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Fractional integral inequalities involving Marichev–Saigo–Maeda fractional integral operator

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Abstract
The aim of this present investigation is establishing Minkowski fractional integral inequalities and certain other fractional integral inequalities by employing the Marichev–Saigo–Maeda (MSM) fractional integral operator. The inequalities presented in this paper are more general than the existing classical inequalities cited.

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1 Introduction

In last few decades, various researchers and mathematician have paid their valuable consideration to fractional integral inequalities (FIIs) and their applications. Recent research focuses on various types of FIIs by employing various types of fractional integral operators (see, e.g., [1–11]). In [12–17] the authors have established various types of inequalities and some other results by utilizing the Saigo fractional integral operator. The reverse Minkowski FIIs are found in [18]. Anber et al. [19] have obtained some FIIs by using the Riemann–Liouville fractional integral. The accompanying essential definitions and properties of the MSM fractional operator, which will be utilized to obtain the main results.

Definition 1.1 A real-valued function $g(\tau)$, $\tau \geq 0$, is said to be in $C_\mu([a,b])$, $\mu \in \mathbb{R}$, if there exists $\sigma \in \mathbb{R}$ such that $\sigma > \mu$ and $\Phi(\tau) = \tau^\sigma \Phi(\tau)$, where $\Phi(\tau) \in C([a,b])$.

Definition 1.2 Let $\nu, \acute{\nu}, \xi, \acute{\xi} \in \mathbb{R}$, and let $\eta > 0$. Then the MSM fractional integral is defined in [20] as

$$\begin{aligned} &\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta} g\right)(x) \\ &\quad = \frac{x^{-\nu}}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} t^{-\nu'} F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right) g(t)\,dt, \quad x \in \mathbb{R}, \end{aligned} \tag{1}$$

where F_3 is the Appell function defined by [21] as

$$F_3(\nu,\nu',\xi,\xi';\eta;x;y)=\sum_{m,n=0}^{\infty}\frac{(\nu)_m(\nu')_n(\xi)_m(\xi')_n}{(\eta)_{m+n}}\frac{x^m y^n}{m!n!},\quad \max\{|x|,|y|\}<1,$$

and $(\nu)_m=\nu(\nu+1)\cdots(\nu+m-1)$ is the Pochhammer symbol.

Lemma 1.1 *Let $\nu,\nu',\xi,\xi'\in\mathbb{R}$, $\eta>0$, and $\rho>\max\{0,(\nu-\nu'-\xi-\eta),(\nu'-\xi')\}$. Then we have the relation*

$$\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}t^{\rho-1}\right)(x)=\frac{\Gamma(\rho)\Gamma(\rho+\eta-\nu-\nu'-\xi)\Gamma(\tau+\xi'-\nu')}{\Gamma(\rho+\xi')\Gamma(\rho+\eta-\nu-\nu')\Gamma(\rho+\eta-\nu'-\xi)}x^{\rho-\nu-\nu'+\eta-1}. \tag{2}$$

Taking $\rho=1$ in Lemma 1.1, we get the relation

$$\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[1]\right)(x)=\frac{\Gamma(1+\eta-\nu-\nu'-\xi)\Gamma(1+\xi'-\nu')}{\Gamma(1+\xi')\Gamma(1+\eta-\nu-\nu')\Gamma(1+\eta-\nu'-\xi)}x^{-\nu-\nu'+\eta}. \tag{3}$$

The details of the integral operator (1) and its properties can be found in [22, 23]. For further applications of MSM fractional integral, we refer the interested readers to [24–28]. For a short history of this operator, see [25, 26, 29].

2 Reverse Minkowski inequalities via MSM fractional integral operator

In this section, we use the MSM fractional integral operator to develop reverse Minkowski integral inequalities. To prove the following reverse Minkowski FII, we first recall the following result.

Theorem 2.1 (see [30]) *If $\nu,\nu',\xi,\xi',\eta\in\mathbb{R}$ are such that $\eta>\max\{\nu,\nu',\xi,\xi'\}>0$, then we have the inequality*

$$F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)>0, \tag{4}$$

provided that $-1<(1-\frac{t}{x})<0$ and $0<(1-\frac{x}{t})<\frac{1}{2}$. Also, if $f(x)>0$, then

$$\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}f\right)(x)>0.$$

Theorem 2.2 *Let $\nu,\nu',\xi,\xi',\eta\in\mathbb{R}$ be such that $\eta>\max\{\nu,\nu',\xi,\xi'\}>0$, $\sigma\geq1$, and let Φ, Ψ be two positive functions on $[0,\infty)$ such that for all $x>0$, $\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi^\sigma(x)]<\infty$ and $\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Psi^\sigma(x)]<\infty$. If $0<m\leq\frac{\Phi(t)}{\Psi(t)}\leq M$, $t\in[0,x]$, then we have the inequality*

$$\begin{aligned}&\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^\sigma(x)\right)^{\frac{1}{\sigma}}+\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Psi^\sigma(x)\right)^{\frac{1}{\sigma}}\\&\quad\leq\frac{1+M(m+2)}{(m+1)(M+1)}\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}(\Phi+\Psi)^\sigma(x)\right)^{\frac{1}{\sigma}},\end{aligned} \tag{5}$$

provided that $-1<(1-\frac{t}{x})<0$ and $0<(1-\frac{x}{t})<\frac{1}{2}$.

