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Lyapunov-type inequality and solution for a fractional differential equation

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Abstract

In this paper, we consider the linear fractional differential equation

$$\begin{cases} {}_0^C\mathscr{D}_t^{\nu} u(t) + q(t)u(t) = 0, & t \in (0,1), \nu \in (1,2], \\ u(0) = \delta u(1), & u'(0) = \gamma u'(1). \end{cases}$$

By obtaining the Green's function we derive the Lyapunov-type inequality for such a boundary value problem. Furthermore, we use the contraction mapping theorem to study the existence of a unique solution for the corresponding nonlinear problem.

MSC: 34A08

Keywords: Fractional differential equation; Green's function; Lyapunov-type inequality; Contraction mapping theorem

1 Introduction

In 1907, Lyapunov [1] stated the following outstanding result.

Theorem 1.1 ([1]) *If the boundary value problem* (*BVP*)

$$\begin{cases} y''(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases}$$

has a nontrivial solution, then we have the following Lyapunov inequality:

$$\int_a^b |q(s)|\, ds > \frac{4}{b-a}. \tag{1}$$

Inequality (1) is very useful in various problems related to differential equations. Since the appearance of Lyapunov's fundamental paper [1], many improvements and generalizations of inequality (1) for integer-order (second- and higher-order) BVPs have appeared in the literature; we refer the reader to the summary reference by Tiryaki [2].

Recently, the studies on Lyapunov's inequality for fractional boundary value problem (FBVP) have begun, in which fractional derivatives (Riemann–Liouville derivative ${}_a^R\mathscr{D}_t^{\nu}$ or

Caputo derivative ${}_a^C\mathscr{D}_t^\nu$) are used instead of the classical ordinary derivative. Such a work was initiated by Ferreira [3] in 2013, who obtained a Lyapunov inequality for the following differential equation with Riemann–Liouville fractional derivative:

$$\left({}_a^R\mathscr{D}_t^\nu y\right)(t) + q(t)y(t) = 0, \quad a < t < b, 1 < \nu \leq 2, \tag{2}$$

subject to the boundary value condition

$$y(a) = y(b) = 0. \tag{3}$$

Next, in 2014, Ferreira [4] obtained a Lyapunov inequality for the following differential equation with Caputo fractional derivative:

$$\left({}_a^C\mathscr{D}_t^\nu y\right)(t) + q(t)y(t) = 0, \quad a < t < b, 1 < \nu \leq 2, \tag{4}$$

subject to boundary value condition (3).

After [3] and [4], many results appeared in the literature; we refer the reader to [5–10], where Lyapunov or Lyapunov-type inequalities are obtained for fractional differential equation subject various boundary value conditions such as

$$\begin{aligned}
&y'(a) = y'(b) = y(c) = 0, \quad a < b, c \in [a,b];\\
&y(a) = y'(a) = 0, \qquad y'(b) = \beta y'(\xi);\\
&y(a) = y'(a) = y''(a) = y''(b) = 0;\\
&y(a) = y'(a) = y(b) = 0.
\end{aligned}$$

Inspired by the works mentioned, in this paper, we aim to investigate the Lyapunov-type inequality for the following fractional differential equations:

$$\begin{cases} {}_0^C\mathscr{D}_t^\nu u(t) + q(t)u(t) = 0, \quad t \in (0,1), \nu \in (1,2],\\ u(0) = \delta u(1), \qquad u'(0) = \gamma u'(1), \end{cases} \tag{5}$$

where δ and γ are real numbers, and $q(t) \in L(0,1)$ is not identically zero on any compact subinterval of $(0,1)$. Furthermore, we obtain the existence of a solution for the corresponding nonlinear problem:

$$\begin{cases} {}_0^C\mathscr{D}_t^\nu u(t) + q(t)f(u(t)) = 0, \quad t \in (0,1), \nu \in (1,2],\\ u(0) = \delta u(1), \qquad u'(0) = \gamma u'(1). \end{cases} \tag{6}$$

BVP (6) was recently studied in [11], but we should point out that only the case of $\delta > 1$ and $0 < \gamma < 1$ was considered in [11]. In this paper, we give a comprehensive discussion on parameters δ and γ.

2 Preliminaries and lemmas

For convenience, we present some definitions and lemmas from fractional calculus theory in the sense of Riemann–Liouville and Caputo.

Definition 2.1 ([12]) Let $\Gamma(\nu) = \int_0^\infty t^{\nu-1}e^{-t}\,dt$, $\nu > 0$, be the gamma function. Then the Riemann–Liouville fractional integral of order ν for $y(t)$ is defined as

$$\bigl({}_a\mathscr{I}_t^\nu y\bigr)(t) := \frac{1}{\Gamma(\nu)}\int_a^t (t-s)^{\nu-1}y(s)\,ds, \quad t \in [a,b].$$

Definition 2.2 ([12]) Let $\nu > 0$ and $n = [\nu] + 1$, where $[\nu]$ denotes the integer part of a number ν. Then the Caputo fractional derivative of order ν for $y(t)$ is defined as

$$\bigl({}_a^C\mathscr{D}_t^\nu y\bigr)(t) := \bigl({}_a\mathscr{I}_t^{n-\nu}y^{(n)}\bigr)(t) = \frac{1}{\Gamma(n-\nu)}\int_a^t \frac{y^{(n)}(s)}{(t-s)^{\nu+1-n}}\,ds, \quad t \in [a,b].$$

