))^m} \frac{\Omega(\vec{y})}{|\vec{y}|^{m(n-1)}} \prod_{i=1}^m f_i(x - y_i) d\vec{y}, \end{aligned}$$

where $|\vec{y}| = |y_1| + \cdots + |y_m|$ and $B(x, t) = \{y \in \mathbb{R}^n : |y - x| \leq t\}$. Finally, the multilinear Marcinkiewicz integral μ is defined by

$$\mu(\vec{f})(x) = \left(\int_0^\infty |F_t(\vec{f})(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

If $m = 1$, it is easy to see that $\mu(\vec{f})$ coincides with $\mu_\Omega(f)$.

Theorem B ([11]) Suppose μ is bounded from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q for some $1 < q_1, \dots, q_m < \infty$ with $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. Then, for $1 < p_1, \dots, p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, there is a $C > 0$ such that

$$\|\mu(\vec{f})\|_{L^p} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}}.$$

Yang [13] introduced the central Campanato spaces on p -adic fields and obtained the behavior of a class of p -adic singular integral operators on these spaces. In fact, the central Campanato space on Euclid spaces can be defined in a similar way. Now we are in a position to define the central Campanato space.

Definition 1.3 ([13]) Suppose $1 \leq p < \infty$, $q > 1$, $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}$. The central Campanato space $\text{CL}^{\alpha,p}(\mathbb{R}^n)$ is defined as

$$\text{CL}^{\alpha,p}(\mathbb{R}^n) := \{f \in L_{\text{loc}}^p(\mathbb{R}^n) : \|f\|_{\text{CL}^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{\text{CL}^{\alpha,p}} = \sup_{r>0} \frac{1}{|B(0,r)|^{\alpha+\frac{1}{p}}} \left(\int_{B(0,r)} |f(x) - f_{B(0,r)}|^p dx \right)^{\frac{1}{p}}.$$

Remark 1.2 When $\alpha = \frac{1}{l} - 1$, the central Campanato space $\text{CL}^{\alpha,p}$ is the central BMO space CMO_l^p , where $0 < l \leq 1 < p < \infty$; see [14] for more about the space of CMO_l^p .

The main results of this paper are as follows.

Theorem 1.1 Let $1 < p < \infty$, $1 \leq q' \leq p$, Ω satisfy (1.1) and the L^q -Dini condition. Suppose that $f \in \text{CL}^{\alpha,p}(\mathbb{R}^n)$ for $-\infty < \alpha n < 0$ and there is a measurable set $E \subset \mathbb{R}^n$ with $|E| > 0$ such that $\mu_\Omega(f)(x) < \infty$ for any $x \in E$. Then $\mu_\Omega(f)(x) < \infty$ a.e. on \mathbb{R}^n and $\|\mu_\Omega(f)\|_{\text{CL}^{\alpha,p}} \leq C \|f\|_{\text{CL}^{\alpha,p}}$, where the constant C is independent of f .

Theorem 1.2 Let Ω be a function defined on $(\mathbb{R}^n)^m$, satisfying (1.2), Lipschitz continuous (1.3) with index replaced by β and (1.4). Suppose μ is bounded from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q for some $1 < q_1, \dots, q_m < \infty$ with $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. Suppose also that $-\infty < \alpha = \alpha_1 + \cdots + \alpha_m < 0$ with $\alpha_1, \dots, \alpha_m < 0$ and $n < p < \infty$. Then, for $1 < p_1, \dots, p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $f_j \in \text{CL}^{\alpha_j, p_j}(\mathbb{R}^n)$ ($j = 1, \dots, m$), $\mu(\vec{f})$ is either infinite everywhere or finite almost everywhere, and in the latter case, there is a constant $C > 0$ such that

$$\|\mu(\vec{f})\|_{\text{CL}^{\alpha, p}} \leq C \prod_{j=1}^m \|f_j\|_{\text{CL}^{\alpha_j, p_j}}.$$

Remark 1.3 Comparing the definition of the Campanato space and the central Campanato space, we can easily see that the selection range of the ball B is different. This leads to the difference in the range of the index about the boundedness of $\mu(\vec{f})$ on Campanato space and central Campanato space.

Throughout this paper we assume that the notation C represents a constant and its values may vary from line to line.

2 Some preliminaries and notations

Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ ($q \geq 1$), the integral modulus $\omega_q(\delta)$ of continuity of order q of Ω is defined by

$$\omega_q(\delta) = \sup_{|\rho| \leq \delta} \left(\int_{\mathbb{S}^{n-1}} |\Omega(\rho x') - \Omega(x')|^q d\sigma(x') \right)^{\frac{1}{q}},$$

where ρ is a rotation on \mathbb{S}^{n-1} , $|\rho| = \|\rho - I\|$. We say that Ω satisfies the L^q -Dini condition, if $\omega_q(\delta)$ satisfies

$$\int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta < \infty.$$

To prove our main results, we begin with some important lemmas.

Lemma 2.1 ([3]) Suppose that $0 < \lambda < n$, and Ω is homogeneous of degree zero and satisfies the L^q -Dini condition for $q > 1$. If there exists a constant $a_0 > 0$, such that $|x| < a_0 R$, then we have

$$\left(\int_{R < |y| < 2R} \left| \frac{\Omega(y-x)}{|y-x|^{n-\lambda}} - \frac{\Omega(y)}{|y|^{n-\lambda}} \right|^q dy \right)^{\frac{1}{q}} \leq CR^{\frac{n}{q}-(n-\lambda)} \left(\frac{|x|}{R} + \int_{\frac{|x|}{2R} < \delta < \frac{|x|}{R}} \frac{\omega_q(\delta)}{\delta} d\delta \right),$$

where the constant $C > 0$ is independent of R and x .

