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On a fractional p-q Laplacian equation with critical nonlinearity

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Abstract

In this paper, we consider the existence of nontrivial solutions for a fractional p-q Laplacian equation with critical nonlinearity in a bounded domain. Our approach is based on variational methods and some analytical techniques.

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1 Introduction

In this paper, we are interested in the existence of nontrivial solutions for the following equation:

$$\begin{cases} (-\Delta)^s_p u + (-\Delta)^s_q u = \lambda |u|^{r-2}u + |u|^{p^*_s-2}u, & x \in \Omega, \\ u = 0, & x \in R^N \setminus \Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain in $\mathbf{R}^N$, $0 < s < 1$, $1 < q < p < r < p^*_s$, λ is a positive constant, $p^*_s = pN/(N-sp)$ is the fractional critical exponent, and $(-\Delta)^s_p$ is the fractional p-Laplacian operator defined on smooth functions as

$$(-\Delta)^s_p u(x) = P.V. \int_{R^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \, dy.$$

The definition is consistent, up to a normalization constant depending on N and s, with the usual definition of the linear fractional Laplacian operator $(-\Delta)^s$ when $p = 2$. When $s = 1$, Eq. (1.1) becomes a local problem of the form

$$-\Delta_p u - \Delta_q u = \lambda |u|^{r-2}u + |u|^{p^*-2}u, \quad (1.2)$$

which has been studied before, and some existence results have been proven under different conditions. For $1 < q < p < r < p^*$, there exists $\lambda^* > 0$ such that for any $\lambda > \lambda^*$, problem (1.2) has a nontrivial solution in $W_0^{1,p}(\Omega)$ (see Yin and Yang [1]), whereas for $1 < r < q < p$, there exits λ_0 such that problem (1.2) has infinitely many solutions in $W_0^{1,p}(\Omega)$ for any

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$\lambda \in (0, \lambda_0)$ (see Li and Zhang [2]). Our result can be viewed as extension on [1] for fractional setting.

As explained in [1], the study of Eq. (1.2) comes from a general reaction–diffusion system

$$u_t = \text{div}\big[H(u)\nabla u\big] + c(x, u),$$

where $H(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. This system has a wide range of applications in physics and related sciences such as biophysics, plasma physics, and chemical reaction design. In applications the function u represents a concentration, $\text{div}[H(u)\nabla u]$ corresponds to the diffusion with diffusion coefficient $H(u)$, whereas $c(x, u)$ is related to source and loss processes. Typically, in chemical and biological applications the reaction term $c(x, u)$ has a polynomial form with respect to the concentration u.

When $p = q = r$, problem (1.1) reduces to the fractional p-Laplacian problem

$$(-\Delta)^s_p u = \lambda |u|^{p-2} u + |u|^{p^*_s - 2} u, \tag{1.3}$$

which has been studied by Mosconi et al. [3], who obtained nontrivial solutions to this Brezis–Nirenberg problem for fractional p-Laplacian operator and extended some well-known results of critical p-Laplacian problems to the fractional setting; see, for example, Azorero and Alonso [4] and Egnell [5]. In fact, there is a rapidly growing literature on problems involving these nonocal operators. For example, the fractional p-eigenvalue problem has been studied by Franzina and Palatucci [6] and Lindgren and Lindqvist [7]. Concerning the existence results for this kind of equations, some well-known existence results for classical Laplace operators have also been extended to the nonlocal fractional setting; see [8–12].

When $p = q$, there also are some recent results on the fractional p-Laplacian operator. In 2017, Mahwin and Bisci [13] proved a Brezis–Nirenberg-type result for the fractional p-Laplacian equation

$$(-\Delta)^s_p u = \lambda g(x, u) + |u|^{p^*_s - 2} u \tag{1.4}$$

in a bounded domain with $p \geq 2$, where g is a subcritical nonlinearity. By variational methods they prove the existence of a local minimizer of the associated functional to (1.4), which turns to be a weak solution of problem (1.4), provided that the constant λ is sufficiently small.

It is worth mentioning that there is also some literature concerning the fractional Laplacian equation with constant γ attached to the critical term,

$$(-\Delta)^s u = \gamma |u|^{2^*_s - 2} u + f(x, u), \tag{1.5}$$

where f satisfies some subcritical conditions; see Fiscella et al. [14]. By variational methods they obtain multiplicity and bifurcation results for (1.5), which generalized those given in [15] to the nonlocal framework of the fractional Laplacian. It is easy to see that the critical term and the subcritical term have different influences on the functional structure.