Proof Using the condition $\frac{\Phi(t)}{\Psi(t)} < M$, $t \in [0,x]$, $x > 0$, we have

$$(M+1)^\sigma \Phi^\sigma(t) \leq M^\sigma (\Phi + \Psi)^\sigma(t). \tag{6}$$

Consider the function

$$\begin{aligned}\mathfrak{F}(x,t) &= \frac{x^{-\nu}(x-t)^{\eta-1}t^{-\nu'}}{\Gamma(\eta)} F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\\ &= \frac{x^{-\nu}(x-t)^{\eta-1}t^{-\nu'}}{\Gamma(\eta)}\left[1+\frac{(\nu')(\xi)}{(\eta)}\left(1-\frac{x}{t}\right)+\frac{(\nu)(\xi)}{(\eta)}\left(1-\frac{t}{x}\right)+\cdots\right].\end{aligned} \tag{7}$$

In view of Theorem 2.1, the function $\mathfrak{F}(x,t)$ is positive for all $t \in (0,x)$, $x > 0$. Therefore multiplying both sides of (6) by $\mathfrak{F}(x,t)$ and then integrating the resulting inequality with respect to t from 0 to x, we have

$$\begin{aligned}&\frac{(M+1)^\sigma x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\Phi^\sigma(t)\,dt\\ &\quad\leq \frac{M^\sigma x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)(\Phi+\Psi)^\sigma(t)\,dt,\end{aligned}$$

which can be written as

$$\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^\sigma(x) \leq \frac{M^\sigma}{(M+1)^\sigma}\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}(\Phi+\Psi)^\sigma(x).$$

Hence it follows that

$$\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^\sigma(x)\right)^{\frac{1}{\sigma}} \leq \frac{M}{(M+1)}\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}(\Phi+\Psi)^\sigma(x)\right)^{\frac{1}{\sigma}}. \tag{8}$$

Now using the condition $m\Psi(t) \leq \Phi(t)$, we have

$$\left(1+\frac{1}{m}\right)\Psi(t) \leq \frac{1}{m}\left(\Phi(t)+\Psi(t)\right),$$

from which it follows that

$$\left(1+\frac{1}{m}\right)^\sigma \Psi^\sigma(t) \leq \left(\frac{1}{m}\right)^\sigma\left(\Phi(t)+\Psi(t)\right)^\sigma. \tag{9}$$

Multiplying both sides of (9) by $\mathfrak{F}(x,t)$ and then integrating the resulting inequality with respect to t from 0 to x, we get

$$\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Psi^\sigma(x)\right)^{\frac{1}{\sigma}} \leq \frac{1}{(m+1)}\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}(\Phi+\Psi)^\sigma(x)\right)^{\frac{1}{\sigma}}. \tag{10}$$

Summing inequalities (8) and (10), we get the desired inequality. □

Theorem 2.3 *Let $\nu,\nu',\xi,\xi',\eta \in \mathbb{R}$ be such that $\eta > \max\{\nu,\nu',\xi,\xi'\} > 0$, $\sigma \geq 1$, and let Φ, Ψ be two positive functions on $[0,\infty)$ such that for all $x > 0$, $\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi^\sigma(x)] < \infty$ and*

$\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Psi^{\sigma}(x)] < \infty$. *If* $0 < m \leq \frac{\Phi(t)}{\Psi(t)} \leq M$, $t \in [0,x]$, *then we have the inequality*

$$\begin{aligned}&\big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma}(x)\big)^{\frac{2}{\sigma}} + \big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Psi^{\sigma}(x)\big)^{\frac{2}{\sigma}}\\ &\quad\geq \left(\frac{(M+1)(m+1)}{M}-2\right)\big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma}(x)\big)^{\frac{1}{\sigma}}\big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Psi^{\sigma}(x)\big)^{\frac{1}{\sigma}},\end{aligned} \tag{11}$$

provided that $-1 < (1-\frac{t}{x}) < 0$ *and* $0 < (1-\frac{x}{t}) < \frac{1}{2}$.

Proof By multiplying inequalities (8) and (10) we have

$$\begin{aligned}&\left(\frac{(M+1)(m+1)}{M}\right)\big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma}(x)\big)^{\frac{1}{\sigma}}\big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Psi^{\sigma}(x)\big)^{\frac{1}{\sigma}}\\ &\quad\leq \big[\big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big(\Phi(x)+\Psi(x)\big)^{\sigma}\big)^{\frac{1}{\sigma}}\big]^2.\end{aligned} \tag{12}$$

Now, applying the Minkowski inequality to the right-hand side of (12), we obtain

$$\begin{aligned}&\big[\big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big(\Phi(x)+\Psi(x)\big)^{\sigma}\big)^{\frac{1}{\sigma}}\big]^2\\ &\quad\leq \big[\big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma}(x)\big)^{\frac{1}{\sigma}} + \big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Psi^{\sigma}(x)\big)^{\frac{1}{\sigma}}\big]^2\\ &\quad\leq \big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma}(x)\big)^{\frac{2}{\sigma}} + \big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Psi^{\sigma}(x)\big)^{\frac{2}{\sigma}}\\ &\qquad + 2\big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma}(x)\big)^{\frac{1}{\sigma}}\big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Psi^{\sigma}(x)\big)^{\frac{1}{\sigma}}.\end{aligned} \tag{13}$$

Thus from inequalities (12) and (13) we get the desired inequality (11). □

3 Fractional integral inequalities via MSM fractional integral operator

This section is devoted to some FIIs involving MSM operator.