By Definitions 2.1 and 2.2 we have

$$\bigl({}_a\mathscr{I}_t^\nu\bigl({}_a^C\mathscr{D}_t^\nu y\bigr)\bigr)(t) = y(t) + C_1 + C_2t + \cdots + C_nt^{n-1}. \tag{7}$$

Lemma 2.1 *A function $u(t)$ is a solution of the boundary value problem (5) if and only if $u(t)$ satisfies*

$$u(t) = \int_0^1 G(t,s)q(s)u(s)\,ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\nu)}\begin{cases} (1-\nu)\frac{\delta\gamma(1-t)+\gamma t}{(1-\gamma)(1-\delta)}(1-s)^{\nu-2} - \frac{\delta}{1-\delta}(1-s)^{\nu-1} - (t-s)^{\nu-1}, \\ \quad 0 \le s \le t \le 1, \\ (1-\nu)\frac{\delta\gamma(1-t)+\gamma t}{(1-\gamma)(1-\delta)}(1-s)^{\nu-2} - \frac{\delta}{1-\delta}(1-s)^{\nu-1}, \\ \quad 0 \le t \le s \le 1. \end{cases} \tag{8}$$

Proof Let $u(t)$ be a solution of (5). Then

$$\bigl({}_0\mathscr{I}_t^\nu\bigl({}_0^C\mathscr{D}_t^\nu u\bigr)\bigr)(t) + {}_0\mathscr{I}_t^\nu\bigl(q(t)u(t)\bigr) = 0.$$

By (7) we obtain

$$u(t) = C_1 + C_2t - \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}q(s)u(s)\,ds. \tag{9}$$

Considering $u(0) = \delta u(1)$, we have

$$C_1 = \delta C_1 + \delta C_2 - \frac{\delta}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-1}q(s)u(s)\,ds;$$

considering $u'(0) = \gamma u'(1)$, we have

$$C_2 = \frac{\gamma(\nu-1)}{\Gamma(\nu)(\gamma-1)}\int_0^1 (1-s)^{\nu-2}q(s)u(s)\,ds, \tag{10}$$

and thus we get

$$C_1 = \frac{\delta\gamma(\nu-1)}{\Gamma(\nu)(1-\delta)(\gamma-1)}\int_0^1 (1-s)^{\nu-2}q(s)u(s)\,ds - \frac{\delta}{\Gamma(\nu)(1-\delta)}\int_0^1 (1-s)^{\nu-1}q(s)u(s)\,ds. \tag{11}$$

Substituting (10) and (11) into (9), we obtain

$$\begin{aligned} u(t) &= \frac{\delta\gamma(\nu-1)}{\Gamma(\nu)(1-\delta)(\gamma-1)}\int_0^1 (1-s)^{\nu-2}q(s)u(s)\,ds \\ &\quad - \frac{\delta}{\Gamma(\nu)(1-\delta)}\int_0^1 (1-s)^{\nu-1}q(s)u(s)\,ds \\ &\quad + \frac{\gamma(\nu-1)t}{\Gamma(\nu)(\gamma-1)}\int_0^1 (1-s)^{\nu-2}q(s)u(s)\,ds - \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}q(s)u(s)\,ds \\ &= \int_0^1 G(t,s)q(s)u(s)\,ds, \end{aligned}$$

where $G(t,s)$ is the Green's function:

$$G(t,s) = \frac{1}{\Gamma(\nu)}\begin{cases} (1-\nu)\frac{\delta\gamma(1-t)+\gamma t}{(1-\gamma)(1-\delta)}(1-s)^{\nu-2} - \frac{\delta}{1-\delta}(1-s)^{\nu-1} - (t-s)^{\nu-1}, \\ \quad 0 \le s \le t \le 1, \\ (1-\nu)\frac{\delta\gamma(1-t)+\gamma t}{(1-\gamma)(1-\delta)}(1-s)^{\nu-2} - \frac{\delta}{1-\delta}(1-s)^{\nu-1}, \\ \quad 0 \le t \le s \le 1. \end{cases}$$

□

Lemma 2.2 *When $\delta \in (0,1)$ and $\gamma \in (0,1)$, Green's function $G(t,s)$ satisfies the following properties*:

(i) $G(t,s) \le 0$, $(t,s) \in [0,1] \times [0,1]$;
(ii) $\max_{0\le t\le 1} |G(t,s)| = -G(1,s) = \frac{(1-s)^{\nu-2}}{\Gamma(\nu)(1-\delta)(1-\gamma)}[\gamma(\nu-1) + (1-\gamma)(1-s)]$ *for* $s \in [0,1]$,
(iii) $\int_0^1 |G(t,s)|\,ds \le \frac{\gamma(\nu-1)+1}{\Gamma(\nu+1)(1-\delta)(1-\gamma)}$.

Proof (i) $G(t,s) \le 0$ is obvious since $\delta \in (0,1)$ and $\gamma \in (0,1)$.