Lemma 2.2 Suppose that $1 < p < \infty$, $-\infty < \alpha < \frac{1}{2}$, and $\eta > \max\{0, \alpha pn\}$. If $f \in \text{CL}^{\alpha, p}$, then there exists $C > 0$ such that, for any ball B_0 centered at origin with side length d and $x_0 \in B_0$,

$$\left(\int_{\mathbb{R}^n} \frac{d^\eta |f(x) - f_{B_0}|^p}{d^{n+\eta} + |x - x_0|^{n+\eta}} dx \right)^{\frac{1}{p}} \leq C d^{n\alpha} \|f\|_{\text{CL}^{\alpha, p}}.$$

Proof Let B_k be a ball with the same center of B_0 and its radius is 2^k times of B_0 . Decomposing \mathbb{R}^n into a geometrically increasing sequence of concentric balls, and using the fact that $|f_{B_k} - f_{B_0}| \leq Ck|B_k|^\alpha \|f\|_{\text{CL}^{\alpha,p}}$, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \frac{d^\eta |f(x) - f_{B_0}|^p}{d^{n+\eta} + |x - x_0|^{n+\eta}} dx \\ &= \sum_{k=0}^{\infty} \int_{B_k \setminus B_{k-1}} \frac{d^\eta |f(x) - f_{B_0}|^p}{d^{n+\eta} + |x - x_0|^{n+\eta}} dx \\ &\leq C \sum_{k=0}^{\infty} \frac{d^\eta}{(2^k d)^{n+\eta}} \left(\int_{B_k} |f(x) - f_{B_k}|^p dx + |f_{B_k} - f_{B_0}|^p |B_k| \right) \\ &\leq C \sum_{k=0}^{\infty} \frac{1}{2^{k\eta}} ((2^k d)^{n\alpha p} \|f\|_{\text{CL}^{\alpha,p}}^p + k^p (2^k d)^{n\alpha p} \|f\|_{\text{CL}^{\alpha,p}}^p) \\ &\leq Cd^{n\alpha p} \|f\|_{\text{CL}^{\alpha,p}}^p. \end{aligned}$$

Then the proof is complete. \square

Lemma 2.3 Suppose that $1 < p < \infty$, $-\infty < \alpha < \frac{1}{2}$, $1 \leq q' < p$ and $\eta > \max\{0, \alpha pn\}$. If $f \in \text{CL}^{\alpha,p}$, then there exists $C > 0$ such that, for any ball B_0 with side length d and centered at origin and $x_0 \in B_0$,

$$\left(\int_{\mathbb{R}^n} \frac{d^\eta |f(x) - f_{B_0}|^{q'}}{d^{n+\eta} + |x - x_0|^{n+\eta}} dx \right)^{\frac{1}{q'}} \leq Cd^{n\alpha} \|f\|_{\text{CL}^{\alpha,p}}.$$

Proof Let B_k be a ball with the same center of B_0 and its radius is 2^k times of B_0 . Decomposing \mathbb{R}^n into a geometrically increasing sequence of concentric balls, and using Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \frac{d^\eta |f(x) - f_{B_0}|^{q'}}{d^{n+\eta} + |x - x_0|^{n+\eta}} dx \\ &\leq \sum_{k=0}^{\infty} \left(\int_{\mathbb{R}^n} \frac{d^\eta |f(x) - f_{B_0}|^p}{d^{n+\eta} + |x - x_0|^{n+\eta}} dx \right)^{\frac{q'}{p}} \\ &\quad \times \left(\int_{B_k \setminus B_{k-1}} \frac{d^\eta}{d^{n+\eta} + |x - x_0|^{n+\eta}} dx \right)^{\frac{p-q'}{p}} \\ &\leq C \sum_{k=0}^{\infty} d^{n\alpha q'} \|f\|_{\text{CL}^{\alpha,p}}^{q'} \left(\frac{d^\eta (2^k d)^n}{(2^{k-1} d)^{n+\eta}} \right)^{\frac{p-q'}{p}} \\ &\leq Cd^{n\alpha q'} \|f\|_{\text{CL}^{\alpha,p}}^{q'}. \end{aligned}$$

Then the proof is complete. \square

Lemma 2.4 Suppose that $1 \leq p < \infty$, $\alpha < 0$, $B = B(0, r)$, $x \in B$ and $t > 8r > 0$. If $f \in \text{CL}^{\alpha, p}$, then, for $0 \leq k \leq k_0$ with $k_0 \in \mathbb{N}$ satisfying $2r \leq 2^{-k_0}t < 4r$, we have

$$\left(\frac{1}{|B(x, 2^{-k}t)|} \int_{B(x, 2^{-k}t)} |f(y) - f_B|^p dy \right)^{\frac{1}{p}} \leq Cr^{n\alpha} \|f\|_{\text{CL}^{\alpha, p}}.$$

Proof Since $x \in B$ and $0 \leq k \leq k_0$ satisfying $2r \leq 2^{-k_0}t < 4r$, we have $B(x, 2^{-k}t) \subset B(0, 2^{-k+1}t)$. Thus

$$\begin{aligned} & \left(\frac{1}{|B(x, 2^{-k}t)|} \int_{B(x, 2^{-k}t)} |f(y) - f_B|^p dy \right)^{\frac{1}{p}} \\ & \leq C \left(\frac{1}{|B(0, 2^{-k+1}t)|} \int_{B(0, 2^{-k+1}t)} |f(y) - f_{B(0, 2^{-k+1}t)}|^p dy \right)^{\frac{1}{p}} \\ & \quad + C \sum_{j=k}^{k_0} |f_{B(0, 2^{-j+1}t)} - f_{B(0, 2^{-j}t)}| + C |f_{B(0, 2^{-k_0}t)} - f_B| \\ & \leq C (2^{-k+1}t)^{n\alpha} \|f\|_{\text{CL}^{\alpha, p}} + C \sum_{j=k}^{k_0} |f_{B(0, 2^{-j+1}t)} - f_{B(0, 2^{-j}t)}| + C |f_{B(0, 2^{-k_0}t)} - f_B|. \end{aligned}$$

For $j = k, \dots, k_0$, we have

$$\begin{aligned} |f_{B(0, 2^{-j+1}t)} - f_{B(0, 2^{-j}t)}| & \leq \frac{1}{|B(0, 2^{-j}t)|} \int_{B(0, 2^{-j}t)} |f(y) - f_{B(0, 2^{-j+1}t)}| dy \\ & \leq 2^n \left(\frac{1}{|B(0, 2^{-j+1}t)|} \int_{B(0, 2^{-j+1}t)} |f(y) - f_{B(0, 2^{-j+1}t)}|^p dy \right)^{\frac{1}{p}} \\ & \leq 2^n (2^{-j+1}t)^{n\alpha} \|f\|_{\text{CL}^{\alpha, p}}. \end{aligned}$$

For the last term, since $2r \leq 2^{-k_0}t \leq 4r$, we have

$$|f_{B(0, 2^{-k_0}t)} - f_B| \leq 4^n (2^{-k_0}t)^{n\alpha} \|f\|_{\text{CL}^{\alpha, p}}.$$