Theorem 3.1 *Let* $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, *and let* Φ, Ψ *be two positive functions on* $[0,\infty)$ *such that* $\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi(x)] < \infty$ *and* $\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Psi(x)] < \infty$. *If* $0 < m \leq \frac{\Phi(t)}{\Psi(t)} \leq M < \infty$, $t \in [0,x]$, $x > 0$, *then we have*

$$\big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi(x)\big)^{\frac{1}{r}}\big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Psi(x)\big)^{\frac{1}{s}} \leq \left(\frac{M}{m}\right)^{\frac{1}{rs}}\big(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(x)\big]^{\frac{1}{r}}\big[\Psi(x)\big]^{\frac{1}{s}}\big), \tag{14}$$

where $\nu,\nu',\xi,\xi',\eta \in \mathbb{R}$ *are such that* $\eta > \max\{\nu,\nu',\xi,\xi'\} > 0$.

Proof Since $\frac{\Phi(t)}{\Psi(t)} \leq M < \infty$, $t \in [0,x]$, $x > 0$, we have

$$\big[\Psi(t)\big]^{\frac{1}{s}} \geq M^{\frac{-1}{s}}\big[\Phi(t)\big]^{\frac{1}{s}}. \tag{15}$$

It follows that

$$\begin{aligned}\big[\Phi(t)\big]^{\frac{1}{r}}\big[\Psi(t)\big]^{\frac{1}{s}} &\geq M^{\frac{-1}{s}}\big[\Phi(t)\big]^{\frac{1}{r}}\big[\Phi(t)\big]^{\frac{1}{s}}\\ &\geq M^{\frac{-1}{s}}\big[\Phi(t)\big]^{\frac{1}{r}+\frac{1}{s}}\\ &\geq M^{\frac{-1}{s}}\big[\Phi(t)\big].\end{aligned} \tag{16}$$

Multiplying by (7) both sides of (16) and then integrating the resulting inequality with respect to t from 0 to x, we get

$$\begin{aligned}&\frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\left[\Phi(t)\right]^{\frac{1}{r}}\left[\Psi(t)\right]^{\frac{1}{s}}\,dt\\&\quad\geq\frac{M^{\frac{-1}{s}}x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\Phi(t)\,dt.\end{aligned}\tag{17}$$

It follows that

$$\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\left[\left[\Phi(t)\right]^{\frac{1}{r}}\left[\Psi(t)\right]^{\frac{1}{s}}\right]\geq M^{\frac{-1}{r}}\left[\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi(t)\right].\tag{18}$$

Consequently, we have

$$\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\left[\left[\Phi(t)\right]^{\frac{1}{r}}\left[\Psi(t)\right]^{\frac{1}{s}}\right]\right)^{\frac{1}{r}}\geq M^{\frac{-1}{rs}}\left[\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi(t)\right]^{\frac{1}{r}}.\tag{19}$$

On the other hand, $m\Psi(t)\leq\Phi(t)$, $t\in[0,x]$, $x>0$, and therefore we have

$$\left[\Phi(t)\right]^{\frac{1}{r}}\geq m^{\frac{1}{r}}\left[\Psi(t)\right]^{\frac{1}{r}}.\tag{20}$$

It follows that

$$\begin{aligned}\left[\Phi(t)\right]^{\frac{1}{r}}\left[\Psi(t)\right]^{\frac{1}{s}}&\geq m^{\frac{1}{r}}\left[\Psi(t)\right]^{\frac{1}{r}}\left[\Psi(t)\right]^{\frac{1}{s}}\\&\geq m^{\frac{1}{r}}\left[\Psi(t)\right]^{\frac{1}{r}+\frac{1}{s}}\\&\geq m^{\frac{1}{r}}\left[\Psi(t)\right].\end{aligned}\tag{21}$$

Multiplying both sides of (21) by (7) and integrating the resulting inequality with respect to t from 0 to x, we get

$$\begin{aligned}&\frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\left[\Phi(t)\right]^{\frac{1}{r}}\left[\Psi(t)\right]^{\frac{1}{s}}\,dt\\&\quad\geq\frac{m^{\frac{1}{r}}x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\Psi(t)\,dt.\end{aligned}\tag{22}$$

Hence we can write

$$\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\left[\left[\Phi(t)\right]^{\frac{1}{r}}\left[\Psi(t)\right]^{\frac{1}{s}}\right]\right)^{\frac{1}{r}}\geq m^{\frac{1}{rs}}\left[\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi(t)\right]^{\frac{1}{s}}.\tag{23}$$

Multiplying (19) and (23), we get the desired inequality. □

Theorem 3.2 *Let* $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, *and let* Φ, Ψ *be two positive functions on* $[0,\infty)$ *such that* $\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi^r(x)]<\infty$ *and* $\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Psi^s(x)]<\infty$. *If* $0<m\leq\frac{\Phi(t)^r}{\Psi(t)^s}\leq M<\infty$, $t\in[0,x]$, $x>0$, *then we have*

$$\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^r(x)\right)^{\frac{1}{r}}\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Psi^s(x)\right)^{\frac{1}{s}}\leq\left(\frac{M}{m}\right)^{\frac{1}{rs}}\left(\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\left[\Phi(x)\right]^{\frac{1}{r}}\left[\Psi(x)\right]^{\frac{1}{s}}\right),\tag{24}$$

where $\nu,\nu',\xi,\xi',\eta\in\mathbb{R}$ *are such that* $\eta>\max\{\nu,\nu',\xi,\xi'\}>0$.