(ii) For $s \in [0,1]$ and $t \in [s,1]$, we have

$$G'_t(t,s) = \frac{1-\nu}{\Gamma(\nu)}\left[\frac{\gamma}{1-\gamma}(1-s)^{\nu-2} + (t-s)^{\nu-2}\right] \le 0,$$

which means

$$G(1,s) \le G(t,s) \le G(s,s) \le 0, \quad s \le t \le 1; \tag{12}$$

for $t \in [0,s]$, we have

$$G'_t(t,s) = \frac{\gamma(1-\nu)}{\Gamma(\nu)(1-\gamma)}(1-s)^{\nu-2} \le 0,$$

which means

$$G(s,s) \le G(t,s) \le G(0,s) \le 0, \quad 0 \le t \le s. \tag{13}$$

Inequalities (12) and (13) show that, for $s \in [0,1]$,

$$G(1,s) \leq G(t,s) \leq G(0,s) \leq 0, \quad 0 \leq t \leq 1.$$

Therefore, for $s \in [0,1]$,

$$\begin{aligned}\max_{0\leq t\leq 1}\left|G(t,s)\right| &= -G(1,s)\\ &= \frac{1}{\Gamma(\nu)}\left[\frac{(\nu-1)\gamma}{(1-\delta)(1-\gamma)}(1-s)^{\nu-2} + \frac{(1-s)^{\nu-1}}{1-\delta}\right]\\ &= \frac{(1-s)^{\nu-2}}{\Gamma(\nu)(1-\delta)(1-\gamma)}\left[\gamma(\nu-1) + (1-\gamma)(1-s)\right].\end{aligned}$$

(iii) By (ii) we have

$$\begin{aligned}\int_0^1 \left|G(t,s)\right| ds &\leq \int_0^1 -G(1,s)\, ds\\ &= \int_0^1 \frac{(1-s)^{\nu-2}}{\Gamma(\nu)(1-\delta)(1-\gamma)}\left[\gamma(\nu-1) + (1-\gamma)(1-s)\right] ds\\ &= \frac{\gamma(\nu-1)+1}{\Gamma(\nu+1)(1-\delta)(1-\gamma)}.\end{aligned}$$

□

Lemma 2.3 *When $\delta \in (1,+\infty)$ and $\gamma \in (0,1)$, Green's function $G(t,s)$ satisfies the following properties*:

(i) $G(t,s) \geq 0$, $(t,s) \in [0,1] \times [0,1]$;

(ii) $\max_{0\leq t\leq 1} |G(t,s)| = G(0,s) = \frac{\delta(1-s)^{\nu-2}}{\Gamma(\nu)(\delta-1)(1-\gamma)}[\gamma(\nu-1) + (1-\gamma)(1-s)]$ *for* $s \in [0,1]$,

(iii) $\int_0^1 |G(t,s)|\, ds \leq \frac{\delta(\gamma\nu+1-\gamma)}{\Gamma(\nu+1)(\delta-1)(1-\gamma)}$.

Proof (i) When $0 \leq t \leq s \leq 1$,

$$G(t,s) = (\nu-1)\frac{\delta\gamma(1-t)+\gamma t}{\Gamma(\nu)(\delta-1)(1-\gamma)}(1-s)^{\nu-2} + \frac{\delta}{\Gamma(\nu)(\delta-1)}(1-s)^{\nu-1} \geq 0.$$

When $0 \leq s \leq t \leq 1$,

$$\begin{aligned}G(t,s) &= (\nu-1)\frac{\delta\gamma(1-t)+\gamma t}{\Gamma(\nu)(\delta-1)(1-\gamma)}(1-s)^{\nu-2} + \frac{(1-s)^{\nu-1}}{\Gamma(\nu)}\left[\frac{1}{\delta-1} + 1 - \left(\frac{t-s}{1-s}\right)^{\nu-1}\right]\\ &\geq 0.\end{aligned}$$

(ii) For $s \in [0,1]$ and $t \in [s,1]$, we have

$$G'_t(t,s) = \frac{1-\nu}{\Gamma(\nu)}\left[\frac{\gamma}{1-\gamma}(1-s)^{\nu-2} + (t-s)^{\nu-2}\right] \leq 0,$$

which means

$$0 \leq G(1,s) \leq G(t,s) \leq G(s,s), \quad s \leq t \leq 1. \tag{14}$$

For $t \in [0,s]$, we have

$$G'_t(t,s) = \frac{\gamma(1-\nu)(1-s)^{\nu-2}}{(1-\gamma)\Gamma(\nu)} \leq 0,$$

which means

$$G(s,s) \leq G(t,s) \leq G(0,s), \quad 0 \leq t \leq s. \tag{15}$$

Inequalities (14) and (15) show us that, for $s \in [0,1]$,

$$\max_{0\leq t\leq 1}\bigl|G(t,s)\bigr| = G(0,s) = \frac{\delta(1-s)^{\nu-2}}{\Gamma(\nu)(\delta-1)(1-\gamma)}\bigl[\gamma(\nu-1)+(1-\gamma)(1-s)\bigr].$$

(iii) From (ii) we have

$$\begin{aligned}\int_0^1 \bigl|G(t,s)\bigr|\,ds &\leq \int_0^1 G(0,s)\,ds \\ &= \int_0^1 \frac{\delta(1-s)^{\nu-2}}{\Gamma(\nu)(\delta-1)(1-\gamma)}\bigl[\gamma(\nu-1)+(1-\gamma)(1-s)\bigr]\,ds \\ &= \frac{\delta(\gamma\nu+1-\gamma)}{\Gamma(\nu+1)(\delta-1)(1-\gamma)}.\end{aligned}$$

□

Lemma 2.4 *When $\delta \in (0,1)$ and $\gamma \in (1, 1+\frac{(\nu-1)\delta}{2-\nu}]$, Green's function $G(t,s)$ satisfies the following properties*:

(i) $G(t,s) \geq 0$, $(t,s) \in [0,1]\times[0,1]$;

(ii) $\max_{0\leq t\leq 1}|G(t,s)| = G(1,s) = \frac{(1-s)^{\nu-2}}{\Gamma(\nu)(1-\delta)(\gamma-1)}[\gamma(\nu-1)-(\gamma-1)(1-s)]$ *for* $s \in [0,1]$,

(iii) $\int_0^1 |G(t,s)|\,ds \leq \frac{\gamma(\nu-1)+1}{\Gamma(\nu+1)(1-\delta)(\gamma-1)}$.

