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Majorization inequalities via Green functions and Fink's identity with applications to Shannon entropy

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Abstract

This paper is devoted to obtain generalized results related to majorization-type inequalities by using well-known Fink's identity and new types of Green functions, introduced by Mehmood et al. (J. Inequal. Appl. 2017:108, 2017). We give a generalized majorization theorem for the class of n-convex functions. We utilize the Csiszár f-divergence and generalized majorization-type inequalities in providing the corresponding generalizations. As an application, we present the obtained results in terms of Shannon entropy and Kullback–Leibler distance.

Keywords: Majorization inequailty; Fink's identity; New Green functions; n-convex functions; Čebyšev functional; Csiszár f-divergence; Kullback–Leibler divergence; Shannon entropy

1 Introduction

Majorization is a powerful and useful mathematical tool, which arises frequently in many different areas of research. Anyone seriously interested in majorization will eventually need to take a good look at Marshall et al.'s classic text [29], which gives a very nearly comprehensive coverage of both history and theory of majorization. It also contains a wealth of additional material including many interesting applications to diverse areas of mathematics. Alberti and Uhlmann [7] have also written a more specified monograph on majorization, which may be of interest to many readers. Moreover, Ando has written an excellent pair [8, 9] of survey papers on majorization, which provide a brief introduction to the subject and cover much of the field's development since the publication of Marshall et al.'s book [29].

In 2018, Latif et al. [27] studied generalized results related to majorization inequality by using Taylor's polynomial in combination with newly introduced Green functions. In the same year, Siddique et al. [34] gave generalized majorization results via Lidstone's polynomial and newly defined Green functions. The theory of majorization often appears in many fields of applications. Khan et al. [20] presented significant material on majorization along with its applications in information theory.

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In this paper, our main goal is obtaining generalized results about majorization by using new Green functions and Fink's identity. We further make connection of majorization with information theory and discuss our generalized majorization inequality in terms of divergences and entropies. The results we obtain in this paper are closely related to the contents of [1–5]. Moreover, some results related to the present topic can also be found in [19].

The following definition of majorization is from [33, p. 319].

Definition 1 For fixed $k \geq 2$, let $\mathbf{x} = (x_1,\ldots,x_k)$ and $\mathbf{y} = (y_1,\ldots,y_k)$ be two nonincreasing k-tuples of real numbers. If

$$\sum_{i=1}^{j} y_i \leq \sum_{i=1}^{j} x_i, \tag{1}$$

for $j = 1, 2, \ldots, k - 1$ and

$$\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i, \tag{2}$$

then we say that $\mathbf{x}$ majorizes $\mathbf{y}$. Symbolically, we write $\mathbf{x} \succ \mathbf{y}$. Hardy et al. [15] introduced this notation and notion of majorization.

The following theorem is called the classical majorization theorem and is given in [29, p. 11] (see also [33, p. 320]).

Theorem A *Consider two nonincreasing real k-tuples $\mathbf{x} = (x_1,\ldots,x_k)$ and $\mathbf{y} = (y_1,\ldots,y_k)$, where $x_i, y_i \in [\vartheta_1, \vartheta_2] \subset \mathbb{R}$ for $i = 1,\ldots,k$. Then $\mathbf{x}$ majorizes $\mathbf{y}$ if and only if following inequality holds:*

$$\sum_{i=1}^{k} f(y_i) \leq \sum_{i=1}^{k} f(x_i), \tag{3}$$

where $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ is a continuous convex function.

The following theorem is regarded as a generalization of the majorization theorem given by Fuchs [14] (see also [33, p. 323] and [29, p. 580]).

Theorem B *Consider two nonincreasing real k-tuples $\mathbf{x} = (x_1,\ldots,x_k)$ and $\mathbf{y} = (y_1,\ldots,y_k)$, where $x_i, y_i \in [\vartheta_1, \vartheta_2] \subset \mathbb{R}$ for $i = 1,\ldots,k$. Let $\mathbf{p} = (p_1,\ldots,p_k)$ be a real k-tuple such that*

$$\sum_{i=1}^{j} p_i y_i \leq \sum_{i=1}^{j} p_i x_i, \tag{4}$$

for $j = 1, 2, \ldots, k - 1$ and

$$\sum_{i=1}^{k} p_i y_i = \sum_{i=1}^{k} p_i x_i. \tag{5}$$

Then

$$\sum_{i=1}^{k} p_i f(y_i) \leq \sum_{i=1}^{k} p_i f(x_i), \tag{6}$$

where $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ is a continuous convex function.

The following theorem represents an integral form of Theorem B and is in fact a simple consequence of Theorem 1 given in [31] (see also [33, p. 328].

Theorem C *Let $\phi, \psi : [a, b] \to [\vartheta_1, \vartheta_2]$ be two continuous decreasing functions, and let $p : [a, b] \to \mathbb{R}$ be any continuous function. If*

$$\int_a^\lambda p(w)\psi(w)\,dw \leq \int_a^\lambda p(w)\phi(w)\,dw \quad \text{for every } \lambda \in [a, b] \tag{7}$$

and

$$\int_a^b p(w)\psi(w)\,dw = \int_a^b p(w)\phi(w)\,dw, \tag{8}$$

then

$$\int_a^b p(w)f\big(\psi(w)\big)\,dw \leq \int_a^b p(w)f\big(\phi(w)\big)\,dw, \tag{9}$$

where $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ is a continuous convex function.