Proof Replacing $\Phi(t)$ and $\Psi(t)$ by $\Phi(t)^r$ and $\Psi(t)^s$, $t \in [0,x]$, $x > 0$, in Theorem 3.1, we get the desired inequality (24). □

Theorem 3.3 *Let Φ and Ψ be two positive functions on $[0,\infty)$ such that Φ is nondecreasing and Ψ is nonincreasing. Then*

$$\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi^{\sigma}(x)\Psi^{\theta}(x)\big] \leq \frac{1}{\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[1]}\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi^{\sigma}(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Psi^{\theta}(x)\big], x > 0, \tag{25}$$

where $\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[1]$ is defined by (3), $\sigma,\theta > 0$, and $\nu,\nu',\xi,\xi',\eta \in \mathbb{R}$ are such that $\eta > \max\{\nu,\nu',\xi,\xi'\} > 0$.

Proof Let $t,\rho \in [0,x]$, $x > 0$. Then for any $\sigma > 0$ and $\theta > 0$, we have

$$\big(\Phi^{\sigma}(t) - \Phi^{\sigma}(\rho)\big)\big(\Psi^{\theta}(t) - \Psi^{\theta}(\rho)\big) \geq 0. \tag{26}$$

It follows that

$$\Phi^{\sigma}(t)\Psi^{\theta}(t) + \Phi^{\sigma}(\rho)\Psi^{\theta}(\rho) \leq \Phi^{\sigma}(\rho)\Psi^{\theta}(t) + \Phi^{\sigma}(t)\Psi^{\theta}(\rho). \tag{27}$$

Multiplying both sides of (27) by (7) and integrating the resulting inequality with respect to t from 0 to x, we get

$$\begin{aligned}
&\frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\Phi^{\sigma}(t)\Psi^{\theta}(t)\,dt \\
&\quad + \Phi^{\sigma}(\rho)\Psi^{\theta}(\rho)\frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)[1]\,dt \\
&\leq \Psi^{\theta}(\rho)\frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\Phi^{\sigma}(t)\,dt \\
&\quad + \Phi^{\sigma}(\rho)\frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\Psi^{\theta}(t)\,dt.
\end{aligned} \tag{28}$$

It follows that

$$\begin{aligned}
&\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi^{\sigma}(x)\Psi^{\theta}(x)\big] + \Phi^{\sigma}(\rho)\Psi^{\theta}(\rho)\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[1] \\
&\quad \leq \Phi^{\sigma}(\rho)\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Psi^{\theta}(x)\big] + \Psi^{\theta}(\rho)\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi^{\sigma}(x)\big].
\end{aligned} \tag{29}$$

Again, multiplying both sides of (29) by $\mathfrak{F}(x,\rho)$, which is obtained by replacing t by ρ in (7), and then integrating with respect to ρ from 0 to x, we obtain

$$\begin{aligned}
&\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi^{\sigma}(x)\Psi^{\theta}(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[1] + \mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi^{\sigma}(x)\Psi^{\theta}(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[1] \\
&\quad \leq \mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi^{\sigma}(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Psi^{\theta}(x)\big] + \mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Psi^{\theta}(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi^{\sigma}(x)\big],
\end{aligned} \tag{30}$$

which completes the proof. □

Theorem 3.4 *Let Φ and Ψ be two positive functions on $[0,\infty)$ such that Φ is nondecreasing and Ψ is nonincreasing. Then*

$$\begin{aligned} &\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[\Phi^{\sigma}(x)\Psi^{\theta}(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[1]+\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[\Phi^{\sigma}(x)\Psi^{\theta}(x)\big]\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}[1] \\ &\quad\le \mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi^{\sigma}(x)\big]\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[\Psi^{\theta}(x)\big]+\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Psi^{\theta}(x)\big]\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[\Phi^{\sigma}(x)\big] \end{aligned} \tag{31}$$

for all $x>0$, $\sigma,\theta>0$, where $\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[1]$ is defined by (3), and $\alpha,\beta,\zeta,\zeta',\lambda,\nu,\acute{\nu},\xi,\xi',\eta\in\mathbb{R}$ are such that $\eta>\max\{\nu,\nu',\xi,\xi'\}>0$ and $\lambda>\max\{\nu,\nu',\xi,\xi'\}>0$.