Proof We first prove (i) and (ii). For $s \in [0,1]$, when $t \in [0,s]$,

$$G'_t(t,s) = \frac{\gamma(\nu-1)(1-s)^{\nu-2}}{\Gamma(\nu)(\gamma-1)} \geq 0,$$

which means that, for $s \in [0,1]$,

$$G(0,s) \leq G(t,s) \leq G(s,s), \quad t \in [0,s]. \tag{16}$$

When $t \in [s,1]$,

$$\begin{aligned} G'_t(t,s) &= \frac{(\nu-1)}{\Gamma(\nu)}\left[\frac{\gamma}{\gamma-1}(1-s)^{\nu-2}-(t-s)^{\nu-2}\right], \\ G''_{tt}(t,s) &= \frac{1}{\Gamma(\nu)}(\nu-1)(2-\nu)(t-s)^{\nu-3} \geq 0. \end{aligned} \tag{17}$$

Letting $G'_t(t,s)=0$, we get $t^* = (\frac{\gamma}{\gamma-1})^{\frac{1}{\nu-2}}(1-s)+s \in [s,1]$. Combining with (17), for $s \in [0,1]$, we have

$$G\bigl(t^*,s\bigr) \leq G(t,s) \leq G(s,s), \quad t \in \bigl[s,t^*\bigr], \tag{18}$$

$$G\bigl(t^*,s\bigr) \leq G(t,s) \leq G(1,s), \quad t \in \bigl[t^*,1\bigr]. \tag{19}$$

Inequalities (16), (18), and (19) show us that, for $s \in [0,1]$,

$$\max_{0 \le t \le 1} \left|G(t,s)\right| = \max\left\{\left|G(0,s)\right|, \left|G(s,s)\right|, \left|G\left(t^*,s\right)\right|, \left|G(1,s)\right|\right\}.$$

Now we prove $G(0,s)$, $G(s,s)$, $G(t^*,s)$, and $G(1,s)$ are all nonnegative.

For $G(0,s)$, we have

$$G(0,s) = \frac{1}{\Gamma(\nu)}\left[\frac{\delta\gamma(\nu-1)}{(1-\delta)(\gamma-1)}(1-s)^{\nu-2} - \frac{\delta}{1-\delta}(1-s)^{\nu-1}\right],$$
$$G_s'(0,s) = \frac{\delta(\nu-1)(1-s)^{\nu-3}}{(1-\delta)\Gamma(\nu)}\left[(1-s) + \frac{\gamma(2-\nu)}{\gamma-1}\right] \ge 0,$$

which means that $G(0,s)$ is increasing for $s \in [0,1]$. Considering that $\gamma < \frac{1}{2-\nu}$ in case of $1 < \gamma \le 1 + \frac{(\nu-1)\delta}{2-\nu}$ and $0 < \delta < 1$, we have, for $s \in [0,1]$,

$$G(0,s) \ge G(0,0) = \frac{\delta}{\Gamma(\nu)(1-\delta)(\gamma-1)}\left[1-\gamma(2-\nu)\right] > 0. \tag{20}$$

Inequalities(16) and (20) show that $G(s,s) > 0$.

For $G(t^*,s)$, we have

$$\begin{aligned} G\left(t^*,s\right) &= \frac{(1-s)^{\nu-2}}{\Gamma(\nu)}\left\{\frac{\delta\gamma(\nu-1)}{(\gamma-1)(1-\delta)} + \left[(2-\nu)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\nu-1}{\nu-2}} + \frac{\delta}{1-\delta}\right](s-1)\right. \\ &\quad \left. + \frac{\gamma(\nu-1)}{\gamma-1}s\right\} \\ &= \frac{(1-s)^{\nu-2}}{\Gamma(\nu)}g(s), \end{aligned} \tag{21}$$

where

$$g(s) = \frac{\delta\gamma(\nu-1)}{(\gamma-1)(1-\delta)} + \left[(2-\nu)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\nu-1}{\nu-2}} + \frac{\delta}{1-\delta}\right](s-1) + \frac{\gamma(\nu-1)}{\gamma-1}s, \quad s \in [0,1].$$

Obviously, $g(s)$ is increasing on $[0,1]$, and thus

$$g(s) \ge g(0) = \frac{\delta\gamma(\nu-1)}{(\gamma-1)(1-\delta)} - (2-\nu)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\nu-1}{\nu-2}} - \frac{\delta}{1-\delta}, \quad s \in [0,1]. \tag{22}$$