So we obtain

$$\begin{aligned} & \left(\frac{1}{|B(x, 2^{-k}t)|} \int_{B(x, 2^{-k}t)} |f(y) - f_B|^p dy \right)^{\frac{1}{p}} \\ & \leq C \sum_{j=k}^{k_0} 2^{-jn\alpha} t^{n\alpha} \|f\|_{\text{CL}^{\alpha, p}} \leq Cr^{n\alpha} \|f\|_{\text{CL}^{\alpha, p}}. \end{aligned}$$

Then the proof is complete. \square

Lemma 2.5 Suppose that $1 \leq p < \infty$, $\alpha < 0$, $B = B(0, r)$, $x \in B$ and $k \in \mathbb{N}$. If $f \in \text{CL}^{\alpha, p}$, then we have

$$\left(\frac{1}{|B(x, 2^k r)|} \int_{B(x, 2^k r)} |f(y) - f_B|^p dy \right)^{\frac{1}{p}} \leq Cr^{n\alpha} \|f\|_{\text{CL}^{\alpha, p}}.$$

Proof For $x \in B$ and $k \in \mathbb{N}$, we have $B(x, 2^k r) \subset B(0, 2^{k+1} r)$. Similar to Lemma 2.4, we have

$$\begin{aligned} & \left(\frac{1}{|B(x, 2^k r)|} \int_{B(x, 2^k r)} |f(y) - f_B|^p dy \right)^{\frac{1}{p}} \\ & \leq C \left(\frac{1}{|B(0, 2^{k+1} r)|} \int_{B(0, 2^{k+1} r)} |f(y) - f_{B(0, 2^{k+1} r)}|^p dy \right)^{\frac{1}{p}} \\ & \quad + C |f_{B(0, 2^{k+1} r)} - f_{B(0, 2^k r)}| + \cdots + |f_{B(0, 2r)} - f_B| \\ & \leq C \sum_{l=0}^{k+1} 2^{\ln \alpha} r^{n\alpha} \|f\|_{CL^{\alpha, p}} \leq C r^{n\alpha} \|f\|_{CL^{\alpha, p}}. \end{aligned}$$

Then the proof is complete. \square

Lemma 2.6 Suppose that $\alpha < 0$, $B = B(0, r)$, $x \in B$ and $t > 8r > 0$, $1 \leq p < \infty$. If $f \in CL^{\alpha, p}$, then we have

$$\int_{8r \leq |x-y| < t} \frac{|f(y) - f_B|}{|x-y|^{n-1}} dy \leq C t r^{n\alpha} \|f\|_{CL^{\alpha, p}}.$$

Proof Let $k_0 \in \mathbb{N}$ satisfying $2r \leq 2^{-k_0} t < 4r$. Using Lemma 2.4, we obtain

$$\begin{aligned} \int_{8r \leq |x-y| < t} \frac{|f(y) - f_B|}{|x-y|^{n-1}} dy & \leq \sum_{k=0}^{k_0-1} \int_{2^{-k-1} t \leq |x-y| \leq 2^{-k} t} \frac{|f(y) - f_B|}{|x-y|^{n-1}} dy \\ & \leq \sum_{k=0}^{k_0-1} \frac{1}{(2^{-k-1} t)^{n-1}} \int_{B(x, 2^{-k} t)} |f(y) - f_B| dy \\ & \leq C \sum_{k=0}^{k_0-1} 2^{-k} t \left(\frac{1}{|B(x, 2^{-k} t)|} \int_{B(x, 2^{-k} t)} |f(y) - f_B|^p dy \right)^{\frac{1}{p}} \\ & \leq C t r^{n\alpha} \|f\|_{CL^{\alpha, p}}. \end{aligned}$$

Then the proof is complete. \square

Lemma 2.7 Suppose that $m \in \mathbb{N}$, $B = B(0, r)$, $x \in B$, $1 \leq p < \infty$, $0 < \beta \leq 1$ and $\gamma < \beta$, $\alpha = \alpha_1 + \cdots + \alpha_m < 0$ with $\alpha_1, \dots, \alpha_m < 0$. If $f_i \in CL^{\alpha_i p_i}$ ($i = 1, \dots, m$), then we have

$$\int_{(B(x, 8r)^m)^c} \frac{r^\beta \prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x - y_j|)^{mn + \beta - \gamma}} d\vec{y} \leq C r^{\gamma + n\alpha} \prod_{i=1}^m \|f_i\|_{CL^{\alpha_i p_i}}.$$

Proof Using Lemma 2.5 and Hölder's inequality, we have

$$\begin{aligned} & \int_{(B(x, 8r)^m)^c} \frac{r^\beta \prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x - y_j|)^{mn + \beta - \gamma}} d\vec{y} \\ & = \sum_{k=0}^{\infty} \int_{(B(x, 2^{k+4}r)^m \setminus (B(x, 2^{k+3}r))^m)} \frac{r^\beta \prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x - y_j|)^{mn + \beta - \gamma}} d\vec{y} \end{aligned}$$

$$\begin{aligned}
&\leq Cr^\beta \sum_{k=0}^{\infty} \frac{1}{(2^{k+3}r)^{mn+\beta-\gamma}} \int_{(B(x, 2^{k+4}r))^m} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| dy \\
&\leq Cr^\gamma \sum_{k=0}^{\infty} \frac{1}{2^{k(\beta-\gamma)}} \prod_{i=1}^m \frac{1}{|B(x, 2^{k+4}r)|} \int_{B(x, 2^{k+4}r)} |f_i(y_i) - (f_i)_B| dy_i \\
&\leq C \sum_{k=0}^{\infty} \frac{1}{2^{k(\beta-\gamma)}} r^{\gamma+n\alpha} \prod_{i=1}^m \|f_i\|_{CL^{\alpha_i, p_i}} \\
&\leq Cr^{\gamma+n\alpha} \prod_{i=1}^m \|f_i\|_{CL^{\alpha_i, p_i}}.
\end{aligned}$$