Proof Multiplying both sides of (29) by

$$\mathfrak{F}(x,\rho)=\frac{x^{-\alpha}}{\Gamma(\lambda)}(x-\rho)^{\lambda-1}\rho^{-\beta}F_3\left(\alpha,\beta,\zeta,\zeta';\lambda;1-\frac{\rho}{x},1-\frac{x}{\rho}\right)$$

and integrating the resulting identity with respect to ρ over $(0,x)$, we have

$$\begin{aligned} &\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi^{\sigma}(x)\Psi^{\theta}(x)\big]\frac{x^{-\alpha}}{\Gamma(\lambda)}\int_0^x(x-\rho)^{\lambda-1}\rho^{-\beta}F_3\left(\alpha,\beta,\zeta,\zeta';\lambda;1-\frac{\rho}{x},1-\frac{x}{\rho}\right)[1]\,d\rho \\ &\quad+\frac{x^{-\alpha}}{\Gamma(\lambda)}\int_0^x(x-\rho)^{\lambda-1}\rho^{-\beta}F_3\left(\alpha,\beta,\zeta,\zeta';\lambda;1-\frac{\rho}{x},1-\frac{x}{\rho}\right) \\ &\quad\times\big[\Phi^{\sigma}(\rho)\Psi^{\theta}(\rho)\big]\,d\rho\,\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[1] \\ &\le\frac{x^{-\alpha}}{\Gamma(\lambda)}\int_0^x(x-\rho)^{\lambda-1}\rho^{-\beta}F_3\left(\alpha,\beta,\zeta,\zeta';\lambda;1-\frac{\rho}{x},1-\frac{x}{\rho}\right) \\ &\quad\times\Phi^{\sigma}(\rho)\,d\rho\,\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Psi^{\theta}(x)\big] \\ &\quad+\frac{x^{-\alpha}}{\Gamma(\lambda)}\int_0^x(x-\rho)^{\lambda-1}\rho^{-\beta}F_3\left(\alpha,\beta,\zeta,\zeta';\lambda;1-\frac{\rho}{x},1-\frac{x}{\rho}\right) \\ &\quad\times\Psi^{\theta}(\rho)\,d\rho\,\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi^{\sigma}(x)\big], \end{aligned} \tag{32}$$

which yields the desired inequality (31). □

Remark 1 Inequalities (25) and (31) may be reversed if

$$\big(\Phi^{\sigma}(t)-\Phi^{\sigma}(\rho)\big)\big(\Psi^{\theta}(t)-\Psi^{\theta}(\rho)\big)\ge 0.$$

Remark 2 Applying Theorem 3.4 to $\alpha=\nu$, $\beta=\nu'$, $\zeta=\xi$, $\zeta'=\xi'$, $\lambda=\eta$, we get Theorem 3.3.

Theorem 3.5 *Let $\Phi\ge 0$ and $\Psi\ge 0$ be two functions on $[0,\infty)$ such that Ψ is nondecreasing. If*

$$\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi(x)\ge\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Psi(x),\quad x>0, \tag{33}$$

then for all $x>0$, $\sigma>0$, $\theta>0$, and $\sigma-\theta>0$, we have

$$\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma-\theta}(x)\le\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma}(x)\Psi^{-\theta}(x), \tag{34}$$

where $\nu,\nu',\xi,\xi',\eta\in\mathbb{R}$ are such that $\eta>\max\{\nu,\nu',\xi,\xi'\}>0$.

Proof Using the arithmetic–geometric inequality, for $\sigma > 0$ and $\theta > 0$, we have

$$\frac{\sigma}{\sigma-\theta}\Phi^{\sigma-\theta}(t) - \frac{\theta}{\sigma-\theta}\Psi^{\sigma-\theta}(t) \le \Phi^{\sigma}(t)\Psi^{-\theta}(t), \quad t \in (0,x), x > 0. \tag{35}$$

Multiplying both sides of (35) by (7) and then integrating with respect to t from 0 to x, we have

$$\begin{aligned}
&\frac{\sigma}{\sigma-\theta}\frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\Phi^{\sigma-\theta}(t)\,dt \\
&\quad - \frac{\theta}{\sigma-\theta}\frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\Psi^{\sigma-\theta}(t)\,dt \\
&\le \frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\Phi^{\sigma}(t)\Psi^{-\theta}(t)\,dt.
\end{aligned}$$

Consequently,

$$\frac{\sigma}{\sigma-\theta}\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma-\theta}(x) - \frac{\theta}{\sigma-\theta}\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Psi^{\sigma-\theta}(x) \le \Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma}(x)\Psi^{-\theta}(x),$$

which can be written as

$$\frac{\sigma}{\sigma-\theta}\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma-\theta}(x) \le \Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma}(x)\Psi^{-\theta}(x) + \frac{\theta}{\sigma-\theta}\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Psi^{\sigma-\theta}(x).$$

It follows that

$$\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma-\theta}(x) \le \frac{\sigma-\theta}{\sigma}\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma}(x)\Psi^{-\theta}(x) + \frac{\theta}{\sigma}\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Psi^{\sigma-\theta}(x). \tag{36}$$

By inequality (33) we have

$$\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma-\theta}(x) \le \frac{\sigma-\theta}{\sigma}\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma}(x)\Psi^{-\theta}(x) + \frac{\theta}{\sigma}\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\Phi^{\sigma-\theta}(x), \tag{37}$$

which gives the required result. □

Theorem 3.6 *Let Φ, Ψ, and h be positive continuous functions on $[0,\infty)$ such that*

$$\bigl(\Psi(t)-\Psi(\rho)\bigr)\left(\frac{\Phi(\rho)}{h(\rho)} - \frac{\Phi(t)}{h(t)}\right) \ge 0, \quad t,\rho \in [0,x), x > 0. \tag{38}$$

Then for all $x > 0$, we have

$$\frac{\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi(x)]}{\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[h(x)]} \ge \frac{\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[(\Psi\Phi)(x)]}{\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[(\Psi h)(x)]}, \tag{39}$$

where $\nu,\nu',\xi,\xi',\eta \in \mathbb{R}$ are such that $\eta > \max\{\nu,\nu',\xi,\xi'\} > 0$.