Let $k(t) = \frac{\delta(\nu-1)}{1-\delta}t - (2-\nu)t^{\frac{\nu-1}{\nu-2}} - \frac{\delta}{1-\delta}$, $t \in [1,+\infty)$. Then

$$k'(t) = \frac{\delta(\nu-1)}{1-\delta} + (\nu-1)t^{\frac{1}{\nu-2}} \ge 0, \tag{23}$$

which mean that $k(t)$ is increasing in $[1,+\infty)$. Letting $k(t_0) = 0$, we get $t_0 = \frac{(1-\delta)(2-\nu)t_0^{\frac{\nu-1}{\nu-2}}+\delta}{\delta(\nu-1)}$ and

$$t_0 - 1 = \frac{(1-\delta)(2-\nu)t_0^{\frac{\nu-1}{\nu-2}} + \delta(2-\nu)}{\delta(\nu-1)} > 0.$$

Then

$$\begin{aligned}\frac{t_0}{t_0-1} &= 1+\frac{1}{t_0-1}\\ &= 1+\frac{(\nu-1)\delta}{\delta(2-\nu)+(1-\delta)(2-\nu)t_0^{\frac{\nu-1}{\nu-2}}}\\ &> 1+\frac{(\nu-1)\delta}{2-\nu}\geq\gamma,\end{aligned}$$

that is, $t_0<\frac{\gamma}{\gamma-1}$. By (23) we obtain $g(0)=k(\frac{\gamma}{\gamma-1})\geq k(t_0)=0$, and thus

$$g(s)\geq g(0)\geq 0. \tag{24}$$

From (21) and (24) it follows that $G(t^*,s)\geq 0$.

Since $G(t^*,s)\geq 0$, by (19) it follows that $G(1,s)\geq 0$

Above all, we conclude that

$$G(t,s)\geq 0,\quad (t,s)\in[0,1]\times[0,1]$$

and

$$\max_{0\leq t\leq 1}\bigl|G(t,s)\bigr| = \max\bigl\{G(s,s),G(1,s)\bigr\}.$$

Since $\gamma<\frac{1}{2-\nu}$ in the case of $1<\gamma\leq 1+\frac{(\nu-1)\delta}{2-\nu}$ and $0<\delta<1$, we get

$$\begin{aligned}G(s,s)-G(1,s) &= \frac{(1-\nu)(1-s)^{\nu-2}}{\Gamma(\nu)}\biggl[\frac{\delta\gamma(1-s)+\gamma s}{(1-\gamma)(1-\delta)}-\frac{\gamma}{(1-\gamma)(1-\delta)}+(1-s)^{\nu-1}\biggr]\\ &= \frac{(1-s)^{\nu-1}}{\Gamma(\nu)(\gamma-1)}\bigl[(2-\nu)\gamma-1\bigr]\leq 0,\end{aligned}$$

so

$$\max_{0\leq t\leq 1}\bigl|G(t,s)\bigr| = G(1,s) = \frac{(1-s)^{\nu-2}}{\Gamma(\nu)(1-\delta)(\gamma-1)}\bigl[\gamma(\nu-1)-(\gamma-1)(1-s)\bigr].$$

(iii) By (ii) we have

$$\begin{aligned}\int_0^1\bigl|G(t,s)\bigr|\,ds &\leq \int_0^1 G(1,s)\,ds\\ &= \int_0^1\frac{(1-s)^{\nu-2}}{\Gamma(\nu)(1-\delta)(\gamma-1)}\bigl[\gamma(\nu-1)-(\gamma-1)(1-s)\bigr]\,ds\\ &= \frac{\gamma(\nu-1)+1}{\Gamma(\nu+1)(1-\delta)(\gamma-1)}.\end{aligned}$$

□

Lemma 2.5 *When $\delta\in(1,+\infty)$ and $\gamma\in(1,\frac{1}{2-\nu}]$, Green's function $G(t,s)$ satisfies the following properties*:

(i) $G(t,s)\leq 0$, $(t,s)\in[0,1]\times[0,1]$;

(ii) *For* $s \in [0,1]$,

$$\begin{aligned}
&\max_{0\le t\le 1}\bigl|G(t,s)\bigr| \\
&\quad \le \frac{(1-s)^{\nu-2}}{\Gamma(\nu)(\delta-1)(\gamma-1)} \\
&\qquad \times \left\{\delta\gamma(\nu-1)-\left[\delta(\gamma-1)-\gamma(2-\nu)(\delta-1)\left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{2-\nu}}\right](1-s)\right\};
\end{aligned}$$

(iii) $\int_0^1 |G(t,s)|\,ds \le \frac{1}{\Gamma(\nu+1)(\delta-1)(\gamma-1)}\{\delta[1+\gamma(\nu-1)]+(\delta-1)\gamma(2-\nu)(\frac{\gamma-1}{\gamma})^{\frac{1}{2-\nu}}\}$.