Then the proof is complete. \square

3 Proof the main results

Proof of Theorem 1.1 Let $1 < p < \infty$, $1 \leq q' \leq p$ and $f \in CL^{\alpha, p}$. We set B be the ball $B(0, d)$, $B^* = 8B = B(0, 8d)$, $B^j = 2^j B = B(0, 2^j d)$ ($j \in \mathbb{Z}$). We first show that $\mu_\Omega(f)(x) < \infty$, a.e. on B . Let

$$f(x) = f_{B^*} + (f(x) - f_{B^*}) \chi_{B^*}(x) + (f(x) - f_{B^*})(1 - \chi_{B^*}(x)) =: f_1 + f_2 + f_3.$$

By Theorem A and (1.1), we know that $\mu_\Omega(f_1)(x) = 0$ on B and

$$\left(\int_B |\mu_\Omega(f_2)(x)|^p dx \right)^{\frac{1}{p}} \leq \left(\int_{B^*} |\mu_\Omega(f_2)(x)|^p dx \right)^{\frac{1}{p}} \leq Cd^{n\alpha + \frac{n}{p}} \|f\|_{CL^{\alpha, p}}.$$

Hence

$$\int_B |\mu_\Omega(f_2)(x)| dx \leq |B|^{\frac{1}{p'}} \left(\int_B |\mu_\Omega(f_2)(x)|^p dx \right)^{\frac{1}{p}} \leq d^{n+n\alpha} \|f\|_{CL^{\alpha, p}}.$$

This shows that $\mu_\Omega(f_2)(x) < \infty$, a.e. on B . Since $|E| > 0$, we have $|B \cap E| > 0$. There exists an $x_0 \in B \cap E$, such that $\mu_\Omega(f)(x_0) < \infty$ and $\mu_\Omega(f_2)(x_0) < \infty$. Then

$$\mu_\Omega(f_3)(x_0) \leq \mu_\Omega(f)(x_0) + \mu_\Omega(f_2)(x_0) < \infty.$$

Fix any $x \in B$, we write

$$\begin{aligned}
&|\mu_\Omega(f_3)(x) - \mu_\Omega(f_3)(x_0)| \\
&\leq \left(\int_0^\infty \left(\int_{|x-y|< t, |x_0-y|>t} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} f_3(y) \right| dy \right)^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
&\quad + \left(\int_0^\infty \left(\int_{|x-y|>t, |x_0-y|<t} \left| \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}} f_3(y) \right| dy \right)^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
&\quad + \left(\int_0^\infty \left(\int_{|x-y|< t, |x_0-y|< t} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}} \right| |f_3(y)| dy \right)^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

We first estimate I_1 . It is obvious that $|y - x| \approx |y - x_0| \approx |y|$, when $x \in B$, $x_0 \in B$ and $y \in (B^*)^c$. Applying Minkowski's inequality, we get

$$\begin{aligned} I_1 &\leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_3(y)| \left(\int_{|x-y|< t, |x_0-y|>t} \frac{dt}{t^3} \right)^{\frac{1}{2}} dy \\ &\leq C \int_{(B^*)^c} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_3(y)| \left| \frac{1}{|x-y|^2} - \frac{1}{|x_0-y|^2} \right|^{\frac{1}{2}} dy \\ &\leq C \int_{(B^*)^c} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_3(y)| \frac{d^{\frac{1}{2}}}{|y|^{\frac{3}{2}}} dy \\ &= C \int_{(B^*)^c} \frac{d^{\frac{1}{2}} |\Omega(x-y)|}{|y|^{n+\frac{1}{2}}} |f_3(y)| dy. \end{aligned}$$

We take $\eta > 0$, such that $\alpha np < \eta < \frac{q'}{2}$. Then $\tau = (\frac{1}{2} - \frac{\eta}{q'})q > 0$. Using Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned} I_1 &\leq C \left(\int_{(B^*)^c} \frac{d^\tau |\Omega(x-y)|^q}{|y|^{n+\tau}} dy \right)^{\frac{1}{q}} \left(\int_{(B^*)^c} \frac{d^\eta |f_3(y)|^{q'}}{|y|^{n+\eta}} dy \right)^{\frac{1}{q'}} \\ &\leq Cd^{n\alpha} \|f\|_{\text{CL}^{\alpha,p}}. \end{aligned}$$

Similarly, we can get $I_2 \leq Cd^{n\alpha} \|f\|_{\text{CL}^{\alpha,p}}$. For the last term I_3 , we use Minkowski's inequality to get

$$\begin{aligned} I_3 &\leq \int_{\mathbb{R}^n} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}} \right| |f_3(y)| \left(\int_{|x-y|< t, |x_0-y|<t} \frac{dt}{t^3} \right)^{\frac{1}{2}} dy \\ &\leq C \int_{\mathbb{R}^n} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}} \right| |f_3(y)| \left(\int_{|x_0-y|<t}^{\infty} \frac{dt}{t^3} \right)^{\frac{1}{2}} dy \\ &\leq C \int_{\mathbb{R}^n} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}} \right| \frac{|f_3(y)|}{|y|} dy \\ &\leq C \sum_{j=3}^{\infty} \int_{2^j d < |y| < 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}} \right| \frac{|f_3(y)|}{|y|} dy \\ &\leq C \sum_{j=3}^{\infty} \frac{1}{2^j d} \int_{2^j d < |y| < 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}} \right| |f_3(y)| dy. \end{aligned}$$

Applying Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned} I_3 &\leq C \sum_{j=3}^{\infty} \frac{1}{2^j d} \left(\int_{2^j d < |y| < 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}} \right|^q dy \right)^{\frac{1}{q}} \\ &\quad \times \left(\int_{2^j d < |y| < 2^{j+1} d} |f_3(y)|^{q'} dy \right)^{\frac{1}{q'}} \\ &\leq C \sum_{j=3}^{\infty} \frac{1}{2^j} \left(\frac{1}{(2^{j+1} d)^n} \int_{2^j d < |y| < 2^{j+1} d} |f_3(y)|^{q'} dy \right)^{\frac{1}{q'}} \end{aligned}$$

$$\begin{aligned}
& + C \sum_{j=3}^{\infty} \left(\int_{\frac{|x-x_0|}{2^{j+1}d}}^{\frac{|x-x_0|}{2^j d}} \frac{\omega_q(\delta)}{\delta} d\delta \right) \left(\frac{1}{(2^{j+1}d)^n} \int_{2^j d < |y| < 2^{j+1}d} |f_3(y)|^{q'} dy \right)^{\frac{1}{q'}} \\
& =: I_{31} + I_{32}.
\end{aligned}$$