For other forms of integral version and generalization of the majorization theorem, see [29, p. 583], [10, 21, 23–26, 28]. In this paper, we present the results for decreasing functions ϕ and ψ that satisfy the conditions of Theorem C, but those results also hold for increasing ϕ and ψ satisfying the inequality

$$\int_\lambda^b p(w)\psi(w)\,dw \leq \int_\lambda^b p(w)\phi(w)\,dw \quad \text{for every } \lambda \in [a, b] \tag{10}$$

and condition (8). For instance, see the example in [29, p. 584].

In 1991, Fink [13] proved a very important result, known as Fink's identity, given as follows.

Theorem D *Let $[\vartheta_1, \vartheta_2] \subset \mathbb{R}$ and $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}, n \geq 1$, be such that $f^{(n-1)}$ is absolutely continuous. Then*

$$
\begin{aligned}
f(x) = {} & \frac{n}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} f(t)\,dt \\
& - \sum_{s=1}^{n-1} \left(\frac{n-s}{s!}\right) \left(\frac{f^{(s-1)}(\vartheta_1)(x-\vartheta_1)^s - f^{(s-1)}(\vartheta_2)(x-\vartheta_2)^s}{\vartheta_2 - \vartheta_1}\right) \\
& + \frac{1}{(n-1)!(\vartheta_2 - \vartheta_1)} \int_{\vartheta_1}^{\vartheta_2} (x-t)^{n-1} s^{[\vartheta_1, \vartheta_2]}(t, x) f^{(n)}(t)\,dt,
\end{aligned} \tag{11}
$$

where

$$s^{[\vartheta_1,\vartheta_2]}(t,x) = \begin{cases} (t - \vartheta_1), & \vartheta_1 \le t \le x \le \vartheta_2, \\ (t - \vartheta_2), & \vartheta_1 \le x < t \le \vartheta_2. \end{cases} \tag{12}$$

As stated in [30], the complete reference about Abel–Gontscharoff polynomial and theorem for "two-point right focal problem" is given in [6].

Remark 1 The Abel–Gontscharoff polynomial, as a special choice of the "two-point right focal" interpolating polynomial for $n = 2$, is

$$f(u) = f(\vartheta_1) + (u - \vartheta_1)f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2} G_{\Omega,2}(u,v)f''(v)\,dv, \tag{13}$$

where $G_{\Omega,2}(u,v) : [\vartheta_1,\vartheta_2] \times [\vartheta_1,\vartheta_2] \to \mathbb{R}$ is the Green function for the "two-point right focal problem" given as

$$G_1(u,v) = G_{\Omega,2}(u,v) = \begin{cases} (\vartheta_1 - v), & \vartheta_1 \le v \le u, \\ (\vartheta_1 - u), & u \le v \le \vartheta_2. \end{cases} \tag{14}$$

Let us define some new types of Green functions $G_l : [\vartheta_1,\vartheta_2] \times [\vartheta_1,\vartheta_2] \to \mathbb{R}$, where $[\vartheta_1,\vartheta_2] \subset \mathbb{R}$ and $l = 2,3,4$, given by Mehmood et al. (see [30]), which are continuous and convex, by keeping in view the Abel–Gontscharoff Green's function for the "two-point right focal problem":

$$G_2(u,v) = \begin{cases} (u - \vartheta_2), & \vartheta_1 \le v \le u, \\ (v - \vartheta_2), & u \le v \le \vartheta_2; \end{cases} \tag{15}$$

$$G_3(u,v) = \begin{cases} (u - \vartheta_1), & \vartheta_1 \le v \le u, \\ (v - \vartheta_1), & u \le v \le \vartheta_2; \end{cases} \tag{16}$$

$$G_4(u,v) = \begin{cases} (\vartheta_2 - v), & \vartheta_1 \le v \le u, \\ (\vartheta_2 - u), & u \le v \le \vartheta_2. \end{cases} \tag{17}$$

Next, we evaluate new generalizations of majorization inequality with the help of the following lemma given by Mehmood et al. [30].

Lemma 1 *Let $f : [\vartheta_1,\vartheta_2] \to \mathbb{R}$ be a twice differentiable function, and let G_l ($l = 1,2,3,4$) be Green functions given in* (14)–(17). *Then along with identity* (13), *we have the following identities*:

$$f(u) = f(\vartheta_2) + (u - \vartheta_2)f'(\vartheta_1) + \int_{\vartheta_1}^{\vartheta_2} G_2(u,v)f''(v)\,dv, \tag{18}$$

$$f(u) = f(\vartheta_2) - (\vartheta_2 - \vartheta_1)f'(\vartheta_2) + (u - \vartheta_1)f'(\vartheta_1) + \int_{\vartheta_1}^{\vartheta_2} G_3(u,v)f''(v)\,dv, \tag{19}$$

$$f(u) = f(\vartheta_1) + (\vartheta_2 - \vartheta_1)f'(\vartheta_1) - (\vartheta_2 - u)f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2} G_4(u,v)f''(v)\,dv. \tag{20}$$

This paper is arranged as follows. Section 2.1 contains some important results obtained by generalizing majorization-type identities by using Fink's identity and new Green functions. We also obtain a generalized majorization theorem for the class of n-convex functions. In Sect. 2.2, we use the Csiszár f-divergence and generalized majorization-type inequalities to obtain new generalized results. Further, we discuss our generalized results in terms of Shannon entropy and Kullback–Leibler distance.