Proof We first prove (i) and (ii). For $s \in [0,1]$ and $t \in [0,s]$,

$$G'_t(t,s)=\frac{\gamma(\nu-1)(1-s)^{\nu-2}}{(\gamma-1)\Gamma(\nu)}\ge 0,$$

which means

$$G(0,s)\le G(t,s)\le G(s,s),\quad t\in[0,s]. \tag{25}$$

When $t \in [s,1]$,

$$\begin{aligned}
G'_t(t,s)&=\frac{(\nu-1)}{\Gamma(\nu)}\left[\frac{\gamma}{\gamma-1}(1-s)^{\nu-2}-(t-s)^{\nu-2}\right],\\
G''_{tt}(t,s)&=\frac{1}{\Gamma(\nu)}(\nu-1)(2-\nu)(t-s)^{\nu-3}\ge 0.
\end{aligned} \tag{26}$$

Letting $G'_t(t,s)=0$, we get $t^*=(\frac{\gamma}{\gamma-1})^{\frac{1}{\nu-2}}(1-s)+s\in[s,1]$. Combining (26), for $s \in [0,1]$, we have

$$G\bigl(t^*,s\bigr)\le G(t,s)\le G(s,s),\quad t\in\bigl[s,t^*\bigr], \tag{27}$$

$$G\bigl(t^*,s\bigr)\le G(t,s)\le G(1,s),\quad t\in\bigl[t^*,1\bigr]. \tag{28}$$

Inequalities (25), (27), and (28) show that, for $s \in [0,1]$,

$$\max_{0\le t\le 1}\bigl|G(t,s)\bigr|=\max\bigl\{\bigl|G(0,s)\bigr|,\bigl|G(s,s)\bigr|,\bigl|G\bigl(t^*,s\bigr)\bigr|,\bigl|G(1,s)\bigr|\bigr\}.$$

We now prove that $G(0,s)$, $G(s,s)$, $G(t^*,s)$, $G(1,s)$ are all nonpositive.

For $G(s,s)$, we have

$$\begin{aligned}
G(s,s)&=\frac{(1-s)^{\nu-2}}{\Gamma(\nu)(\delta-1)(\gamma-1)}\bigl\{\bigl[(2-\nu)\gamma-1\bigr]\delta+\bigl[(\nu-1)\delta\gamma-(\gamma-1)\delta-(\nu-1)\gamma\bigr]s\bigr\}\\
&=\frac{(1-s)^{\nu-2}}{\Gamma(\nu)(\delta-1)(\gamma-1)}L(s),
\end{aligned}$$

where

$$L(s)=\bigl[(2-\nu)\gamma-1\bigr]\delta+\bigl[(\nu-1)\delta\gamma-(\gamma-1)\delta-(\nu-1)\gamma\bigr]s,\quad s\in[0,1].$$

We have

$$L(0) = \delta\big[(2-\nu)\gamma - 1\big] \le 0, \qquad L(1) = -(\nu-1)\gamma < 0,$$

which means that $L(s) \le 0$, $s \in [0,1]$, and therefore

$$G(s,s) \le 0, \quad s \in [0,1].$$

Inequalities $G(0,s) \le 0$ and $G(t^*,s) \le 0$ follow from (25), (27), and $G(s,s) \le 0$.
For $G(1,s)$, we have

$$G(1,s) = \frac{1}{\Gamma(\nu)}\left[\frac{\gamma(\nu-1)}{(1-\delta)(\gamma-1)}(1-s)^{\nu-2} - \frac{1}{1-\delta}(1-s)^{\nu-1}\right],$$
$$G'_s(1,s) = \frac{-(\nu-1)(1-s)^{\nu-3}}{\Gamma(\nu)(\delta-1)(\gamma-1)}\big[(2-\nu)\gamma + (\gamma-1)(1-s)\big] \le 0,$$

and thus, for $s \in [0,1]$,

$$G(1,s) \le G(1,0) = \frac{1}{\Gamma(\nu)(\delta-1)(\gamma-1)}\big[\gamma(2-\nu) - 1\big] \le 0.$$

Above all, we get that $G(t,s) \le 0$ and, for $s \in [0,1]$,

$$\max_{0\le t\le 1}\big|G(t,s)\big| = \max\big\{-G(0,s), -G\big(t^*,s\big)\big\}.$$

We can easily compute that

$$-G\big(t^*,s\big) = \frac{(1-s)^{\nu-2}}{\Gamma(\nu)}h_1(s),$$

where

$$\begin{aligned} h_1(s) = {} & \left\{\frac{\delta[1-\gamma(2-\nu)]}{(\delta-1)(\gamma-1)} + (2-\nu)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\nu-1}{\nu-2}}\right\} \\ & - \left\{\frac{\delta[1-\gamma(2-\nu)]}{(\delta-1)(\gamma-1)} + (2-\nu)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\nu-1}{\nu-2}} - \frac{\gamma(\nu-1)}{(\delta-1)(\gamma-1)}\right\}s, \end{aligned}$$

and

$$-G(0,s) = \frac{(1-s)^{\nu-2}}{\Gamma(\nu)}h_2(s),$$

where

$$h_2(s) = \frac{\delta[1-\gamma(2-\nu)]}{(\delta-1)(\gamma-1)} + \frac{\delta}{\delta-1}s.$$

Obviously,

$$h_1(0) = \frac{\delta[1-\gamma(2-\nu)]}{(\delta-1)(\gamma-1)} + (2-\nu)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\nu-1}{\nu-2}} > \frac{\delta[1-\gamma(2-\nu)]}{(\delta-1)(\gamma-1)} = h_2(0) > 0,$$
$$h_2(1) = \frac{\delta\gamma(\nu-1)}{(\delta-1)(\gamma-1)} > \frac{\gamma(\nu-1)}{(\delta-1)(\gamma-1)} = h_1(1) > 0.$$