To estimate I_{31} , we take $\eta > 0$. Using Lemma 2.3, it yields

$$\begin{aligned}
I_{31} & \leq C \sum_{j=3}^{\infty} \frac{1}{2^j} \left(\int_{2^j d < |y| < 2^{j+1}d} \frac{(2^{j+1}d)^\eta |f_3(y)|^{q'}}{|y|^{n+\eta} + (2^{j+1}d)^{n+\eta}} dy \right)^{\frac{1}{q'}} \\
& \leq C \sum_{j=3}^{\infty} \frac{1}{2^j} \left(\left(\int_{2^j d < |y| < 2^{j+1}d} \frac{(2^{j+1}d)^\eta |f(y) - f_{B_j}|^{q'}}{|y|^{n+\eta} + (2^{j+1}d)^{n+\eta}} dy \right)^{\frac{1}{q'}} + |f_{B_j} - f_{B^*}| \right) \\
& \leq C \sum_{j=3}^{\infty} \frac{j}{2^j} (2^j d)^{n\alpha} \|f\|_{\text{CL}^{\alpha,p}} \\
& \leq C d^{n\alpha} \|f\|_{\text{CL}^{\alpha,p}}.
\end{aligned}$$

As for I_{32} , we have

$$\begin{aligned}
I_{32} & = C \sum_{j=3}^{\infty} \left(\int_{\frac{|x-x_0|}{2^{j+1}d}}^{\frac{|x-x_0|}{2^j d}} \frac{\omega_q(\delta)}{\delta} d\delta \right) \left(\frac{1}{(2^{j+1}d)^n} \int_{2^j d < |y| < 2^{j+1}d} |f(y) - f_{B^*}|^{q'} dy \right)^{\frac{1}{q'}} \\
& \leq C \sum_{j=3}^{\infty} \left(\int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta \right) \left(\frac{1}{(2^{j+1}d)^n} \int_{2^j d < |y| < 2^{j+1}d} |f(y) - f_{B^*}|^{q'} dy \right)^{\frac{1}{q'}} \\
& \leq C \sum_{j=3}^{\infty} \left(\frac{1}{(2^{j+1}d)^n} \int_{2^j d < |y| < 2^{j+1}d} |f(y) - f_{B_{j+1}}|^{q'} dy \right)^{\frac{1}{q'}} \\
& \quad + C \sum_{j=3}^{\infty} |f_{B_{j+1}} - f_{B^*}| \\
& =: I_{32}^1 + I_{32}^2.
\end{aligned}$$

Since $q' \leq p$ and using Hölder's inequality, we obtain

$$\begin{aligned}
I_{32}^1 & \leq C \sum_{j=3}^{\infty} \left(\frac{1}{|B_{j+1}|} \int_{B_{j+1}} |f(y) - f_{B_{j+1}}|^p dy \right)^{\frac{1}{p}} \\
& \leq \sum_{j=3}^{\infty} |B_{j+1}|^\alpha \|f\|_{\text{CL}^{\alpha,p}} \\
& \leq C d^{n\alpha} \|f\|_{\text{CL}^{\alpha,p}}.
\end{aligned}$$

For the last term I_{32}^2 , it is easy to get

$$\begin{aligned}
I_{32}^2 & \leq C \sum_{j=3}^{\infty} (j+1) (2^{j+1}d)^{n\alpha} \|f\|_{\text{CL}^{\alpha,p}} \\
& \leq C d^{n\alpha} \|f\|_{\text{CL}^{\alpha,p}}.
\end{aligned}$$

Summarizing the above estimate, we conclude that

$$|\mu_{\Omega}(f_3)(x) - \mu_{\Omega}(f_3)(x_0)| \leq Cd^{n\alpha} \|f\|_{CL^{\alpha,p}}.$$

Thus we have

$$\begin{aligned} \mu_{\Omega}(f)(x) &\leq \mu_{\Omega}(f_1)(x) + \mu_{\Omega}(f_2)(x) \\ &+ |\mu_{\Omega}(f_3)(x) - \mu_{\Omega}(f_3)(x_0)| + \mu_{\Omega}(f_3)(x_0) < \infty, \quad \text{a.e. on } B. \end{aligned}$$

Because B is any ball centered at the origin, we get $\mu_{\Omega}(f)(x) < \infty$, a.e. on \mathbb{R}^n . Finally, we show that $\|\mu_{\Omega}(f)\|_{CL^{\alpha,p}} \leq C\|f\|_{CL^{\alpha,p}}$. In fact, from the above proof we find that there exists an $x_0 \in B$, such that $\mu_{\Omega}(f_3)(x_0) < \infty$. Repeating the above proof, we obtain

$$\begin{aligned} &\left(\int_B |\mu_{\Omega}f(x) - \mu_{\Omega}f_3(x_0)|^p dx \right)^{\frac{1}{p}} \\ &\leq C \left(\int_B |\mu_{\Omega}f_2(x)|^p dx \right)^{\frac{1}{p}} + \left(\int_B |\mu_{\Omega}f_3(x) - \mu_{\Omega}f_3(x_0)|^p dx \right)^{\frac{1}{p}} \\ &\leq Cd^{n\alpha+\frac{n}{p}} \|f\|_{CL^{\alpha,p}}. \end{aligned}$$

Taking the supremum over all such B , the proof is complete. \square

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2 It suffices to verify that, for any $f_j \in CL^{\alpha_j, p_j}(\mathbb{R}^n)$ ($j = 1, \dots, m$), if there exists $y_0 \in \mathbb{R}^n$ such that $\mu(\vec{f})(y_0) < \infty$, then, for any ball $B = B(0, r) \subset \mathbb{R}^n$ with $y_0 \in B$,

$$\left(\frac{1}{|B|} \int_B |\mu(\vec{f})(x) - (\mu(\vec{f}))_B|^p dx \right)^{\frac{1}{p}} \leq C|B|^{\alpha} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}.$$

For any $r > 0$, we denote

$$\mu^r(\vec{f})(x) = \left(\int_0^{8r} \left| \frac{1}{t^m} \int_{(B(0,t))^m} \frac{\mathcal{Q}(\vec{y})}{|\vec{y}|^{m(n-1)}} \prod_{i=1}^m f_i(x - y_i) d\vec{y} \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

and

$$\mu^{\infty}(\vec{f})(x) = \left(\int_{8r}^{\infty} \left| \frac{1}{t^m} \int_{(B(0,t))^m} \frac{\mathcal{Q}(\vec{y})}{|\vec{y}|^{m(n-1)}} \prod_{i=1}^m f_i(x - y_i) d\vec{y} \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