2 Main results

This section contains results about the generalization of majorization inequality. For simplicity, we define some notations used in this paper. Let $\mathbf{x} = (x_1, \dots, x_k), \mathbf{y} = (y_1, \dots, y_k)$, and $\mathbf{p} = (p_1, \dots, p_k)$ be such that $x_i, y_i \in [\vartheta_1, \vartheta_2] \subset \mathbb{R}$ and $p_i \in \mathbb{R}$ for $i = 1, 2, \dots, k$. We denote the majorization difference for a function f as

$$\mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, f(\cdot)\big) := \sum_{i=1}^{k} p_i f(x_i) - \sum_{i=1}^{k} p_i f(y_i). \tag{21}$$

Let $p : [a, b] \to \mathbb{R}$ and $\phi, \psi : [a, b] \to [\vartheta_1, \vartheta_2]$ be continuous functions. We define the integral majorization difference for a function f as

$$\widetilde{\mathbb{D}}\big(p, \phi, \psi, f(\cdot)\big) := \int_a^b p(w)\big(f\big(\phi(w)\big) - f\big(\psi(w)\big)\big)\, dw. \tag{22}$$

2.1 Majorized results using Fink's identity

First, we present two equivalent statements of majorization inequality between newly defined Green functions and continuous convex functions.

Theorem 1 *Let $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be a continuous convex function, and let $\boldsymbol{x} = (x_1, \dots, x_k)$, $\boldsymbol{y} = (y_1, \dots, y_k)$, and $\boldsymbol{p} = (p_1, \dots, p_k)$ be k-tuples such that $x_i, y_i \in [\vartheta_1, \vartheta_2]$ and $p_i \in \mathbb{R}$ for $i = 1, 2, \dots, k$ and condition (5) is satisfied. Let G_l ($l = 1, 2, 3, 4$) be the Green functions given in (14)–(17). Then*

$$\sum_{i=1}^{k} p_i f(y_i) \leq \sum_{i=1}^{k} p_i f(x_i) \tag{23}$$

if and only if

$$\sum_{i=1}^{k} p_i G_l(y_i, v) \leq \sum_{i=1}^{k} p_i G_l(x_i, v) \tag{24}$$

for all $v \in [\vartheta_1, \vartheta_2]$.

Proof Here we only give the proof for $l = 4$, as the idea of the proof is the same for each $l = 1, 2, 3, 4$.

Let inequality (23) hold. Then $G_4(\cdot, v)$ ($v \in [\vartheta_1, \vartheta_2]$), being continuous and convex, satisfies inequality (23), that is, inequality (24) holds for $l = 4$.

Conversely, let f be a convex function such that $f \in C^2([\vartheta_1, \vartheta_2])$ and (24) holds. Then by (20) we can write

$$f(x_i) = f(\vartheta_1) + (\vartheta_2 - \vartheta_1)f'(\vartheta_1) - (\vartheta_2 - x_i)f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2} G_4(x_i, v)f''(v)\, dv, \tag{25}$$

$$f(y_i) = f(\vartheta_1) + (\vartheta_2 - \vartheta_1)f'(\vartheta_1) - (\vartheta_2 - y_i)f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2} G_4(y_i, v)f''(v)\, dv. \tag{26}$$

Using (25) and (26) along with (5), we have

$$\sum_{i=1}^{k} p_i f(x_i) - \sum_{i=1}^{k} p_i f(y_i) = \int_{\vartheta_1}^{\vartheta_2} \left[\sum_{i=1}^{k} p_i G_4(x_i, v) - \sum_{i=1}^{k} p_i G_4(y_i, v) \right] f''(v)\, dv. \tag{27}$$

Now the convexity of f implies that $f''(v) \geq 0$ for all $v \in [\vartheta_1, \vartheta_2]$. Hence using (24) in (27), for $l = 4$, we get (23).

Note that in this proof the requirement of the existence of the second derivative of f is not necessary ([33], p. 172). We can directly eliminate this condition because it is possible to approximate uniformly continuous convex functions by convex polynomials.   □

Now we present some useful results obtained by generalizing majorization-type identities by using Fink's identity.

Theorem 2 *Let $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for $n \geq 3$. Let $\boldsymbol{x} = (x_1, \ldots, x_k)$, $\boldsymbol{y} = (y_1, \ldots, y_k)$, and $\boldsymbol{p} = (p_1, \ldots, p_k)$ with $x_i, y_i \in [\vartheta_1, \vartheta_2] \subset \mathbb{R}$ and $p_i \in \mathbb{R}$ for $i = 1, 2, \ldots, k$. Also, let $s^{[\vartheta_1, \vartheta_2]}(t, x)$ be as defined in (12), and let G_l ($l = 1, 2, 3, 4$) be the Green functions given in (14)–(17). Then*

$$\mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, f(\cdot)\big)$$

$$= f'(\xi_l) \sum_{i=1}^{k} p_i(x_i - y_i) + \sum_{s=2}^{n-1} \left(\frac{n-s}{(s-2)!(\vartheta_2 - \vartheta_1)} \right) \int_{\vartheta_1}^{\vartheta_2} \mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, G_l(\cdot, v)\big)$$

$$\times \big(f^{(s-1)}(\vartheta_2)(v - \vartheta_2)^{s-2} - f^{(s-1)}(\vartheta_1)(v - \vartheta_1)^{s-2} \big)\, dv + \frac{1}{(n-3)!(\vartheta_2 - \vartheta_1)}$$

$$\times \int_{\vartheta_1}^{\vartheta_2} f^{(n)}(t) \left(\int_{\vartheta_1}^{\vartheta_2} \big(\mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, G_l(\cdot, v)\big)\big)(v - t)^{n-3} s^{[\vartheta_1, \vartheta_2]}(t, v)\, dv \right) dt, \tag{28}$$

where $\xi_1, \xi_4 = \vartheta_2$ and $\xi_2, \xi_3 = \vartheta_1$.

Proof Here we only give the proof for $l = 4$, as the idea of the proof is the same for each $l = 1, 2, 3, 4$.