So, if we make a line $H(s)$ through $(0, h_1(0))$ and $(1, h_2(1))$, that is,

$$H(s) = \frac{\delta[1-\gamma(2-\nu)]}{(\delta-1)(\gamma-1)} + (2-\nu)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\nu-1}{\nu-2}} + \left[\frac{\delta}{\delta-1} - (2-\nu)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\nu-1}{\nu-2}}\right]s,$$

then we have

$$0 \le h_1(s), h_2(s) \le H(s), \quad s \in [0,1].$$

Therefore, for $s \in [0,1]$,

$$\begin{aligned}
&\max_{0\le t\le 1}\left|G(t,s)\right| \\
&\quad = \max\left\{-G(0,s), -G\left(t^*,s\right)\right\} \\
&\quad = \frac{(1-s)^{\nu-2}}{\Gamma(\nu)}\max\left\{h_1(s), h_2(s))\right\} \\
&\quad \le \frac{(1-s)^{\nu-2}}{\Gamma(\nu)}H(s) \\
&\quad = \frac{(1-s)^{\nu-2}}{\Gamma(\nu)(\delta-1)(\gamma-1)}\left\{\delta\gamma(\nu-1) - \left[\delta(\gamma-1) - \gamma(2-\nu)(\delta-1)\left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{2-\nu}}\right](1-s)\right\}.
\end{aligned}$$

(iii) easily s follows by (ii):

$$\begin{aligned}
&\int_0^1 \left|G(t,s)\right| ds \\
&\quad \le \int_0^1 \max_{0\le t\le 1}\left|G(t,s)\right| ds \\
&\quad \le \int_0^1 \frac{(1-s)^{\nu-2}}{\Gamma(\nu)(\delta-1)(\gamma-1)} \\
&\qquad \times \left\{\delta\gamma(\nu-1) - \left[\delta(\gamma-1) - \gamma(2-\nu)(\delta-1)\left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{2-\nu}}\right](1-s)\right\} ds \\
&\quad = \frac{1}{\Gamma(\nu+1)(\delta-1)(\gamma-1)}\left\{\delta\left[1+\gamma(\nu-1)\right] + (2-\nu)(\delta-1)\gamma\left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{2-\nu}}\right\}.
\end{aligned}$$
□

3 Main result

Theorem 3.1 *Suppose the boundary value problem* (5) *has a nonzero solution* $u(t)$.

(i) *If* $\delta \in (0,1)$ *and* $\gamma \in (0,1)$, *then*

$$\int_0^1 (1-s)^{\nu-2}\left[\gamma(\nu-1) + (1-\gamma)(1-s)\right]\left|q(s)\right| ds > \Gamma(\nu)(1-\delta)(1-\gamma);$$

(ii) *If* $\delta \in (1,+\infty)$ *and* $\gamma \in (0,1)$, *then*

$$\int_0^1 (1-s)^{\nu-2}\big[\gamma(\nu-1)+(1-\gamma)(1-s)\big]\big|q(s)\big|\,ds > \frac{\Gamma(\nu)(\delta-1)(1-\gamma)}{\delta};$$

(iii) *If* $\delta \in (0,1)$ *and* $\gamma \in (1, 1+\frac{(\nu-1)\delta}{2-\nu}]$, *then*

$$\int_0^1 (1-s)^{\nu-2}\big[\gamma(\nu-1)-(\gamma-1)(1-s)\big]\big|q(s)\big|\,ds > \Gamma(\nu)(1-\delta)(\gamma-1);$$

(iv) *if* $\delta \in (1,+\infty)$ *and* $\gamma \in (1, \frac{1}{2-\nu}]$, *then*

$$\begin{aligned}&\int_0^1 (1-s)^{\nu-2}\left\{\delta\gamma(\nu-1)-\left[\delta(\gamma-1)-\gamma(2-\nu)(\delta-1)\left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{2-\nu}}\right](1-s)\right\}\\ &\quad\times\big|q(s)\big|\,ds\\ &\quad> \Gamma(\nu)(\delta-1)(\gamma-1).\end{aligned}$$

Proof Let $u(t)$ be a nonzero solution of the boundary value problem (5). By Lemma 2.1 we have

$$u(t) = \int_0^1 G(t,s)q(s)u(s)\,ds.$$

Let $m = \max_{t\in[0,1]} |u(t)|$. Then

$$\big|u(t)\big| \le \int_0^1 \big|G(t,s)\big|\big|q(s)\big|\big|u(s)\big|\,ds \le m\int_0^1 \big|G(t,s)\big|\big|q(s)\big|\,ds.$$

Next, since $|G(t,s)||q(s)| \le \max_{0\le t\le 1}|G(t,s)||q(s)|$, but $|G(t,s)||q(s)| \not\equiv \max_{0\le t\le 1}|G(t,s)| \times |q(s)|$, we have

$$\int_0^1 \big|G(t,s)\big|\big|q(s)\big|\,ds < \int_0^1 \max_{0\le t\le 1}\big|G(t,s)\big|\big|q(s)\big|\,ds,$$

which means

$$\big|u(t)\big| < m\int_0^1 \max_{0\le t\le 1}\big|G(t,s)\big|\big|q(s)\big|\,ds,$$

that is,

$$1 < \int_0^1 \max_{0\le t\le 1}\big|G(t,s)\big|\big|q(s)\big|\,ds. \tag{29}$$

Substituting Lemma 2.2(ii), Lemma 2.3(ii), Lemma 2.4(ii), and Lemma 2.5(ii) into (29), we easily get statements (i), (ii), (iii), and (iv) of Theorem 3.1. □

By Theorem 3.1 we have the following conclusions.