Since \mathcal{Q} satisfies the vanishing condition (1.4), for any $x \in B$,

$$\begin{aligned} \mu^r(\vec{f})(x) &= \mu^r((f_1 - (f_1)_B)\chi_{10B}, \dots, (f_m - (f_m)_B)\chi_{10B})(x) \\ &\leq \mu((f_1 - (f_1)_B)\chi_{10B}, \dots, (f_m - (f_m)_B)\chi_{10B})(x). \end{aligned}$$

We have $\alpha_1 + \dots + \alpha_m = \alpha$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ with $1 < p_1, \dots, p_m < \infty$. Using Theorem B, we get

$$\begin{aligned} \left(\int_B |\mu^r(\vec{f})(x)|^p dx \right)^{\frac{1}{p}} &\leq C \prod_{j=1}^m \left(\int_{10B} |f_j(y_j) - (f_j)_{10B}|^{p_j} dy_j \right)^{\frac{1}{p_j}} \\ &\leq C \prod_{j=1}^m |10B|^{\alpha_j + \frac{1}{p_j}} \|f_j\|_{CL^{\alpha_j, p_j}} \\ &\leq C|B|^{\alpha + \frac{1}{p}} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}. \end{aligned}$$

We notice that

$$\begin{aligned} &\left(\frac{1}{|B|} \int_B |\mu(\vec{f})(x) - (\mu(\vec{f}))_B|^p dx \right)^{\frac{1}{p}} \\ &\leq C \left(\frac{1}{|B|} \int_B \left| \mu(\vec{f})(x) - \inf_{y \in B} \mu(\vec{f})(y) \right|^p dx \right)^{\frac{1}{p}} \\ &\leq C \left(\frac{1}{|B|} \int_B |\mu^r(\vec{f})(x)|^p dx \right)^{\frac{1}{p}} + \left(\frac{1}{|B|} \int_B \sup_{y \in B} |\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(y)|^p dx \right)^{\frac{1}{p}} \\ &\leq C|B|^\alpha \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}} + \left(\frac{1}{|B|} \int_B \sup_{y \in B} |\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(y)|^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

So the proof of Theorem 1.2 reduces to proving that, for any $x, z \in B$,

$$|\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(z)| \leq C|B|^\alpha \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}.$$

It is easy to see that

$$\begin{aligned} |\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(z)| &= \left| \left(\int_{8r}^\infty |F_t(\vec{f})(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} - \left(\int_{8r}^\infty |F_t(\vec{f})(z)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right| \\ &\leq \left(\int_{8r}^\infty |F_t(\vec{f})(x) - F_t(\vec{f})(z)| \left| |F_t(\vec{f})(x) + F_t(\vec{f})(z)| \frac{dt}{t} \right|^{\frac{1}{2}} \right). \end{aligned}$$

For any $z \in B$, $t_1, \dots, t_m > r$, we have $B(z, t_i) \subset B(0, 2t_i)$ ($i = 1, \dots, m$). If $n < p < \infty$ and $-\infty < \alpha < 0$, the vanishing condition of Ω and Hölder's inequality allow us to obtain

$$\begin{aligned} &\left| \int_{\prod_{i=1}^m B(z, t_i)} \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right| \\ &\leq C \left(\int_{\prod_{i=1}^m B(z, t_i)} \left| \prod_{j=1}^m (f_j(y_j) - (f_j)_{B(0, 2t_j)}) \right|^p d\vec{y} \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{\prod_{i=1}^m B(z, t_i)} \frac{d\vec{y}}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)p'}} \right)^{\frac{1}{p'}} \end{aligned}$$

$$\begin{aligned}
&\leq C \left(\int_{\prod_{i=1}^m B(0, 2t_i)} \left| \prod_{j=1}^m (f_j(y_j) - (f_j)_{B(0, 2t_j)}) \right|^p d\vec{y} \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_{\prod_{i=1}^m B(z, t_i)} \frac{d\vec{y}}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)p'}} \right)^{\frac{1}{p'}} \\
&\leq C \prod_{j=1}^m t_j^{\frac{n}{p'} - n + 1} \prod_{j=1}^m \left(\prod_{i \neq j} t_i^{\frac{n}{p_i}} t_j^{\frac{n}{p_j} + n\alpha_j} \right) \|f_j\|_{CL^{\alpha_j, p_j}} \\
&= C \prod_{j=1}^m t_j^{\frac{n}{p'} - n + 1} \prod_{j=1}^m t_j^{\frac{n}{p} + n\alpha_j} \|f_j\|_{CL^{\alpha_j, p_j}} \\
&= C \prod_{j=1}^m t_j^{1 + n\alpha_j} \|f_j\|_{CL^{\alpha_j, p_j}}.
\end{aligned}$$

For $z \in B$, $t > 8r$, $B(z, t)$ can be decomposed into the following disjoint union:

$$\begin{aligned}
(B(z, t))^m &= \left\{ \bigcup_{i=1}^m ((B(z, t) \setminus B(z, 8r))^{i-1} \times B(z, 8r) \times (B(z, t) \setminus B(z, 8r))^{m-i}) \right\} \\
&\cup (B(z, t) \setminus B(z, 8r))^m \cup (B(z, 8r))^m.
\end{aligned}$$

Set $B_i(z, t, r) := (B(z, t) \setminus B(z, 8r))^{i-1} \times B(z, 8r) \times (B(z, t) \setminus B(z, 8r))^{m-i}$. We write

$$\begin{aligned}
t^m F_t(\vec{f})(z) &= \int_{(B(z, t))^m} \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \\
&= \int_{(B(z, t) \setminus B(z, 8r))^m} \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \\
&\quad + \sum_{l=1}^m \int_{B_l(z, t, r)} \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \\
&\quad + \int_{(B(z, 8r))^m} \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y}.
\end{aligned}$$

According to the above estimate, we have

$$\begin{aligned}
|t^m F_t(\vec{f})(z)| &\leq C t^{m+n\alpha} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}; \\
\left| \int_{B_l(z, t, r)} \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right| &\leq C t^{m-1+n(\alpha-\alpha_l)} r^{1+n\alpha_l} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}; \\
\left| \int_{(B(z, 8r))^m} \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right| &\leq C r^{m+n\alpha} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}.
\end{aligned}$$