Using (20) in the majorization difference (21), we get

$$\mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, f(\cdot)\big) = \sum_{i=1}^{k} p_i(x_i - y_i)f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2} \mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, G_4(\cdot, v)\big)f''(v)\, dv. \tag{29}$$

From Fink's identity (11) we have

$$
f''(x) = \sum_{s=2}^{n-1} \left(\frac{n-s}{(s-2)!} \right) \left(\frac{f^{(s-1)}(\vartheta_2)(x-\vartheta_2)^{s-2} - f^{(s-1)}(\vartheta_1)(x-\vartheta_1)^{s-2}}{\vartheta_2 - \vartheta_1} \right)
$$
$$
+ \frac{1}{(n-3)!(\vartheta_2 - \vartheta_1)} \int_{\vartheta_1}^{\vartheta_2} (x-t)^{n-3} s^{[\vartheta_1,\vartheta_2]}(t,x) f^{(n)}(t)\, dt. \tag{30}
$$

Using (30) in (29), we get

$$
\mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, f(\cdot)\big)
$$
$$
= \sum_{i=1}^{k} p_i (x_i - y_i) f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2} \mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, G_4(\cdot, v)\big) \sum_{s=2}^{n-1} \left(\frac{n-s}{(s-2)!} \right)
$$
$$
\times \left(\frac{f^{(s-1)}(\vartheta_2)(v-\vartheta_2)^{s-2} - f^{(s-1)}(\vartheta_1)(v-\vartheta_1)^{s-2}}{\vartheta_2 - \vartheta_1} \right) dv + \frac{1}{(n-3)!(\vartheta_2 - \vartheta_1)}
$$
$$
\times \int_{\vartheta_1}^{\vartheta_2} \mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, G_4(\cdot, v)\big) \left(\int_{\vartheta_1}^{\vartheta_2} (v-t)^{n-3} s^{[\vartheta_1,\vartheta_2]}(t,v) f^{(n)}(t)\, dt \right) dv. \tag{31}
$$

Rearranging (31) and applying Fubini's theorem in the last term, we obtain (28) for $l = 4$. $\qquad\square$

An integral version of Theorem 2 is as follows.

Theorem 3 *Let $[\vartheta_1, \vartheta_2] \subset \mathbb{R}$, and let $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for $n \geq 3$. Let $p : [a,b] \to \mathbb{R}$ and $\phi, \psi : [a,b] \to [\vartheta_1, \vartheta_2]$ be continuous functions. Also, let $s^{[\vartheta_1,\vartheta_2]}(t,x)$ be as defined in (12), and let G_l ($l = 1, 2, 3, 4$) be the Green functions given in (14)–(17). Then*

$$
\widetilde{\mathbb{D}}\big(p, \phi, \psi, f(\cdot)\big)
$$
$$
= f'(\xi_l) \int_a^b p(w)\big(\phi(w) - \psi(w)\big)\, dw + \sum_{s=2}^{n-1} \left(\frac{n-s}{(s-2)!(\vartheta_2 - \vartheta_1)} \right)
$$
$$
\times \int_{\vartheta_1}^{\vartheta_2} \widetilde{\mathbb{D}}\big(p, \phi, \psi, G_l(\cdot, v)\big)\big(f^{(s-1)}(\vartheta_2)(v-\vartheta_2)^{s-2} - f^{(s-1)}(\vartheta_1)(v-\vartheta_1)^{s-2}\big)\, dv
$$
$$
+ \frac{1}{(n-3)!(\vartheta_2 - \vartheta_1)}
$$
$$
\times \int_{\vartheta_1}^{\vartheta_2} f^{(n)}(t) \left(\int_{\vartheta_1}^{\vartheta_2} \widetilde{\mathbb{D}}\big(p, \phi, \psi, G_l(\cdot, v)\big)(v-t)^{n-3} s^{[\vartheta_1,\vartheta_2]}(t,v)\, dv \right) dt, \tag{32}
$$

where $\xi_1, \xi_4 = \vartheta_2$ and $\xi_2, \xi_3 = \vartheta_1$.

Proof Using (20) in the integral majorization difference (22) and performing similar steps as in the proof of Theorem 2, we get the desired result for $l = 4$. Similarly, the result can be proved for $l = 1, 2, 3$. $\qquad\square$

Theorem 4 *Let all the assumptions of Theorem* 2 *hold, and let f be an n-convex function. If*

$$\int_{\vartheta_1}^{\vartheta_2} \mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, G_l(\cdot, v)\big)(v - t)^{n-3} s^{[\vartheta_1, \vartheta_2]}(t, v)\, dv \geq 0, \tag{33}$$

then

$$\mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, f(\cdot)\big) \geq f'(\xi_l) \sum_{i=1}^{k} p_i(x_i - y_i) + \sum_{s=2}^{n-1} \left(\frac{n-s}{(s-2)!(\vartheta_2 - \vartheta_1)} \right) \int_{\vartheta_1}^{\vartheta_2} \mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, G_l(\cdot, v)\big)$$

$$\times \big(f^{(s-1)}(\vartheta_2)(v - \vartheta_2)^{s-2} - f^{(s-1)}(\vartheta_1)(v - \vartheta_1)^{s-2}\big)\, dv, \tag{34}$$

where $\xi_1, \xi_4 = \vartheta_2$ *and* $\xi_2, \xi_3 = \vartheta_1$.