Motivated by the papers mentioned, we tend to investigate the existence of a nontrivial solution for problem (1.1). To our knowledge, not many critical results for fractional p-q Laplacian are present. We denote by X the fractional space $W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\Omega)|u = 0, x \in R^N \setminus \Omega\}$ equipped with the norm

$$\|u\|_p^p := [u]_{s,p}^p = \int \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy, \tag{1.6}$$

where $Q = \mathbf{R}^{2N} \setminus (C\Omega \times C\Omega)$ with $C\Omega = \mathbf{R}^N \setminus \Omega$. By the results of [16] there is a continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^s(\Omega)$ for $s \in [1, p_s^*]$ and compact for $s \in [1, p_s^*)$. For more details on fractional Sobolev spaces, we refer to Palatucci et al. [16] and references therein.

Our approach to studying problem (1.1) is variational and uses critical point theorems. The main difficulty in dealing with this problem is the fact that in general the associated energy functional does not satisfy the Palais–Smale condition. Hence we cannot directly use the standard variational methods. To overcome this, we prove that the corresponding functional satisfies the Palais–Smale condition on a certain range. We also mention that there is a local weak lower semicontinuity result for the corresponding energy functional of problem (1.1), which leads to the existence of a critical point under certain conditions. At last, when $p = 2$, the spectrum result of the fractional operator ensures a suitable decomposition of the functional space, which leads to a multiplicity result. It is worth noting that our results can also be generalized by the abstract result proposed by Devillanova and Solimini [17, 18]. Our main results read as follows.

Theorem 1.1 *If $1 \leq q < p < r < p_s^*$, then there exists $\lambda^* > 0$ such that for any $\lambda > \lambda^*$, problem* (1.1) *has a nontrivial solution in $W_0^{s,p}(\Omega)$.*

Theorem 1.2 *If $2 \leq q < p$ and $1 < r < p_s^*$, then there exists an open interval Λ such that, for every $\lambda \in \Lambda$, problem* (1.1) *admits a weak solution in $W_0^{s,p}(\Omega)$.*

Theorem 1.3 *If $1 \leq q < p = 2$ and $r \in (2, 2_s^*)$, then for any $k \in N$, there exists $\lambda_k \in (0, +\infty]$ such that problem* (1.1) *admits at least k pairs of nontrivial solutions for any $\lambda > \lambda_k$.*

The present paper is organized as follows. Section 2 is devoted to the functional structure and Palais–Smale condition of problem (1.1). In Sect. 3, we prove our results.

2 Preliminaries

In this section, we give some preliminary results about the functional structure of problem (1.1). The fact that u is a weak solution of the problem (1.1) is equivalent to being a critical point of the functional

$$I_\lambda(u) := \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|u\|_q^q - \frac{\lambda}{r}\int_\Omega |u|^r \, dx - \frac{1}{p_s^*}\int_\Omega |u|^{p_s^*} \, dx. \tag{2.1}$$

It is trivial that $I_\lambda(u) \in C^1(X, R)$ and for any $v \in X$, the weak solution satisfies $\langle I'_\lambda(u), v \rangle = 0$, that is,

$$
\int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \big(v(x) - v(y)\big) \, dx \, dy
$$
$$
+ \int_Q \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))}{|x - y|^{N+sq}} \big(v(x) - v(y)\big) \, dx \, dy
$$
$$
= \lambda \int_\Omega |u|^{p-2} uv \, dx + \int_\Omega |u|^{p^*_s} uv \, dx. \tag{2.2}
$$

We denote by S the best fractional Sobolev constant:

$$
S = \inf_{u \in W_0^{s,p}(\Omega) \setminus 0} \frac{\|u\|_p^p}{|u|_{p^*}^p}. \tag{2.3}
$$

Now we define the PS sequence and condition in $W_0^{s,p}(\Omega)$.

Definition 2.1 Let $c \in R$, let X be a Banach space, and let $I_\lambda \in C^1(X, R)$. Then $\{u_k\}$ is called a $(PS)_c$ sequence in X if $I(u_k) = c + o(1)$ and $I'_\lambda(u_k) = o(1)$ in X' as $k \to \infty$, where X' is the dual of X. The functional I_λ satisfies $(PS)_c$ condition in X if every $(PS)_c$ sequence in X for I_λ contains a convergent subsequence.

We first show that I_λ possesses the mountain pass geometry.