For $x, z \in B$, $t \geq 8r$, we set

$$H_t(\vec{f})(x, z) =: \left| \int_{(B(z,t) \setminus B(z,8r))^m} \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right. \\ \left. - \int_{(B(x,t) \setminus B(x,8r))^m} \frac{\Omega(x - y_1, \dots, x - y_m)}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right|.$$

Consequently, we have

$$\begin{aligned} & |\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(z)| \\ & \leq C \left(\int_{8r}^\infty |F_t(\vec{f})(x) - F_t(\vec{f})(z)| \frac{dt}{t^{1-n\alpha}} \right)^{\frac{1}{2}} \prod_{j=1}^m \|f_j\|_{\text{CL}^{\alpha_j, p_j}}^{\frac{1}{2}} \\ & \leq C \prod_{j=1}^m \|f_j\|_{\text{CL}^{\alpha_j, p_j}} \left(\int_{8r}^\infty \left(\sum_{j=1}^m t^{m-1+n(\alpha-\alpha_j)} r^{1+n\alpha_j} + r^{m+n\alpha} \right) \frac{dt}{t^{m+1-n\alpha}} \right)^{\frac{1}{2}} \\ & \quad + C \prod_{j=1}^m \|f_j\|_{\text{CL}^{\alpha_j, p_j}}^{\frac{1}{2}} \left(\int_{8r}^\infty |H_t(\vec{f})(x, z)| \frac{dt}{t^{m+1-n\alpha}} \right)^{\frac{1}{2}} \\ & \leq Cr^{n\alpha} \prod_{j=1}^m \|f_j\|_{\text{CL}^{\alpha_j, p_j}} + C \prod_{j=1}^m \|f_j\|_{\text{CL}^{\alpha_j, p_j}}^{\frac{1}{2}} \left(\int_{8r}^\infty |H_t(\vec{f})(x, z)| \frac{dt}{t^{m+1-n\alpha}} \right)^{\frac{1}{2}}. \end{aligned}$$

Fixing x, z and for $t > 0$, we introduce some notations:

$$\begin{aligned} \mathcal{E}(x, t) &= \{y \in \mathbb{R}^n : 8r \leq |x - y| < t, 8r \leq |z - y| < t\}; \\ \mathcal{E}(z, t) &= \{y \in \mathbb{R}^n : 8r \leq |z - y| < t, 8r \leq |x - y| < t\}; \\ \Gamma(x, t) &= \{y \in \mathbb{R}^n : 8r \leq |x - y| < t, |z - y| \geq t\}; \\ \Gamma(z, t) &= \{y \in \mathbb{R}^n : 8r \leq |z - y| < t, |x - y| \geq t\}; \\ \Lambda(x, t) &= \{y \in \mathbb{R}^n : 8r \leq |x - y| < t, |z - y| < 8r\}; \\ \Lambda(z, t) &= \{y \in \mathbb{R}^n : 8r \leq |z - y| < t, |x - y| < 8r\}; \\ \vec{\Theta}(x, t) &= \Theta_1(x, t) \times \dots \times \Theta_m(x, t), \Theta_i(x, t) \in \{\mathcal{E}(x, t), \Gamma(x, t), \Lambda(x, t)\}; \\ \vec{\Theta}(z, t) &= \Theta_1(z, t) \times \dots \times \Theta_m(z, t), \Theta_i(z, t) \in \{\mathcal{E}(z, t), \Gamma(z, t), \Lambda(z, t)\}; \\ \mathcal{E}(x, t) &= \mathcal{E}(z, t) =: \mathcal{E}(t). \end{aligned}$$

For y , we denote

$$\begin{aligned} \mathcal{E}(x, y) &= \{t > 0 : 8r \leq |x - y| < t, 8r \leq |z - y| < t\}; \\ \mathcal{E}(z, y) &= \{t > 0 : 8r \leq |z - y| < t, 8r \leq |x - y| < t\}; \\ \Gamma(x, y) &= \{t > 0 : 8r \leq |x - y| < t, |z - y| \geq t\}; \\ \Gamma(z, y) &= \{t > 0 : 8r \leq |z - y| < t, |x - y| \geq t\}; \\ \Lambda(x, y) &= \{t > 0 : 8r \leq |x - y| < t, |z - y| < 8r\}; \end{aligned}$$

$$\Lambda(z, y) = \{t > 0 : 8r \leq |z - y| < t, |x - y| < 8r\};$$

$$\Theta_i(x, y_i) \in \{\Xi(x, y_i), \Gamma(x, y_i)\}, \quad i = 1, \dots, m;$$

$$\Theta_i(z, y_i) \in \{\Xi(z, y_i), \Gamma(z, y_i)\}, \quad i = 1, \dots, m.$$

It is easy to see that

$$B(x, t) \setminus B(x, 8r) = \Xi(x, t) \cup \Gamma(x, t) \cup \Lambda(x, t)$$

and

$$B(z, t) \setminus B(z, 8r) = \Xi(z, t) \cup \Gamma(z, t) \cup \Lambda(z, t).$$

We write

$$\begin{aligned} H_t(\vec{f})(x, z) &\leq \int_{(\Xi(t))^m} \left| \frac{\Omega(x - y_1, \dots, x - y_m)}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)}} - \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \right| \\ &\quad \times \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\ &\quad + \int_{(\Lambda(x, t))^m} \frac{|\Omega(x - y_1, \dots, x - y_m)|}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\ &\quad + \int_{(\Lambda(z, t))^m} \frac{|\Omega(z - y_1, \dots, z - y_m)|}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\ &\quad + \int_{\tilde{\Theta}(x, t), \exists \Theta_i(x, t) = \Gamma(x, t)} \frac{|\Omega(x - y_1, \dots, x - y_m)|}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\ &\quad + \int_{\tilde{\Theta}(z, t), \exists \Theta_i(z, t) = \Gamma(z, t)} \frac{|\Omega(z - y_1, \dots, z - y_m)|}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\ &\quad + \sum_{l=1}^{m-1} \int_{(\Xi(x, t))^l} \int_{(\Lambda(x, t))^{m-l}} \frac{|\Omega(x - y_1, \dots, x - y_m)|}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\ &\quad + \sum_{l=1}^{m-1} \int_{(\Xi(z, t))^l} \int_{(\Lambda(z, t))^{m-l}} \frac{|\Omega(z - y_1, \dots, z - y_m)|}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\ &=: \sum_{i=1}^5 H_{t,i}(\vec{f})(x, z) + \sum_{l=1}^{m-1} H_{t,6}^l(\vec{f})(x, z) + \sum_{l=1}^{m-1} H_{t,7}^l(\vec{f})(x, z). \end{aligned}$$