Moreover, (34) *holds in the reverse direction if inequality* (33) *is reversed.*

Proof Since f is an n-convex function, we have $f^{(n)}(x) \geq 0$ for all $x \in [\vartheta_1, \vartheta_2]$. Therefore using (33) in (28), we have (34). Moreover, if inequality (33) is reversed, then using (28), we obtain (34) in the reverse direction. □

Theorem 5 *Let all the assumptions of Theorem* 3 *be satisfied with n-convex f. If*

$$\int_{\vartheta_1}^{\vartheta_2} \widetilde{\mathbb{D}}\big(p, \phi, \psi, G_l(\cdot, v)\big)(v - t)^{n-3} s^{[\vartheta_1, \vartheta_2]}(t, v)\, dv \geq 0, \tag{35}$$

then

$$\widetilde{\mathbb{D}}\big(p, \phi, \psi, f(\cdot)\big)$$

$$\geq f'(\xi_l) \int_a^b p(w)\big(\phi(w) - \psi(w)\big)\, dw + \sum_{s=2}^{n-1} \left(\frac{n-s}{(s-2)!(\vartheta_2 - \vartheta_1)} \right)$$

$$\times \int_{\vartheta_1}^{\vartheta_2} \widetilde{\mathbb{D}}\big(p, \phi, \psi, G_l(\cdot, v)\big)\big(f^{(s-1)}(\vartheta_2)(v - \vartheta_2)^{s-2} - f^{(s-1)}(\vartheta_1)(v - \vartheta_1)^{s-2}\big)\, dv, \tag{36}$$

where $\xi_1, \xi_4 = \vartheta_2$ *and* $\xi_2, \xi_3 = \vartheta_1$.

Moreover, (36) *holds in the reverse direction if inequality* (35) *is reversed.*

Proof Using (35) in (32) and performing similar steps as in the proof of Theorem 4, we get required result. □

A refinement of the weighted majorization-type inequality for two nonincreasing real k-tuples is given in the following corollary.

Corollary 1 *Let all the assumptions of Theorem* 2 *be satisfied. Let* $\mathbf{x} = (x_1, \ldots, x_k)$ *and* $\mathbf{y} = (y_1, \ldots, y_k)$ *be two nonincreasing real k-tuples such that* (4) *and* (5) *hold.*

(i) *Let $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ is an n-convex function with even n > 3. Then*

$$
\mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, f(\cdot)\big) \geq \sum_{s=2}^{n-1} \left(\frac{n-s}{(s-2)!(\vartheta_2 - \vartheta_1)} \right) \int_{\vartheta_1}^{\vartheta_2} \mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, G_l(\cdot, \nu)\big)
$$
$$
\times \big(f^{(s-1)}(\vartheta_2)(\nu - \vartheta_2)^{s-2} - f^{(s-1)}(\vartheta_1)(\nu - \vartheta_1)^{s-2} \big)\, d\nu. \tag{37}
$$

(ii) *If inequality* (37) *holds and the function $\varphi : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ defined by*

$$
\varphi(\cdot) = \sum_{s=2}^{n-1} \left(\frac{n-s}{(s-2)!(\vartheta_2 - \vartheta_1)} \right) \int_{\vartheta_1}^{\vartheta_2} G_l(\cdot, \nu)
$$
$$
\times \big(f^{(s-1)}(\vartheta_2)(\nu - \vartheta_2)^{s-2} - f^{(s-1)}(\vartheta_1)(\nu - \vartheta_1)^{s-2} \big)\, d\nu \tag{38}
$$

is convex, then the right-hand side of (37) *is nonnegative, that is,* (6) *is satisfied.*

Proof (i): Since G_l is continuous and convex for $l = 1, 2, 3, 4$, by Theorem B we can write

$$
\mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, G_l(\cdot, \nu)\big) \geq 0. \tag{39}
$$

Note that for $\vartheta_1 \leq \nu \leq t$, if $n > 3$ is even, then we have

$$
\int_{\vartheta_1}^{t} \mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, G_l(\cdot, \nu)\big)(\nu - t)^{n-3} s^{[\vartheta_1, \vartheta_2]}(t, \nu)\, d\nu \geq 0, \tag{40}
$$

and for $t \leq \nu \leq \vartheta_2$ and even $n > 3$, we have

$$
\int_{t}^{\vartheta_2} \mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, G_l(\cdot, \nu)\big)(\nu - t)^{n-3} s^{[\vartheta_1, \vartheta_2]}(t, \nu)\, d\nu \geq 0. \tag{41}
$$

Combining (40) and (41), we get (33) for even $n > 3$. Also, f is n-convex for even $n > 3$, so using Theorem 4 and (5), we have (37).

(ii): Clearly, we can write (37) in the following form:

$$
\mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, f(\cdot)\big) \geq \mathbb{D}\big(\mathbf{p}, \mathbf{x}, \mathbf{y}, \varphi(\cdot)\big). \tag{42}
$$

Now the use of the convex function φ in (6) leads us to the nonnegativity of the right-hand side of (42), which gives the required result. □

A refinement of the integral majorization-type inequality for two decreasing funtions ϕ and ψ is given in the following corollary, which is in fact the integral version of Corollary 1.