Proof Since Φ, Ψ, and h are positive continuous functions on $[0,\infty)$, by (38) we have

$$\Psi(t)\frac{\Phi(\rho)}{h(\rho)} + \Psi(\rho)\frac{\Phi(t)}{h(t)} - \Psi(\rho)\frac{\Phi(\rho)}{h(\rho)} - \Psi(t)\frac{\Phi(t)}{h(t)} \ge 0, \quad t,\rho \in [0,x), x > 0. \tag{40}$$

Multiplying (40) by $h(t)h(\rho)$, we get

$$\Psi(t)\Phi(\rho)h(t)+\Psi(\rho)\Phi(t)h(\rho)-\Psi(\rho)\Phi(\rho)h(t)-\Psi(t)\Phi(t)h(\rho)\geq 0. \tag{41}$$

Multiplying both sides of (41) by (7) and then integrating with respect to t from 0 to x, we get

$$\begin{aligned}
&\Phi(\rho)\frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\Psi(t)h(t)\,dt\\
&\quad+\Psi(\rho)h(\rho)\frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\Phi(t)\,dt\\
&\quad-\Phi(\rho)\Psi(\rho)\frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)h(t)\,dt\\
&\quad-h(\rho)\frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\Phi(t)\Psi(t)\,dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\Phi(\rho)\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi h)(x)\big]+\Psi(\rho)h(\rho)\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(x)\big]\\
&\quad-\Psi(\rho)\Phi(\rho)\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[h(x)\big]-h(\rho)\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi\Phi)(x)\big]\geq 0.
\end{aligned} \tag{42}$$

Again, multiplying both sides of (42) by $\mathfrak{F}(x,\rho)$ and then integrating with respect to ρ, we have

$$\begin{aligned}
&\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi h)(x)\big]+\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi h)(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(x)\big]\\
&\quad-\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi\Phi)(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[h(x)\big]-\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[h(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi\Phi)(x)\big]\geq 0.
\end{aligned}$$

It follows that

$$\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi h)(x)\big]\leq\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi\Phi)(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[h(x)\big], \tag{43}$$

which gives the desired result. □

Theorem 3.7 *Let Φ, Ψ, and h be positive continuous functions on $[0,\infty)$ such that*

$$\big(\Psi(t)-\Psi(\rho)\big)\left(\frac{\Phi(\rho)}{h(\rho)}-\frac{\Phi(t)}{h(t)}\right)\geq 0,\quad t,\rho\in[0,x),x>0. \tag{44}$$

Then for all $x>0$, we have

$$\frac{\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi(x)]\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}[(\Psi h)(x)]+\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}[\Phi(x)]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[(\Psi h)(x)]}{\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}[(\Psi\Phi)(x)]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[h(x)]+\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[(\Psi\Phi)(x)]\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}[h(x)]}\geq 1, \tag{45}$$

where $\alpha,\beta,\zeta,\zeta',\lambda,\nu,\acute{\nu},\xi,\xi',\eta\in\mathbb{R}$ are such that $\eta>\max\{\nu,\nu',\xi,\xi'\}>0$ and $\lambda>\max\{\nu,\nu',\xi,\xi'\}>0$.

Proof Multiplying both sides of (29) by

$$\mathfrak{F}(x,\rho)=\frac{x^{-\alpha}}{\Gamma(\lambda)}(x-\rho)^{\lambda-1}\rho^{-\beta}F_3\left(\alpha,\beta,\zeta,\zeta';\lambda;1-\frac{\rho}{x},1-\frac{x}{\rho}\right)$$

and integrating the resulting identity with respect to ρ over $(0,x)$, we get

$$\begin{aligned}&\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[\Phi(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi h)(x)\big]+\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[(\Psi h)(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(x)\big]\\&\quad-\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[(\Psi\Phi)(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[h(x)\big]-\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[h(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi\Phi)(x)\big]\geq 0.\end{aligned}$$

It follows that

$$\begin{aligned}&\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[\Phi(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi h)(x)\big]+\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[(\Psi h)(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(x)\big]\\&\quad\geq\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[(\Psi\Phi)(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[h(x)\big]+\mathfrak{I}_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[h(x)\big]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi\Phi)(x)\big],\end{aligned}$$

which completes the proof. □

Remark 3 Applying Theorem 3.7 to $\alpha=\nu$, $\beta=\nu'$, $\zeta=\xi$, $\zeta'=\xi'$, $\lambda=\eta$, we get Theorem 3.6.

Theorem 3.8 *Let Φ and h be two positive continuous functions on $[0,\infty)$ such that $\Phi\leq h$. If $\frac{\Phi}{h}$ is decreasing and Φ is increasing on $[0,\infty)$, then for all $x>0$ and $\sigma\geq 1$, we have*

$$\frac{\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi(x)]}{\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[h(x)]}\geq\frac{\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi^{\sigma}(x)]}{\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[h^{\sigma}(x)]},\tag{46}$$

where $\nu,\nu',\xi,\xi',\eta\in\mathbb{R}$ are such that $\eta>\max\{\nu,\nu',\xi,\xi'\}>0$.