Lemma 2.2 Let $1 < q < p < r < p^*$. Then for any $\lambda > 0$, we have:
  (i) *there exist constants* $\rho, \beta > 0$ *such that* $I_\lambda(u) > \beta$ *for* $u \in X$ *with* $\|u\|_p = \rho$;
  (ii) *there exists* $u_0 \in X$ *such that* $I_\lambda(u_0) < \beta$ *and* $\|u_0\|_p > \rho$.

Proof (i) By the Hölder inequality and fractional Sobolev inequality we have

$$
I_\lambda(u) = \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|u\|_q^q - \frac{\lambda}{r}\int_\Omega |u|^r \, dx - \frac{1}{p^*_s}\int_\Omega |u|^{p^*_s} \, dx
$$
$$
\geq \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|u\|_q^q - \frac{\lambda}{r}|\Omega|^{\frac{p^*-r}{p^*}}|u|_{p^*}^r - \frac{1}{p^*}|u|_{p^*}^{p^*}
$$
$$
\geq \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|u\|_q^q - \frac{\lambda}{r}|\Omega|^{\frac{p^*-r}{p^*}} S^{-\frac{r}{p}}\|u\|_p^r - \frac{1}{p^*}S^{-\frac{p^*}{p}}\|u\|_p^{p^*}.
$$

Since $1 < q < p < r < p^*$, there exits two constants $\rho, \beta > 0$ such that $I_\lambda(u) > \beta$ for all $u \in X$ with $\|u\|_p = \rho$.
  (ii) Fixing any $u \in X$, we deduce that

$$
I_\lambda(tu) = \frac{t^p}{p}\|u\|_p^p + \frac{t^q}{q}\|u\|_q^q - \frac{\lambda t^r}{r}\int_\Omega |u|^r \, dx - \frac{t^{p^*}}{p^*}\int_\Omega |u|^{p^*} \, dx. \tag{2.4}
$$

Since $I_\lambda(tu) \to -\infty$ as $t \to +\infty$, we can choose $t_0 > 0$ such that $\|t_0 u\|_p > \rho$ and $I_\lambda(t_0 u) < 0$. Let $u_0 = t_0 u$. Then (ii) holds. $\qquad \square$

We denote by c_λ the mountain pass level:

$$c_\lambda := \inf\left\{\sup_{t \geq 0} I_\lambda(tu), u \in X\right\}.$$

Then we have the following result.

Lemma 2.3 *Let $1 \leq q < p < r < p^*$. Then for any $\lambda > 0$, I_λ satisfies the $(PS)_c$ conditions for all $c \in (0, \frac{1}{N}S^{\frac{N}{p}})$. Moreover, we have $c_\lambda \in (0, \frac{1}{N}S^{\frac{N}{p}})$ when $\lambda > \lambda^*$ for some positive constant λ^* if $1 \leq q < p < r < p^*$.*

Proof Let $\{u_k\}$ be a $(PS)_c$ sequence of I at the level c, that is,

$$I(u_k) = c + o(1); \qquad \langle I'(u_k), u_k\rangle = o(1)\|u_k\|. \tag{2.5}$$

We first check that $\{u_k\}$ is bounded in X. First, from (2.5) we have

$$
\begin{aligned}
c + o(1)\|u\|_p^p &= I(u_k) - \frac{1}{p}\langle I'(u_k), u_k\rangle \\
&= \left(\frac{1}{q} - \frac{1}{p}\right)\|u\|_q^q + \left(\frac{1}{p} - \frac{1}{r}\right)\theta\int_\Omega |u|^r\,dx + \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_\Omega |u|^{p^*}\,dx \\
&\geq \left(\frac{1}{p} - \frac{1}{r}\right)\theta\int_\Omega |u|^r\,dx + \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_\Omega |u|^{p^*}\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
c + o(1) &= I(u_k) \\
&= \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|u\|_q^q - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx - \frac{\theta}{r}\int_\Omega |u|^r\,dx \\
&\geq \frac{1}{p}\|u\|_p^p - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx - \frac{\theta}{r}\int_\Omega |u|^r\,dx \\
&\geq \frac{1}{p}\|u\|_p^p - C' + o(1)\|u\|_p^p.
\end{aligned}
$$

Thus $\{u_k\}$ is bounded in $W_0^{s,p}(\Omega)$. Taking if necessary a subsequence, we can assume that there exists $u \in W_0^{s,p}(\Omega)$ such that

$$
\begin{aligned}
&u_k \rightharpoonup u \quad \text{in } W_0^{s,p}(\Omega), \\
&u_k \to u \quad \text{in } L^s(\Omega), 1 \leq s < p^*, \\
&u_k \to u \quad \text{a.e. in } \Omega.
\end{aligned}
\tag{2.6}
$$