Corollary 2 *Let all the assumptions of Theorem 3 hold. Let ϕ and ψ be two decreasing functions such that* (7) *and* (8) *are satisfied.*

(i) *If $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ is an n-convex function, where n > 3 is even, then*

$$
\widetilde{\mathbb{D}}\big(p, \phi, \psi, f(\cdot)\big) \geq \sum_{s=2}^{n-1} \left(\frac{n-s}{(s-2)!(\vartheta_2 - \vartheta_1)} \right) \int_{\vartheta_1}^{\vartheta_2} \widetilde{\mathbb{D}}\big(p, \phi, \psi, G_l(\cdot, \nu)\big)
$$
$$
\times \big(f^{(s-1)}(\vartheta_2)(\nu - \vartheta_2)^{s-2} - f^{(s-1)}(\vartheta_1)(\nu - \vartheta_1)^{s-2} \big)\, d\nu. \tag{43}
$$

(ii) *If inequality* (43) *holds and the function $\widetilde{\varphi}$ defined by*

$$
\widetilde{\varphi}(\cdot) = \sum_{s=2}^{n-1} \left(\frac{n-s}{(s-2)!(\vartheta_2 - \vartheta_1)} \right) \int_{\vartheta_1}^{\vartheta_2} G_l(\cdot, v)
$$
$$
\times \left(f^{(s-1)}(\vartheta_2)(v - \vartheta_2)^{s-2} - f^{(s-1)}(\vartheta_1)(v - \vartheta_1)^{s-2} \right) dv \tag{44}
$$

is convex, then the right-hand side of (43) *is nonnegative, that is,* (9) *is satisfied.*

Proof Similar to the proof of Corollary 1. □

Remarks 1

  (i) We can give upper bounds like Grüss- and Ostrowski-type inequalities for our generalized identities. We can also obtain Lagrange- and Cauchy-type mean value theorems by using linear functionals deduced from our generalized results (see, e.g., [27, 32, 34]).

  (ii) We can use an elegant method introduced by Jakšetić and Pečarć [17, 18] (see also [22, 30]) to obtain n-exponential convexity, exponential convexity, and log-convexity with the help of linear functionals deduced from our generalized results on a given family with the same property for both discrete and integral cases. For more detail, see [32].

2.2 Csiszár f-divergence for majorization

In this section, we investigate generalized majorization-type inequality (37) in terms of divergences and entropies. We use Csiszár f-divergence and generalized majorization-type inequalities to obtain new generalized results. Moreover, we also discuss results related to Shannon entropy and Kullback–Leibler (K-L) distance.

The notion of f-divergence is introduced by Csiszár in [11]. For details, see [12].

Definition 2 Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex function. If $\mathbf{r} = (r_1, \ldots, r_k)$ and $\mathbf{w} = (w_1, \ldots, w_k)$ are two positive probability distributions, then the f-divergence functional is

$$
I_f(\mathbf{r}, \mathbf{w}) := \sum_{i=1}^{k} w_i f\left(\frac{r_i}{w_i} \right).
$$

Note that in the f-divergence functional, nonnegative probability distributions can also be used by defining

$$
f(0) := \lim_{t \to 0^+} f(t); \qquad 0f\left(\frac{0}{0} \right) := 0; \qquad 0f\left(\frac{a}{t} \right) := 0, \quad a > 0.
$$

Horváth et al. [16] considered the following functionality based on the previous definition.

Definition 3 Let $J \subset \mathbb{R}$ be an interval, and let $f : J \to \mathbb{R}$ be an n-convex function. Let $\mathbf{r} = (r_1, \ldots, r_k)$ be a real k-tuple, and let $\mathbf{w} = (w_1, \ldots, w_k)$ be a positive real k-tuple such that $\frac{r_i}{w_i} \in J, i = 1, 2, \ldots, k$. Then

$$
\widetilde{I}_f(\mathbf{r}, \mathbf{w}) := \sum_{i=1}^{k} w_i f\left(\frac{r_i}{w_i} \right).
$$

Let $\mathbf{r} = (r_1, \ldots, r_k)$ and $\mathbf{w} = (w_1, \ldots, w_k)$ be two k-tuples. We use the following notation:

$$\frac{\mathbf{r}}{\mathbf{w}} := \left(\frac{r_1}{w_1}, \frac{r_2}{w_2}, \ldots, \frac{r_k}{w_k} \right) \quad \text{and} \quad \widetilde{I}_{G_l}(\mathbf{r}, \mathbf{w}, v) := \sum_{i=1}^{k} w_i G_l\left(\frac{r_i}{w_i}, v \right).$$

The following theorem connects the generalized majorization-type inequality given in Corollary 1 and the Csiszár f-divergence.

Theorem 6 *Let $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for $n > 3$. Also, let $s^{[\vartheta_1, \vartheta_2]}(t, x)$ be as defined in (12), and let G_l ($l = 1, 2, 3, 4$) be the Green functions defined in (14)–(17). Also, let $\mathbf{q} = (q_1, \ldots, q_k), \mathbf{r} = (r_1, \ldots, r_k)$, and $\mathbf{w} = (w_1, \ldots, w_k)$ with $q_i, r_i \in \mathbb{R}$ and $w_i \in \mathbb{R}_+$ for $i = 1, 2, \ldots, k$. Let*

$$\sum_{i=1}^{j} r_i \leq \sum_{i=1}^{j} q_i, \tag{45}$$

for $j = 1, 2, \ldots, k - 1$ and

$$\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} q_i, \tag{46}$$

with $\frac{q_i}{w_i}, \frac{r_i}{w_i} \in [\vartheta_1, \vartheta_2]$ ($i = 1, 2, \ldots, k$). If $\frac{\mathbf{q}}{\mathbf{w}}$ and $\frac{\mathbf{r}}{\mathbf{w}}$ are decreasing and f is an n-convex function for even $n > 3$, then

$$\widetilde{I}_f(\mathbf{q}, \mathbf{w}) \geq \widetilde{I}_f(\mathbf{r}, \mathbf{w}) + \sum_{s=2}^{n-1} \left(\frac{n - s}{(s-2)!(\vartheta_2 - \vartheta_1)} \right) \int_{\vartheta_1}^{\vartheta_2} \left(\widetilde{I}_{G_l}(\mathbf{q}, \mathbf{w}, v) - \widetilde{I}_{G_l}(\mathbf{r}, \mathbf{w}, v) \right)$$

$$\times \left(f^{(s-1)}(\vartheta_2)(v - \vartheta_2)^{s-2} - f^{(s-1)}(\vartheta_1)(v - \vartheta_1)^{s-2} \right) dv. \tag{47}$$