Theorem 3.2

(i) *when* $\delta \in (0,1)$ *and* $\gamma \in (0,1)$, *if*

$$\int_0^1 (1-s)^{\nu-2}\big[\gamma(\nu-1)+(1-\gamma)(1-s)\big]\big|q(s)\big|\,ds \le \Gamma(\nu)(1-\delta)(1-\gamma),$$

then the boundary value problem (5) *has no nonzero solution.*

(ii) *when* $\delta \in (1,+\infty)$ *and* $\gamma \in (0,1)$, *if*

$$\int_0^1 (1-s)^{\nu-2}\big[\gamma(\nu-1)+(1-\gamma)(1-s)\big]\big|q(s)\big|\,ds \le \frac{\Gamma(\nu)(\delta-1)(1-\gamma)}{\delta},$$

then the boundary value problem (5) *has no nonzero solution.*

(iii) *when* $\delta \in (0,1)$ *and* $\gamma \in (1, 1+\frac{(\nu-1)\delta}{2-\nu}]$, *if*

$$\int_0^1 (1-s)^{\nu-2}\big[\gamma(\nu-1)-(\gamma-1)(1-s)\big]\big|q(s)\big|\,ds \le \Gamma(\nu)(1-\delta)(\gamma-1),$$

then the boundary value problem (5) *has no nonzero solution.*

(iv) *when* $\delta \in (1,+\infty)$ *and* $\gamma \in (1,\frac{1}{2-\nu}]$, *if*

$$\begin{aligned}&\int_0^1 (1-s)^{\nu-2}\Bigg\{\delta\gamma(\nu-1)\\ &\quad-\Bigg[\delta(\gamma-1)-\gamma(2-\nu)(\delta-1)\bigg(\frac{\gamma-1}{\gamma}\bigg)^{\frac{1}{2-\nu}}\Bigg](1-s)\Bigg\}\big|q(s)\big|\,ds\\ &\le \Gamma(\nu)(\delta-1)(\gamma-1),\end{aligned}$$

then the boundary value problem (5) *has no nonzero solution.*

Now we consider the existence of solutions to the following nonlinear boundary value problem:

$$\begin{cases} {}_0^C\mathscr{D}_t^{\nu}u(t)+f(t,u(t))=0,\\ u(0)=\delta u(1), \qquad u'(0)=\gamma u'(1). \end{cases} \tag{30}$$

Theorem 3.3 *Let* $f:[0,1]\times R\to R$ *be continuous and satisfy the following Lipschitz condition with Lipschitz constant* L:

$$\big|f(t,u_1)-f(t,u_2)\big| \le L|u_1-u_2| \tag{31}$$

for all $(t,u_1),(t,u_2)\in[0,1]\times R$. *If*

$$L < \begin{cases} \frac{\Gamma(\nu+1)(1-\delta)(1-\gamma)}{\gamma(\nu-1)+1}, & \delta\in(0,1),\gamma\in(0,1),\\ \frac{\Gamma(\nu+1)(\delta-1)(1-\gamma)}{\delta(\gamma\nu+1-\gamma)}, & \delta\in(1,+\infty),\gamma\in(0,1),\\ \frac{\Gamma(\nu+1)(1-\delta)(\gamma-1)}{\gamma(\nu-1)+1}, & \delta\in(0,1),\gamma\in(1,1+\frac{\delta(\nu-1)}{2-\nu}],\\ \frac{\Gamma(\nu+1)(\delta-1)(\gamma-1)}{\delta[1+\gamma(\nu-1)]+(\delta-1)\gamma(2-\nu)(\frac{\gamma-1}{\gamma})^{\frac{1}{2-\nu}}}, & \delta\in(1,+\infty],\gamma\in(1,\frac{1}{2-\nu}], \end{cases} \tag{32}$$

then (30) *has a unique solution.*

Proof Let E be the Banach space $C[0,1]$ with norm $\|u\| = \max_{t\in[0,1]} |u(t)|$.

By Lemma 2.1, $u \in E$ is a solution of (30) if and only if it satisfies the integral equation

$$u(t) = \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds.$$

Define the operator $T : E \to E$ by

$$Tu(t) = \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds.$$

Then T is completely continuous. We claim that T has a unique fixed point in E. In fact, for any $u_1, u_2 \in E$, we have

$$\begin{aligned} \bigl|Tu_1(t) - Tu_2(t)\bigr| &\le \int_0^1 \bigl|G(t,s)\bigr|\bigl|f\bigl(s,u_1(s)\bigr) - f\bigl(s,u_2(s)\bigr)\bigr|\,ds \\ &\le L\int_0^1 \bigl|G(t,s)\bigr|\bigl|u_1(s) - u_2(s)\bigr|\,ds \\ &\le L\int_0^1 \bigl|G(t,s)\bigr|\,ds\|u_1 - u_2\|. \end{aligned} \tag{33}$$

Substituting of Lemma 2.2(iii), Lemma 2.3(iii), Lemma 2.4(iii), and Lemma 2.5(iii) into (33), we conclude that T is a contraction mapping and thus obtain the desired result. □

4 Conclusion

In this paper, we study a linear fractional differential equation. Firstly, by obtaining the Green's function we derive a Lyapunov-type inequality for such a boundary value problem. Furthermore, we use the contraction mapping theorem to study the existence of a unique solution for the corresponding nonlinear problem.

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