Noting that the sequences

$$\left\{\frac{|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))}{|x - y|^{\frac{N+ps}{p}}}\right\}_{k \in N} \quad \text{and} \quad \left\{\frac{|u_k(x) - u_k(y)|^{q-2}(u_k(x) - u_k(y))}{|x - y|^{\frac{N+qs}{q}}}\right\}_{k \in N}$$

are bounded in $L^{p'}(\Omega)$ and $L^{q'}(\Omega)$, by the pointwise convergence $u_k \to u$ we have

$$\frac{|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))}{|x - y|^{\frac{N+ps}{p}}} \rightharpoonup^{L^{p'}(\Omega)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{\frac{N+ps}{p}}},$$

$$\frac{|u_k(x) - u_k(y)|^{q-2}(u_k(x) - u_k(y))}{|x - y|^{\frac{N+qs}{q}}} \rightharpoonup^{L^{q'}(\Omega)} \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))}{|x - y|^{\frac{N+qs}{q}}}.$$

Thus for any $v \in W_0^{s,p}(\Omega)$, we have

$$\lim_{n \to \infty} \langle I'_\lambda(u_n), v \rangle = \langle I'_\lambda(u), v \rangle = 0, \tag{2.7}$$

that is, u is a critical point of I_λ. Then we get

$$I_\lambda(u) = \left(\frac{1}{q} - \frac{1}{p}\right) \|u\|_q^q + \left(\frac{1}{p} - \frac{1}{r}\right) \lambda \int_\Omega |u|^r \, dx + \frac{1}{N} \int_\Omega |u|^{p^*} \, dx \geq 0. \tag{2.8}$$

It now suffices to show that $u_k \to u$ in $W_0^{s,p}(\Omega)$. Let $v_k = u_k - u$. The fractional form of the Brezis–Lieb lemma leads to

$$\begin{aligned}
I_\lambda(u_k) &= \frac{1}{p} \|u\|_p^p + \frac{1}{p} \|v_k\|_p^p + \frac{1}{q} \|v_k\|_q^q + \frac{1}{q} \|u\|_q^q \\
&\quad - \frac{\lambda}{r} \int_\Omega |u|^r - \frac{1}{p^*} \int_\Omega |v_k|^{p^*} - \frac{1}{p^*} \int_\Omega |u|^{p^*} + o(1) \\
&= c + o(1)
\end{aligned}$$

and

$$\langle I'(u_k), u_k \rangle = \|v_k\|_p^p + \|u\|_p^p + \|v_k\|_q^q + \|u-\|_q^q - \lambda \int_\Omega |u|^r - \int_\Omega |v_n|^{p^*} - \int_\Omega |u|^{p^*} + o(1) = o(1),$$

where $o(1) \to 0$ as $k \to \infty$. From this and from (2.7) we have

$$\|v_k\|_p^p + \|v_k\|_q^q = \int_\Omega |v_k|^{p^*} \, dx + o(1). \tag{2.9}$$

Without loss of generality, we assume that

$$\|v_k\|_p^p = a + o(1), \qquad \|v_k\|_q^q = b + o(1),$$

and thus (2.9) implies

$$|v_k|_{p^*}^{p^*} = a + b + o(1).$$

By the fractional Sobolev inequality we have

$$a \geq S(a+b)^{\frac{p}{p^*}} \geq Sa^{\frac{p}{p^*}}. \tag{2.10}$$

If $a = 0$, then we complete the proof. Otherwise, $a \geq S^{N/p}$. Combining this with (2.1) and $1 \leq q < p < r < p^*$, as $n \to \infty$, we have

$$
\begin{aligned}
c &= \frac{a}{p} + \frac{b}{q} - \frac{a+b}{p^*} + \frac{1}{p} \|u\|_p^p + \frac{1}{q} \|u\|_q^q - \frac{\lambda}{r} \int_\Omega |u|^r \, dx - \frac{1}{p_s^*} \int_\Omega |u|^{p_s^*} \, dx \\
&= \left(\frac{1}{p} - \frac{1}{p^*} \right) a + \left(\frac{1}{q} - \frac{1}{p^*} \right) b + \left(\frac{1}{q} - \frac{1}{p} \right) \|u\|_q^q \\
&\quad + \left(\frac{1}{p} - \frac{1}{r} \right) \lambda \int_\Omega |u|^r \, dx + \left(\frac{1}{p} - \frac{1}{p^*} \right) \int_\Omega |u|^{p^*} \, dx + o(1) \\
&\geq \frac{1}{N} a \\
&\geq \frac{1}{N} S^{N/p},
\end{aligned}
$$

which contradicts the assumption on c. Thus we have $a = 0$, and I_λ satisfies the $(PS)_c$ conditions when $c \in (0, \frac{1}{N} S^{N/p})$. So we try to show that $c_\lambda \in (0, \frac{1}{N} S^{\frac{N}{p}})$.