Proof Take $x_i = \frac{q_i}{w_i}, y_i = \frac{r_i}{w_i}$, and $p_i = w_i > 0 (i = 1, 2, \ldots, k)$. Then conditions (45) and (46) imply conditions (4) and (5). So using these substitutions in (37), we get (47). □

Theorem 7 *Let $g : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be a function. If for $f(x) := xg(x), x \in [\vartheta_1, \vartheta_2]$, all conditions of Theorem 6 hold, then*

$$\widehat{I}_g(\mathbf{q}, \mathbf{w}) := \sum_{i=1}^{k} q_i g\left(\frac{q_i}{w_i} \right) \geq \widehat{I}_g(\mathbf{r}, \mathbf{w}) + \sum_{s=2}^{n-1} \left(\frac{n - s}{(s-2)!(\vartheta_2 - \vartheta_1)} \right)$$

$$\times \int_{\vartheta_1}^{\vartheta_2} \left(\widetilde{I}_{G_l}(\mathbf{q}, \mathbf{w}, v) - \widetilde{I}_{G_l}(\mathbf{r}, \mathbf{w}, v) \right) \left(\frac{(xg)^{(s-1)}(\vartheta_2)}{(v - \vartheta_2)^{2-s}} - \frac{(xg)^{(s-1)}(\vartheta_1)}{(v - \vartheta_1)^{2-s}} \right) dv. \tag{48}$$

Proof Following the proof of Theorem 6 for $f(x) := xg(x)$, we get (48). □

The notion of entropic measure of disorder and the theory of majorization are closely related. Next, we present two special cases for majorization relations with connection to entropic inequalities.

In the first case, we discuss generalized majorization-type inequality with the entropy of a discrete probability distribution.

Definition 4 Let $\mathbf{r} = (r_1,\ldots,r_k)$ be a positive probability distribution. Then the Shannon entropy of $\mathbf{r}$ is defined as

$$H(\mathbf{r}) := -\sum_{i=1}^{k} r_i \log r_i.$$

Note that the definition does not provide any problem for zero probability case, because $\lim_{x\to 0} x \log x = 0$.

Corollary 3 *Let $\boldsymbol{q} = (q_1,\ldots,q_k)$ and $\boldsymbol{r} = (r_1,\ldots,r_k)$ be positive real k-tuples, and $\boldsymbol{w} = (w_1,\ldots,w_k)$ be a positive probability distribution such that conditions* (45) *and* (46) *hold with $\frac{q_i}{w_i}, \frac{r_i}{w_i} \in [\vartheta_1,\vartheta_2] (i = 1,2,\ldots,k)$. If* log *has base b greater than* 1 *and $\frac{q}{w}$ and $\frac{r}{w}$ are decreasing, then for the Shannon entropy of $\boldsymbol{w}$, we have the following estimate*:

$$H(\boldsymbol{w}) \leq \sum_{i=1}^{k} w_i \log\left(\frac{r_i}{w_i}\right) - \sum_{s=2}^{n-1}\left(\frac{n-s}{(s-2)!(\vartheta_2 - \vartheta_1)}\right)\int_{\vartheta_1}^{\vartheta_2}\left(\widetilde{I}_{G_l}(\boldsymbol{q},\boldsymbol{w},v) - \widetilde{I}_{G_l}(\boldsymbol{r},\boldsymbol{w},v)\right)$$

$$\times \left(\frac{(-1)^{s-1}(s-2)!}{\vartheta_2^{s-1}\ln b}(v - \vartheta_2)^{s-2} - \frac{(-1)^{s-1}(s-2)!}{\vartheta_1^{s-1}\ln b}(v - \vartheta_1)^{s-2}\right)dv. \qquad (49)$$

If log *has base b between* 0 *and* 1, *then inequality* (49) *is reversed.*

Proof Take $f(x) := -\log x$, which is an n-convex function for even $n > 3$ and $q_i = 1$ ($i = 1,2,\ldots,k$). Then using Theorem 6, we get (49). Moreover, for odd $n > 3$, inequality (49) is reversed. □

Corollary 4 *Let $\boldsymbol{q} = (q_1,\ldots,q_k)$ and $\boldsymbol{r} = (r_1,\ldots,r_k)$ be positive probability distributions such that conditions* (45) *and* (46) *hold with $q_i, r_i \in [\vartheta_1,\vartheta_2]$ ($i = 1,2,\ldots,k$). If* log *has base b greater than* 1 *and $\boldsymbol{q}$ and $\boldsymbol{r}$ are decreasing, then the relation between the Shannon entropies of $\boldsymbol{q}$ and $\boldsymbol{r}$ is given by the following estimate*:

$$H(\boldsymbol{q}) \leq H(\boldsymbol{r}) - \sum_{s=2}^{n-1}\left(\frac{n-s}{(s-2)!(\vartheta_2 - \vartheta_1)}\right)\int_{\vartheta_1}^{\vartheta_2}\left(\widetilde{I}_{G_l}(\boldsymbol{q},\boldsymbol{w},v) - \widetilde{I}_{G_l}(\boldsymbol{r},\boldsymbol{w},v)\right)$$

$$\times \left((x\log x)^{(s-1)}(\vartheta_2)(v - \vartheta_2)^{s-2} - (x\log x)^{(s-1)}(\vartheta_1)(v - \vartheta_1)^{s-2}\right)dv, \qquad (50)$$

where for $d = 1,2$, $(x\log x)'(\vartheta_d) = \frac{1}{\ln b}(1 + \ln \vartheta_d)$ and $(x\log x)^{(s-1)}(\vartheta_d) = \frac{(-1)^{s-1}(s-3)!}{\vartheta_d^{s-2}\ln b}$, $s \geq 3$. If log *has base b between* 0 *and* 1, *then inequality* (50) *is reversed.*