For $x, z \in B$ and Ω satisfying Lipschitz continuous condition, applying Lemma 2.7, we get

$$\begin{aligned} &\int_{8r}^{\infty} |H_{t,1}(\vec{f})(x, z)| \frac{dt}{t^{m+1-n\alpha}} \\ &\leq C \int_{((B(x, 8r))^c)^m} \frac{|x - z|^{\beta}}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)+\beta}} \end{aligned}$$

$$\begin{aligned}
& \times \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \int_{\frac{1}{m}(\sum_{j=1}^m |x-y_j|)}^t \frac{dt}{t^{m+1-n\alpha}} d\vec{y} \\
& \leq C \int_{((B(x,8r))^c)^m} \frac{r^\beta \prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x-y_j|)^{mn+\beta-n\alpha}} d\vec{y} \\
& \leq Cr^{2n\alpha} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}.
\end{aligned}$$

When $x, z \in B$, $8r \leq |x-y| < t$ and $|z-y| < 8r$, we have $|x-y| < |x-z| + |z-y| < 2r + 8r = 10r$. Then $8r < |x-y| < 10r$. Applying Lemma 2.6, we get

$$\begin{aligned}
|H_{t,2}(\vec{f})(x,z)| & \leq C \int_{(B(x,10r) \setminus B(x,8r))^m} \frac{\prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x-y_j|)^{m(n-1)}} d\vec{y} \\
& \leq \prod_{i=1}^m \int_{8r \leq |x-y_i| \leq 10r} \frac{|f_i(y_i) - (f_i)_B|}{|x-y_i|^{n-1}} dy_i \\
& \leq Cr^{m+n\alpha} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}},
\end{aligned}$$

which leads to

$$\int_{8r}^\infty |H_{t,2}(\vec{f})(x,z)| \frac{dt}{t^{m+1-n\alpha}} \leq Cr^{2n\alpha} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}.$$

Similarly,

$$\int_{8r}^\infty |H_{t,3}(\vec{f})(x,z)| \frac{dt}{t^{m+1-n\alpha}} \leq Cr^{2n\alpha} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}.$$

Now we estimate $H_{t,4}(\vec{f})$. For any $x, z \in B$,

$$\left| \bigcap_{j=1}^m \Theta_j(x, y_j) \right| \leq |\Theta_i(x, y_i)| = |\Gamma(x, y_i)| \leq ||z - y_i| - |x - y_i|| \leq |z - x| \leq 2r.$$

And for any $t \in \bigcap_{j=1}^m \Theta_j(x, y_j)$, $t > \frac{1}{m}(\sum_{j=1}^m |x - y_j|)$. Applying Lemma 2.7, we get

$$\begin{aligned}
& \int_{8r}^\infty |H_{t,4}(\vec{f})(x,z)| \frac{dt}{t^{m+1-n\alpha}} \\
& \leq C \int_{((B(x,8r))^c)^m} \frac{\prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x-y_j|)^{m(n-1)}} d\vec{y} \\
& \quad \times \int_{\bigcap_{j=1}^m \Theta_j(x, y_j)} \frac{dt}{t^{m+1-n\alpha}} \\
& \leq Cr \int_{((B(x,8r))^c)^m} \frac{\prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x-y_j|)^{mn+1-n\alpha}} d\vec{y} \\
& \leq Cr^{2n\alpha} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}.
\end{aligned}$$

Similarly,

$$\int_{8r}^{\infty} |H_{t,5}(\vec{f})(x,z)| \frac{dt}{t^{m+1-n\alpha}} \leq Cr^{2n\alpha} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}.$$

As for $H_{t,6}^l(\vec{f})$, suppose $y_1, \dots, y_l \in \Xi(t)$, $y_{l+1}, \dots, y_m \in \Lambda(x,t)$. Using Lemma 2.6, we have

$$\begin{aligned} H_{t,6}^l(\vec{f})(x,z) &\leq \prod_{j=1}^l \int_{8r \leq |x-y_j| < t} \frac{|f_j(y_j) - (f_j)_B|}{|x-y_j|^{n-1}} dy_j \\ &\quad \times \prod_{j=l+1}^m \int_{8r \leq |x-y_j| \leq 10r} \frac{|f_j(y_j) - (f_j)_B|}{|x-y_j|^{n-1}} dy_j \\ &\leq C \prod_{j=1}^l (tr^{n\alpha_j}) \|f_j\|_{CL^{\alpha_j, p_j}} \prod_{j=l+1}^m r^{1+n\alpha_j} \|f_j\|_{CL^{\alpha_j, p_j}} \\ &= Ct^l r^{m-l+n\alpha} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}. \end{aligned}$$

So we obtain

$$\begin{aligned} \int_{8r}^{\infty} |H_{t,6}^l(\vec{f})(x,z)| \frac{dt}{t^{m+1-n\alpha}} &\leq Cr^{m-l+n\alpha} \int_{8r}^{\infty} t^{l-m-1+n\alpha} dt \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}} \\ &\leq Cr^{2n\alpha} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}. \end{aligned}$$

Similarly,

$$\int_{8r}^{\infty} |H_{t,7}^l(\vec{f})(x,z)| \frac{dt}{t^{m+1-n\alpha}} \leq Cr^{2n\alpha} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}.$$

Combining the above estimates, we have

$$\left(\int_{8r}^{\infty} |H_t(\vec{f})(x,z)| \frac{dt}{t^{m+1-n\alpha}} \right)^{\frac{1}{2}} \leq Cr^{n\alpha} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}^{\frac{1}{2}}.$$

Consequently, we have

$$|\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(z)| \leq Cr^{n\alpha} \prod_{j=1}^m \|f_j\|_{CL^{\alpha_j, p_j}}.$$

Then the proof is complete. \square

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Competing interests

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Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details

¹School of Mathematics, University of Chinese Academy of Sciences, Beijing, 100049, China. ²School of Mathematics and Statistics, Xinyang Normal University, Xinyang, 464000, China.

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