We now choose a nonnegative $u_0 \in W_0^{s,p}(\Omega)$ with $|u_0|_{p^*} = 1$. Since $\lim_{t \to \infty} I_\lambda(t u_0) = -\infty$ and $\lim_{t \to 0} I_\lambda(t u_0) = 0$, there exists a $t_\lambda > 0$ such that $\sup_{t \geq 0} I_\lambda(t u_0) = I_\lambda(t_\lambda u_0)$, and thus t_λ satisfies

$$
0 = t_\lambda^{p-1} \|u_0\|_p^p + t_\lambda \|u_0\|_q^q - \lambda t_\lambda^{r-1} |u_0|_r^r - t_\lambda^{p^*-1}.
$$

Then we get

$$
t_\lambda^{p-r} \|u_0\|_p^p + t_\lambda^{q-r} \|u_0\|_q^q - t_\lambda^{p^*-r} = \lambda |u_0|_r^r.
$$

Since $1 \leq q < p < r < p^*$, we get $t_\lambda \to 0$ as $\lambda \to \infty$. Then there exists $\lambda^* > 0$ such that for any $\lambda > \lambda^* > 0$, we have

$$
\sup_{t \geq 0} I_\lambda(t u_0) < \frac{1}{N} S^{\frac{N}{p}},
$$

that is,

$$
c_\lambda \in \left(0, \frac{1}{N} S^{\frac{N}{p}} \right) \quad \text{for } \lambda > \lambda^*. \tag{2.11}
$$

This completes the proof.   □

Next, we prove the local weak lower semicontinuity of I_λ. From now on we denote the best constant of the continuous Sobolev embedding $W_0^{s,p}(\Omega) \hookrightarrow L^{p_s^*(\Omega)}$ as

$$
S' = \sup_{u \in W_0^{s,p}(\Omega) \backslash 0} \frac{|u|_{p^*}}{\|u\|_p}. \tag{2.12}
$$

Lemma 2.4 *Let $t > 1$. Denote by $\overline{B(0, \rho)}$ the closed ball centered at 0 and with radius $\rho > 0$ in the fractional Sobolev space $W_0^{s,t}(\Omega)$. Then there exists a positive constant $\overline{\rho}$ such that the functional I_λ is weakly lower semicontinuous on $\overline{B(0, \overline{\rho})}$.*

Proof It suffices to prove that I_0 is weakly lower semicontinuous. Let $\{u_j\}$ be a weakly convergent sequence in $\subset \overline{B(0, \rho)}$, that is, there exists $u' \in \overline{B(0, \rho)}$ satisfying

$$
\begin{aligned}
&u_j \rightharpoonup u' \quad \text{in } W_0^{s,t}(\Omega), \\
&u_j \to u' \quad \text{in } L^s(\Omega), 1 \le s < t^*, \\
&u_j \to u' \quad \text{a.e. in } \Omega.
\end{aligned}
\tag{2.13}
$$

We try to check that

$$
I := \liminf_{j \to +\infty}\big(I_0(u_j) - I_0(u')\big) \ge 0,
$$

that is,

$$
\liminf_{j \to +\infty}\left\{\frac{1}{p}\big(\|u_j\|_p^p - \|u'\|_p^p\big) + \frac{1}{q}\big(\|u_j\|_q^q - \|u'\|_q^q\big) - \frac{1}{p_s^*}\big(|u_j|_{p_s^*}^{p_s^*} - |u'|_{p_s^*}^{p_s^*}\big)\right\} \ge 0.
\tag{2.14}
$$

Since $2 \le q < p$, by the elementary inequality

$$
|b|^p - |a|^p \ge p|a|^{p-2}a(b-a) + 2^{1-p}|a-b|^p, \quad a, b \in R,
$$

from (2.13) we derive that

$$
\frac{1}{p}\big(\|u_j\|_p^p - \|u'\|_p^p\big) + \frac{1}{q}\big(\|u_j\|_q^q - \|u'\|_q^q\big) \ge C\|u_j - u'\|_p^p.
\tag{2.15}
$$