Proof By taking $\Psi=\Phi^{\sigma-1}$ in Theorem 3.6 we have

$$\frac{\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi(x)]}{\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[h(x)]}\geq\frac{\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[(\Phi\Phi^{\sigma-1})(x)]}{\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}[(h\Phi^{\sigma-1})(x)]}.\tag{47}$$

Since $\Phi\leq h$, we can write

$$h\Phi^{\sigma-1}(x)\leq h^{\sigma}(x).\tag{48}$$

Multiplying both sides of (48) by (7) and integrating the resulting inequality with respect to t from 0 to x, we have

$$\begin{aligned}&\frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\Phi^{\sigma-1}h(t)\,dt\\&\quad\leq\frac{x^{-\nu}}{\Gamma(\eta)}\int_0^x(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)h^{\sigma}(t)\,dt,\end{aligned}$$

which implies that

$$\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\big(h\Phi^{\sigma-1}\big)(x)\big]\leq\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\big(h^{\sigma}(x)\big].\tag{49}$$

From (49) we can write

$$\frac{1}{\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[(h\Phi^{\sigma-1})(x)]} \geq \frac{1}{\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[h^{\sigma}(x)]},$$

and so we have

$$\frac{\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[(\Phi\Phi^{\sigma-1})(x)]}{\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[(h\Phi^{\sigma-1})(x)]} \geq \frac{\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi^{\sigma}(x)]}{\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[h^{\sigma}(x)]}. \tag{50}$$

Hence from (47) and (50) we get the desired result. □

Theorem 3.9 *Let Φ and h be two positive continuous functions on $[0,\infty)$ such that $\Phi \leq h$. If $\frac{\Phi}{h}$ is decreasing and Φ is increasing on $[0,\infty)$, then for all $x > 0$ and $\sigma \geq 1$, we have*

$$\frac{\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi(x)]\Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}[h^{\sigma}(x)] + \Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}[\Phi(x)]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[h^{\sigma}(x)]}{\Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}[\Phi^{\sigma}(x)]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[h(x)] + \Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi^{\sigma}(x)]\Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}[h(x)]} \geq 1, \tag{51}$$

where $\alpha,\beta,\zeta,\zeta',\lambda,\nu,\acute{\nu},\xi,\xi',\eta \in \mathbb{R}$ are such that $\eta > \max\{\nu,\nu',\xi,\xi'\} > 0$ and $\lambda > \max\{\nu,\nu',\xi,\xi'\} > 0$.

Proof Taking $\Psi = \Phi^{\sigma-1}$ in Theorem (3.7), we have

$$\frac{\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi(x)]\Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}[(h\Phi^{\sigma-1})(x)] + \Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}[\Phi(x)]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[(h\Phi^{\sigma-1})(x)]}{\Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}[\Phi^{\sigma}(x)]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[h(x)] + \Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi^{\sigma}(x)]\Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}[h(x)]} \geq 1. \tag{52}$$

Now since $\Phi \leq h$, we have

$$h\Phi^{\sigma-1}(x) \leq h^{\sigma}(x). \tag{53}$$

Multiplying both sides of (53) by

$$\mathfrak{F}(x,\rho) = \frac{x^{-\alpha}}{\Gamma(\lambda)}(x-\rho)^{\lambda-1}\rho^{-\beta}F_3\left(\alpha,\beta,\zeta,\zeta';\lambda;1-\frac{\rho}{x},1-\frac{x}{\rho}\right)$$

and integrating the resulting identity with respect to ρ over $(0,x)$, we get

$$\Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\left(h\Phi^{\sigma-1}\right)(x)\right] \leq \Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\sigma}(x)\right]. \tag{54}$$

Now multiplying both sides of (54) by $\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[\Phi(x)]$, we have

$$\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\left[\Phi(x)\right]\Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\left(h\Phi^{\sigma-1}\right)(x)\right] \leq \Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\left[\Phi(x)\right]\Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\sigma}(x)\right]. \tag{55}$$

Similarly, we have

$$\Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\Phi(x)\right]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\left[\left(h\Phi^{\sigma-1}\right)(x)\right] \leq \Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\Phi(x)\right]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\sigma}(x)\right]. \tag{56}$$

Hence by (55) and (56) we have

$$\begin{aligned}&\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(x)\big]\Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[\big(h\Phi^{\sigma-1}\big)(x)\big]+\Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[\Phi(x)\big]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\big(h\Phi^{\sigma-1}\big)(x)\big]\\&\quad\le\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(x)\big]\Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[h^{\sigma}(x)\big]+\Im_{0,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\big[\Phi(x)\big]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[h^{\sigma}(x)\big].\end{aligned}\tag{57}$$

By (52) and (57) we get the desired result. □

Theorem 3.10 *Let Φ, Ψ, and h be positive continuous functions on $[0,\infty)$ such that*

$$\big(\Phi(t)-\Phi(\rho)\big)\big(\Psi(t)-\Psi(\rho)\big)\big(h(t)+h(\rho)\big)\ge 0,\quad t,\rho\in(0,x),x>0.\tag{58}$$

Then for all $x>0$, we have

$$\begin{aligned}&\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Phi\Psi h)(x)\big]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[1]+\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Phi\Psi)(x)\big]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[h(x)\big]\\&\quad\ge\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Psi(x)\big]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Phi h)(x)\big]+\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(x)\big]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi h)(x)\big],\end{aligned}\tag{59}$$

where $\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[1]$ is defined by (3), and $\nu,\nu',\xi,\xi',\eta\in\mathbb{R}$ are such that $\eta>\max\{\nu,\nu',\xi,\xi'\}>0$.