Proof Take $g(x) := \log x$, so that $xg(x) := x\log x$ is an n-convex function for even $n > 3$ and $w_i = 1$ ($i = 1,2,\ldots,k$). Then by Theorem 7 we get (50). Moreover, for odd $n > 3$, inequality (50) is reversed. □

In the second case, we study generalized majorization-type inequality in terms of the K-L distance or relative entropy between two probability distributions.

Definition 5 Let $\mathbf{r} = (r_1,\ldots,r_k)$ and $\mathbf{w} = (w_1,\ldots,w_k)$ be two positive probability distributions. Then the K-L distance between them is defined by

$$L(\mathbf{r},\mathbf{w}) := \sum_{i=1}^{k} r_i \log\left(\frac{r_i}{w_i}\right).$$

Corollary 5 *Let $\boldsymbol{q} = (q_1,\ldots,q_k), \boldsymbol{r} = (r_1,\ldots,r_k),$ and $\boldsymbol{w} = (w_1,\ldots,w_k)$ be positive real k-tuples such that conditions (45) and (46) hold with $\frac{q_i}{w_i}, \frac{r_i}{w_i} \in [\vartheta_1, \vartheta_2](i = 1,2,\ldots,k)$. If log has base b greater than 1 and $\frac{\boldsymbol{q}}{\boldsymbol{w}}$ and $\frac{\boldsymbol{r}}{\boldsymbol{w}}$ are decreasing, then*

$$\sum_{i=1}^{k} w_i \log\left(\frac{q_i}{w_i}\right) \leq \sum_{i=1}^{k} w_i \log\left(\frac{r_i}{w_i}\right) - \sum_{s=2}^{n-1}\left(\frac{n-s}{(s-2)!(\vartheta_2 - \vartheta_1)}\right)$$
$$\times \int_{\vartheta_1}^{\vartheta_2}\left(\widetilde{I}_{G_l}(\boldsymbol{q},\boldsymbol{w},\nu) - \widetilde{I}_{G_l}(\boldsymbol{r},\boldsymbol{w},\nu)\right)$$
$$\times \left(\frac{(-1)^{s-1}(s-2)!}{\ln b(\nu - \vartheta_2)^{2-s}\vartheta_2^{s-1}} - \frac{(-1)^{s-1}(s-2)!}{\ln b(\nu - \vartheta_1)^{2-s}\vartheta_1^{s-1}}\right)d\nu. \tag{51}$$

If log *has base b between* 0 *and* 1, *then inequality* (51) *is reversed.*

Proof Take $f(x) := -\log x$, which is an *n*-convex function for $n > 3$. Then by Theorem 6 we get (51). Moreover, for odd $n > 3$, inequality (51) is reversed.   □

Corollary 6 *Let $\boldsymbol{q} = (q_1,\ldots,q_k), \boldsymbol{r} = (r_1,\ldots,r_k),$ and $\boldsymbol{w} = (w_1,\ldots,w_k)$ be positive probability distributions such that conditions (45) and (46) hold with $\frac{q_i}{w_i}, \frac{r_i}{w_i} \in [\vartheta_1, \vartheta_2](i = 1,2,\ldots,k)$. If* log *has base b greater than* 1 *and $\frac{\boldsymbol{q}}{\boldsymbol{w}}$ and $\frac{\boldsymbol{r}}{\boldsymbol{w}}$ are decreasing, then the relation between the K-L distance of $(\boldsymbol{r},\boldsymbol{w})$ and $(\boldsymbol{q},\boldsymbol{w})$ is given by the following estimate:*

$$L(\boldsymbol{q},\boldsymbol{w}) \geq L(\boldsymbol{r},\boldsymbol{w}) + \sum_{s=2}^{n-1}\left(\frac{n-s}{(s-2)!(\vartheta_2 - \vartheta_1)}\right)\int_{\vartheta_1}^{\vartheta_2}\left(\widetilde{I}_{G_l}(\boldsymbol{q},\boldsymbol{w},\nu) - \widetilde{I}_{G_l}(\boldsymbol{r},\boldsymbol{w},\nu)\right)$$
$$\times \left((x\log x)^{(s-1)}(\vartheta_2)(\nu - \vartheta_2)^{s-2} - (x\log x)^{(s-1)}(\vartheta_1)(\nu - \vartheta_1)^{s-2}\right)d\nu, \tag{52}$$

where for $d = 1,2$, $(x\log x)'(\vartheta_d) = \frac{1}{\ln b}(1 + \ln \vartheta_d)$ and $(x\log x)^{(s-1)}(\vartheta_d) = \frac{(-1)^{s-1}(s-3)!}{\vartheta_d^{s-2}\ln b}, s \geq 3$. If log *has base b between* 0 *and* 1, *then inequality* (52) *is reversed.*

Proof Take $g(x) := \log x$, so that $xg(x) := x\log x$ is an *n*-convex function for even $n > 3$. Then by Theorem 7 we get (52). Moreover, for odd $n > 3$, inequality (52) is reversed.   □

Authors' contributions

All the authors contributed equally in this research. All authors read and approved the final manuscript.

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