On the other hand, the Brezis–Lieb lemma leads to

$$
\liminf_{j \to +\infty}\big(|u_j|_{p_s^*}^{p_s^*} - |u'|_{p_s^*}^{p_s^*}\big) = \liminf_{j \to +\infty}|u_j - u'|_{p_s^*}^{p_s^*}.
\tag{2.16}
$$

Hence we have

$$
\liminf_{j \to +\infty}\big(I_0(u_j) - I_0(u')\big) \ge \liminf_{j \to +\infty}\left\{C\|u_j - u'\|_p^p - \frac{1}{p_s^*}|u_j - u'|_{p_s^*}^{p_s^*}\right\}.
\tag{2.17}
$$

Finally, by continuous embedding and owing to $\{u_j - u'\} \subset \overline{B(0, 2\rho)}$, we obtain

$$
I \ge \liminf_{j \to +\infty}\left(C - \frac{S'^{p_s^*}\|u_j - u'\|_p^{p_s^*-p}}{p_s^*}\right)\|u_j - u'\|_p^p.
$$

Thus for ρ sufficiently small such that

$$
0 < \rho \le \frac{1}{2}\left(\frac{Cp_s^*}{S'^{p_s^*}}\right)^{1/(p_s^*-p)},
$$

the functional I_λ is weakly semicontinuous on $\overline{B_\rho}$, provided that

$$
\overline{\rho} \in \left(0, \frac{1}{2}\left(\frac{Cp_s^*}{S'^{p_s^*}}\right)^{1/(p_s^*-p)}\right].
\tag{2.18}
$$

The proof is now complete.   □

3 Main theorems

To prove the first existence result, we need the following general version of the mountain pass lemma.

Lemma 3.1 *Let $I \in C^1(X, R)$ be a functional on Banach space X. Assume that there exist $\beta, \rho > 0$ such that*

   (i) $I(u) > \beta$ *for all $u \in X$ with $\|u\|_p = \rho$.*

   (ii) $I(0) = 0$, *and $I(v_0) < \beta$ for some $v_0 \in X$ with $\|v_0\|_p > \rho$.*

Set $\alpha := \inf\{\max_{t \geq 0} I(tu), u \in X \setminus 0\}$. Tthen there exists a sequence $\{u_n\} \subset X$ such that $I(u_n) \to \alpha$ and $I'(u_n) \to 0$ in X^* as $n \to \infty$.

Proof of Theorem 1.1 From Lemmas 2.2, 2.3, and 3.1 we obtain the existence of a critical point of I_λ in $W_0^{s,p}(\Omega)$ when $\lambda > \lambda^*$.   □

Next, we define the auxiliary function

$$h(\rho) := \frac{\rho - S'^{p_s^*} \rho^{p_s^* - 1} - C' \rho^{q-1}}{S'^r |\Omega|^{(p_s^* - r)/p_s^*} \rho^{r-1}}, \quad \rho \geq 0, \tag{3.1}$$

where $|\Omega|$ denotes the Lebesgue measure of the domain Ω, S' is the critical Sobolev constant given in (2.8), and C' is the embedding constant satisfying $\|u\|_q^q \leq C' \|u\|_p^q$. By the weak lower semicontinuity result in Lemma 2.4 we can prove the existence of a critical point of the energy functional by a direct minimization approach.

Proof of Theorem 1.2 Let $\rho_{\max}$ be the global maximum point of the rational function defined in (3.1), set $\rho_0 := \min\{\rho_{\max}, \overline{\rho}\}$, where $\overline{\rho}$ is defined in Lemma 2.4, and take

$$\lambda \in \Lambda := \big(0, h(\rho_0)\big).$$

Hence there exists $\rho_{0,\lambda} \in (0, \rho_0)$ such that

$$\lambda < \frac{\rho_{0,\lambda} - S'^{p_s^*} \rho_{0,\lambda}^{p_s^* - 1} - C' \rho_{0,\lambda}^{q-1}}{S'^r |\Omega|^{(p_s^* - r)/p_s^*} \rho_{0,\lambda}^{r-1}}. \tag{3.2}$$

Then let $0 < \varepsilon < \rho_{0,\lambda}$ and for $0 < \zeta < \eta$, set

$$\Theta_\lambda(u) := -\frac{1}{q} \int_\Omega \frac{|u(x) - u(y)|^q}{|x - y|^{n+qs}} \, dx + \frac{1}{p_s^*} \int_\Omega |u|^{p_s^*} \, dx + \frac{\lambda}{r} \int_\Omega |u|^r \, dx$$

and

$$\Phi_\lambda(\eta, \zeta) := \sup_{\overline{B(0,\eta)}} \Theta_\lambda(u) - \sup_{\overline{B(0,\eta-\zeta)}} \Theta_\lambda(u). \tag{3.3}$$