Proof By the assumption stated in Theorem 3.10, for any t and ρ, we have

$$\begin{aligned}&\Phi(t)\Psi(t)h(t)+\Phi(t)\Psi(\rho)h(\rho)-\Phi(t)\Psi(\rho)h(t)-\Phi(t)\Psi(\rho)h(\rho)-\Phi(\rho)\Psi(t)h(t)\\&\quad-\Phi(\rho)\Psi(t)h(\rho)+\Phi(\rho)\Psi(\rho)h(t)+\Phi(\rho)\Psi(\rho)h(\rho)\ge 0.\end{aligned}\tag{60}$$

Multiplying both sides of (60) by (7) and integrating the resulting inequality with respect to t from 0 to x, we get

$$\begin{aligned}&\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Phi\Psi h)(t)\big]+\Psi(\rho)h(\rho)\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(t)\big]-\Psi(\rho)\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Phi h)(t)\big]\\&\quad-\Psi(\rho)h(\rho)\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(t)\big]-\Phi(\rho)\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi h)(t)\big]-\Phi(\rho)h(\rho)\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Psi(t)\big]\\&\quad+\Phi(\rho)\Psi(\rho)\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[h(t)\big]+\Phi(\rho)\Psi(\rho)h(\rho)\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[1]\ge 0.\end{aligned}\tag{61}$$

Again, multiplying both sides of (59) by

$$\mathfrak{F}(x,\rho)=\frac{x^{-\alpha}}{\Gamma(\lambda)}(x-\rho)^{\lambda-1}\rho^{-\beta}F_3\biggl(\alpha,\beta,\zeta,\zeta';\lambda;1-\frac{\rho}{x},1-\frac{x}{\rho}\biggr)$$

and integrating the resulting identity with respect to ρ over $(0,x)$, we get

$$\begin{aligned}&\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Phi\Psi h)(t)\big]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[1]+\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi h)(x)\big]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(t)\big]\\&\qquad+\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Phi\Psi)(x)\big]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[h(t)\big]+\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Phi\Psi h)(x)\big]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}[1]\\&\quad\ge\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi h)(x)\big]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(t)\big]+\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Psi(x)\big]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Phi h)(t)\big]\\&\qquad+\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi(x)\big]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[(\Psi h)(t)\big]+\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Phi h(x)\big]\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta}\big[\Psi(t)\big],\end{aligned}\tag{62}$$

which completes the proof. □

Theorem 3.11 *Let Φ, Ψ, and h be positive continuous functions on* $[0,\infty)$ *such that*

$$\bigl(\Phi(t)-\Phi(\rho)\bigr)\bigl(\Psi(t)+\Psi(\rho)\bigr)\bigl(h(t)+h(\rho)\bigr)\geq 0,\quad t,\rho\in(0,x), x>0. \tag{63}$$

Then for all $x>0$, we have

$$\begin{aligned}&\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\bigl[\Phi(x)\bigr]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\bigl[(\Psi h)(x)\bigr]+\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\bigl[(\Phi h)(x)\bigr]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\bigl[\Psi(x)\bigr]\\&\quad\geq\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\bigl[(\Psi h)(x)\bigr]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\bigl[\Phi(x)\bigr]+\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\bigl[h(x)\bigr]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\eta}\bigl[(\Phi\Psi)(x)\bigr],\end{aligned} \tag{64}$$

where $\nu,\nu',\xi,\xi',\eta\in\mathbb{R}$ are such that $\eta>\max\{\nu,\nu',\xi,\xi'\}>0$.

Proof By the assumption stated in Theorem 3.11, for any t and ρ, we have

$$\begin{aligned}&\Phi(t)\Psi(t)h(t)+\Phi(t)\Psi(\rho)h(\rho)+\Phi(t)\Psi(\rho)h(t)+\Phi(t)\Psi(\rho)h(\rho)\\&\quad\geq\Phi(\rho)\Psi(t)h(t)+\Phi(\rho)\Psi(t)h(\rho)+\Phi(\rho)\Psi(\rho)h(t)+\Phi(\rho)\Psi(\rho)h(\rho)\geq 0.\end{aligned} \tag{65}$$

Applying a procedure similar to that of Theorem 3.10, we get the proof of Theorem 3.11. □

4 Concluding remarks

In this present paper, we introduced certain inequalities by employing the (MSM) fractional integral operator. The inequalities obtained are more general than the existing classical inequalities. The MSM operator (1) turns to the Saigo fractional integral operator [22] due to the relation $\mathfrak{I}_{0,x}^{\nu,0,\xi,\xi',\eta}(x)=\mathfrak{I}_{0,x}^{\eta,\nu-\eta,-\xi}(x)$ ($\gamma\in\mathbb{C}$). Thus the inequalities obtained in this paper reduce to the integral inequalities involving the Saigo fractional integral operators, recently defined by Chinchane and Pachpatte [31].

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