From Remark 3 in [13] we know that if

$$\limsup_{\varepsilon \to 0^+} \frac{\Phi_\lambda(\varrho, \varepsilon)}{\varepsilon} < \varrho \tag{3.4}$$

for some $\varrho > 0$, then there exists $w \in B(0, \varrho)$ such that

$$I_\lambda(w) < \frac{\varrho^p}{p} - \Phi_\lambda(u) \tag{3.5}$$

for every $u \in \overline{B(0, \varrho)}$. Next, we denote

$$\Psi_\lambda(\varepsilon, \rho_{0,\lambda}) := \frac{\Phi_\lambda(\rho_{0,\lambda}, \varepsilon)}{\varepsilon}. \tag{3.6}$$

By continuous embedding and rescaling u it is easy to prove that

$$\limsup_{\varepsilon \to 0^+} \Psi_\lambda(\varepsilon, \rho_{0,\lambda}) < \rho_{0,\lambda}. \tag{3.7}$$

Thus by (3.5) there exists $w_\lambda \in W_0^{s,p}(\Omega)$ such that

$$I_\lambda(w_\lambda) < \frac{\rho_{0,\lambda}^p}{p} - \Phi_\lambda, \quad \forall u \in \overline{B(0, \rho_{0,\lambda})}. \tag{3.8}$$

Since $\rho_{0,\lambda} < \overline{\rho}$, by Lemma 2.4 the energy functional I_λ is weakly lower semicontinuous on $\overline{B(0, \rho_{0,\lambda})}$, and the restriction $I_\lambda|_{\overline{B(0, \rho_{0,\lambda})}}$ has a global minimum $u_{0,\lambda} \in \overline{B(0, \rho_{0,\lambda})}$. On the other hand, if $\|u_{0,\lambda}\| = \rho_{0,\lambda}$, then by (3.7) we have

$$I_\lambda(u_{0,\lambda}) > I_\lambda(w_\lambda),$$

which is a contradiction. Thus $u_{0,\lambda}$ is a local minimum for the energy functional, which is a weak solution of problem (1.1). Thus completes the proof. □

At last, we give a multiplicity result for problem (1.1) when $1 \le q < p = 2$, based on a suitable decomposition of the functional space H_0^s. We first recall that H_0^s is a Hilbert space with the inner product

$$\langle u, v \rangle = \int_Q \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy \tag{3.9}$$

and the norm $\|u\| = \langle u, u \rangle$. Denote by $\{\lambda_j\}_{j \in N}$ the sequence of the eigenvalues of the eigenvalue problem

$$\begin{cases} (-\Delta)^s u = \lambda u, & x \in \Omega, \\ u = 0, & x \in R^n \setminus \Omega, \end{cases} \tag{3.10}$$

with

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \le \lambda_{j+1} \le \cdots \tag{3.11}$$

and eigenfunctions e_j corresponding to λ_j. Also, we can normalize $\{e_j\}_{j \in N}$ to construct an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H_0^s(\Omega)$. For details on the spectrum theory of the fractional Laplacian, we refer to [6] and [7]. Then we set, for any $j \in N$,

$$\mathbb{P}_{j+1} = \left\{ u \in H_0^s(\Omega) : \langle u, e_i \rangle = 0 \text{ for } i = 1, \dots, j \right\},$$

where $\mathbb{P}_1 = H_0^s(\Omega)$ as defined in Servadei and Valdinoci [19]. We also denote by

$$\mathbb{H}_j = \text{span}\{e_1, \ldots, e_j\}$$

the linear subspace generated by the first j eigenfunctions of $(-\Delta)^s$. It is trivial that $H_0^s(\Omega)$ is the direct sum of the above two subspaces, that is,

$$H_0^s(\Omega) = \mathbb{P}_{j+1} \oplus \mathbb{H}_j \tag{3.12}$$

for $j \in N$. We need the following version of the symmetric mountain pass theorem for multiplicity result (see Rabinowitz and Ambrosetti [20]).

Lemma 3.2 *Let $E = V \oplus X$, where E is a real Banach space, and V is a finite-dimensional space. Suppose that $I \in C^1(E, R)$ is a functional satisfying the following conditions:*
- (I_1) *$I(u) = I(-u)$ and $I(0) = 0$;*
- (I_2) *there exists a constant $\rho > 0$ such that $I|\partial B_\rho \cap X \geq 0$;*
- (I_3) *there exist a subspace $W \subset E$ with $\dim V < \dim W < +\infty$ and $M > 0$ such that $\max_{u \in W} I(u) < M$;*
- (I_4) *for $M > 0$ from (I_3), $I(u)$ satisfies the $(PS)_c$ condition for $0 \leq c \leq M$.*

Then there exist at least $\dim W - \dim V$ pairs of nontrivial critical points of I.

Since I_λ is even and $I_\lambda(0) = 0$, condition (I_1) is always satisfied. We try to check (I_2) and (I_3). We first consider $V = \mathbb{H}_j$ and $X = \mathbb{P}_{j+1}$ with $j \in N$ chosen as follows.

Lemma 3.3 *There exist $j \in N$ and $\rho, \alpha > 0$ such that $I_\lambda \geq \alpha$ for any $u \in \mathbb{P}_{j+1}$ with $\|u\| = \rho$.*

Proof Since $r \in (2, 2^*)$, by Lemma 4.1 of Fiscella et al. [14], for any $\delta > 0$, there exists $j \in N$ such that $|u|_r^r \leq \delta \|u\|^r$ for $u \in \mathbb{P}_{j+1}$. Thus for constant $c > 0$, we have

$$I_\lambda(u) \geq \frac{1}{2}\|u\|^2 + \frac{1}{q}\|u\|_q^q - \frac{\lambda}{r}\delta\|u\|^r - \frac{c}{p^*}\|u\|^{2^*}.$$

For $1 \leq q < p < r < p^*$, it is clear that there exist $\alpha, \rho > 0$ such that $I_\lambda \geq \alpha$ for all $u \in \mathbb{P}_{j+1}$ with $\|u\| = \rho$. □

Lemma 3.4 *Let $l \in N$. Then there exist a subspace W of $H_0^s(\Omega)$ and a constant $M > 0$ such that $\dim W = l$ and $\max_{u \in W} I(u) < M$.*

Proof By decomposition argument as before, we can take $W = \text{span}\{e_1, e_2, \ldots, e_l\}$ and $\dim W = l$. Let us choose a nonnegative $u_0 \in W$ with $|u_0|_{p^*} = 1$. Since $\lim_{t \to \infty} I_\lambda(tu_0) = -\infty$ and $\lim_{t \to 0} I_\lambda(tu_0) = 0$, there exists $t_\lambda > 0$ such that $\sup_{t \geq 0} I_\lambda(tu_0) = I_\lambda(t_\lambda u_0)$, and then t_λ satisfies

$$0 = t_\lambda^{p-1}\|u_0\|_p^p + t_\lambda\|u_0\|_q^q - \lambda t_\lambda^{r-1}|u_0|_r^r - t_\lambda^{p^*-1}. \tag{3.13}$$

Then we get

$$t_\lambda^{p-r}\|u_0\|_p^p + t_\lambda^{q-r}\|u_0\|_q^q - t_\lambda^{p^*-r} = \lambda|u_0|_r^r. \tag{3.14}$$

Since $1 \leq q < p < r < p^*$, we get $t_\lambda \to 0$ as $\lambda \to \infty$. Thus for any constant $M > 0$, there exists $\lambda^* > 0$ such that for any $\lambda > \lambda(M) > 0$, we have

$$\sup_{t \geq 0} I_\lambda(tu_0) < M, \tag{3.15}$$

that is, $\max_{u \in W} I(u) < M$, concluding the proof. $\qquad\square$

Proof of Theorem 1.2 By Lemmas 3.2 and 3.3 we have that I_λ satisfies (I_2) in $X = \mathbb{P}_{j+1}$ and for any $l \in N$, there is a subspace $W \subset H_0^s(\Omega)$ with dim $W = l + j$, and I_λ satisfies (I_3) with $M > 0$ for $\lambda > \lambda(M) > 0$. By Lemma 2.3 we can take λ^* sufficiently large to ensure that I_λ satisfies (I_4) for any $\lambda > \lambda^*$. Thus we can apply Lemma 3.2 to conclude that I_λ admits k pairs of nontrivial critical points for $\lambda > 0$ sufficiently large. This completes the proof. $\quad\square$

Abbreviations
P.V., principle value; $(PS)_c$, Palais–Smale condition.

Availability of data and materials
Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This work was carried out in collaboration between both authors. ZY designed the study and guided the research. ZZ performed the analysis and wrote the first draft of the manuscript. ZY and ZZ managed the analysis of the study. Both authors read and approved the final manuscript.

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